LEAST ORDER, STABLE SOLUTION OF THE EXACT MODEL MATCHING PROBLEM

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Abstract. The set of all solutions of the minimal design problem is presented in parametric form. Then conditions are placed on the parameters, which conditions define the set of stable solutions of the minimal design problem. Solutions to the nonminimal model matching problem are also presented in parametric form, and it is shown how solutions of successively higher order can be searched for a stable solution. In this way a solution to the model matching problem can be found which has the lowest order consistent with a stability constraint.

Keywords. Linear systems; multivariable control systems; multivariable system design; stability; system order; system theory.

1. INTRODUCTION

It has been shown how to reformulate such problems as the minimal order system inverse problem (Wang and Davison, 1973a, 1973b), the minimal order dynamic observer problem (Wang and Davison, 1973b; Wolovich and co-workers, 1976) and the model following problem (Wolovich and co-workers 1976; Morse, 1973; Wolovich, 1974a) as a problem known as the Minimal Design Problem (MDP). The MDP can be stated thus: given a $p\times m$ rational transfer matrix, $T_1(s)$, of rank p and a $p\times q$ rational transfer matrix, $T_2(s)$, find an $m\times q$ proper, rational transfer matrix, $T(s)$, of minimal dynamic order (or McMillan degree) such that

$$T_1(s)T(s) = T_2(s) \tag{1.1}$$

If $T_1(s)$ has rank less than p, one can of course still study (1.1). Obviously, the row nullspace of $T_1(s)$ would have to be a subspace of the row nullspace of $T_2(s)$, in which case the problem can be reduced to one where rank T_1 is the number of rows of T_1. The problem has been solved in various ways (Wang and Davison, 1973a, 1973b; Wolovich and co-workers, 1975, 1976; Morse, 1976; Moore and Silverman, 1972). It should be noted that if $p \geq m$ the MDP has either no solution or a unique solution which can easily be found. Normally then $p < m$. The first contribution of this paper is to show how all solutions of the MDP can be obtained and expressed in parametric form.

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It is also of interest to solve the MDP with the further constraint that the solution be stable. This extra constraint may be essential for some problems such as the dynamic observer problem or it may merely be a desirable design criterion. Some progress has been made towards solving this stable MDP (Wolovich and co-workers, 1975, 1976; Anderson and Scott, 1977) but the results have been incomplete. This paper indicates how the parameters can be constrained so that from amongst all solutions of the MDP the stable ones can be selected.

If there is no stable solution to the MDP it is desirable to find a stable solution of (1.1) of least order. The question of whether or not there exists a stable solution of (1.1) is readily answered by Wolovich and co-workers (1975); however, no guide is available as to the least possible order of such a solution. In this paper an algorithm is presented which determines the least order stable solution to (1.1).

In essence, the algorithm constructs a parametric representation of all solutions of a certain degree (in fact, with a certain set of column degrees). Then one seeks to choose the parameters in order to have a stable solution. The problem of finding a proper solution to (1.1) without the constraint on minimality of order is known as the exact model matching problem (Wolovich, 1974a).

The system descriptions used are almost all matrix fraction descriptions. While model matching has been viewed with state-variable concepts, (Moore and Silverman, 1972), questions of minimality and stability have not been well resolved.

Section 2 discusses a special form of matrix which is used to present, in parametric form, all minimal bases (Forney, 1975) of a vector space. The particular vector space in which we shall be interested is the kernel of $[T_1(s) \vdots -T_2(s)]$. Minimal bases of this vector space may be used to solve the MDP as described in Section 3. In Section 4 methods are examined which reduce the number of parameters required to express the solutions.

Section 5 leads on to a discussion of the least order stable exact model matching problem.

Perhaps the most important concepts of the paper are the parameterisation procedures for solutions, not just of the minimal design problem, but also of the model matching problem, with the elimination of excess parameters.

Before proceeding, it will be helpful to define a few terms which will be used later. The definitions given are based on those given by Forney (1975).

The degree of an n-tuple $(g_1, g_2, \ldots, g_n)^T$ of polynomials, is the greatest degree of its components g_j, $1 < j < n$.

If G is an $n \times k$ polynomial matrix with columns c_i, the i-th index or column degree of G is defined as the degree of c_i, $1 < i < k$, and the order of G is defined as

$$\sum_{i=1}^{k} \text{degree } (c_i).$$

Note: (1) A matrix whose columns constitute a basis of a vector space will also be referred to as a basis of the vector space.

(2) All vector spaces referred to in this paper will be vector spaces of n-tuples over F(s), the field of rational functions of s.

If V is a k-dimensional vector space of n-tuples over F(s), then a minimal basis of V is an n×k polynomial matrix G such that G is a basis of V and G has least order among all polynomial bases of V.

The invariant dynamical indices (or just indices) of a vector space V of n-tuples over F(s), are the indices of any minimal basis of V. The invariant dynamical indices will be denoted by $\nu(i)$, i = 1, 2, ..., dim V.

If G is a n×k polynomial matrix with indices $\nu(i)$, $1 \le i \le k$, its high order coefficient matrix, denoted $[G]_h$, is the n×k real matrix whose i-th column consists of the coefficients of $x^{\nu(i)}$ in the i-th column of G.

Note: Since we will usually be referring to indices of basis matrices which are the same as the indices of the vector space, the alternative usages of the term index (and indices) will not create confusion. In the event of mention of the index of a matrix, which is not a minimal basis, being required, we will use the term column degree.

2. THE ν-FORM MATRIX AND THE PARAMETRIC REPRESENTATION OF MINIMAL BASES

Let all minimal bases of a k-dimensional vector space V be ordered according to increasing indices $\nu(1) \le \nu(2) \le \ldots \le \nu(k)$, and let

$$\nu(1)=\ldots=\nu(r_1)<\nu(r_1+1)=\ldots=\nu(r_1+r_2)<\nu(r_1+r_2+1) \ldots<\nu(k)$$

That is, the columns of the matrix of basis vectors are arranged in groups of r_i columns with the same index, the group of least index being first etc. For a particular vector space V the $\nu(i)$ are invariant (Forney, 1975) and hence so are the r_i. Let the number of groups of columns with equal indices be s, then $\nu(r_1+r_2+ \ldots +r_s) = \nu(k)$.
Define

$$\sigma_i = \sum_{j=1}^{i-1} r_j + 1, \quad i = 1, 2, \ldots s \tag{2.1}$$

So there are r_1 columns in the basis with index $\nu(\sigma_1) = \nu(1)$, r_2 columns with index $\nu(\sigma_2) = \nu(r_1+1)$, ..., r_i columns with index $\nu(\sigma_i)$.

A ν-form matrix corresponding to the indices of V is a k×k upper block triangular, unimodular matrix with the following structure.

$$N = \begin{bmatrix} D_1 & U_{12} & \ldots & U_{1s} \\ 0 & D_2 & \ldots & U_{2s} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & D_s \end{bmatrix} \tag{2.2}$$

The diagonal blocks D_i are real, non-singular $r_i \times r_i$ matrices. The entries above the diagonal blocks are polynomials with bounded degrees, those in the $r_i \times r_j$ block U_{ij} having degree less than or equal to $\nu(\sigma_j) - \nu(\sigma_i)$. The ν-form matrix has been termed a "properly indexed unimodular matrix" by Wolovich (1974b) but we use the name ν-form for brevity. Note that a ν-form matrix may correspond to any ordered set of integers $\{\nu(i)\}$, but in this paper ν-form matrices will always correspond to the indices of an appropriate vector space. We will now set out some of the properties of ν-form matrices.

Lemma 1: The ν-form matrices possess the following group structure. The product of two ν-form matrices corresponding to the same set of indices $\{\nu(i)\}$ is a ν-form matrix corresponding to $\{\nu(i)\}$, and any ν-form matrix corresponding to a set of indices $\{\nu(i)\}$ has an inverse and this inverse is a ν-form matrix corresponding to $\{\nu(i)\}$.

Proof: These results are established by Wolovich (1974b).

Lemma 2: Any minimal basis of a vector space V can be reduced to "Forney's echelon form" (Forney, 1975) upon post multiplication by an appropriate ν-form matrix corresponding to the indices of V.

Proof: It can readily be shown that the algorithm presented by Forney (1975) for constructing the Forney echelon form minimal basis from any given minimal basis, is equivalent to right multiplication of the given minimal basis by a ν-form matrix. The details are omitted.

We now wish to present a theorem which says that if we can compute a minimal basis $\bar{G}$ of V, then every minimal basis G of V can be expressed parametrically as $G = \bar{G}N$ where N is a ν-form parameter matrix. The parameters are the real elements of the diagonal blocks of N and also the coefficients of the polynomials above the diagonal blocks. There is a finite number of parameters because the polynomials have bounded degrees which depend on the indices of V.

Theorem 1: If $\bar{G}$ is any degree ordered minimal basis of a k-dimensional vector space V of n-vectors over the field of rational functions of s, then $G = \bar{G}N$ is a degree ordered minimal basis of V for any ν-form matrix N corresponding to the indices of V. Conversely, any minimal basis G can be expressed in the form $G = \bar{G}\bar{N}$ for some ν-form matrix $\bar{N}$.

Proof: Let $G = \bar{G}N$ where N is any ν-form matrix. Now, since $\bar{G}$ is a polynomial basis of V and N is a unimodular polynomial matrix, then G is a polynomial basis of V (not necessarily minimal), (Forney, 1975). It remains to be shown that G is degree ordered and minimal.

Let $\bar{G} = \left[\bar{G}_1 \vdots \bar{G}_2 \vdots \cdots \vdots \bar{G}_s\right]$ where the $\bar{G}_i$ are $n \times r_i$ blocks whose entries have degree less than or equal to $\nu(\sigma_i)$. Let N be defined as in (2.2). Then the j-th block of G is the $n \times r_j$ matrix

$$G_j = \sum_{i=1}^{j-1} \bar{G}_i U_{ij} + \bar{G}_j D_j, \quad j = 1, 2, \ldots, s \quad (2.3)$$

where $G = \left[G_1 \vdots G_2 \vdots \ldots \vdots G_s\right]$.

The entries of $\bar{G}_i$ have degrees less than or equal to $\nu(\sigma_i)$ and the entries of U_{ij} have degrees less than or equal to $\nu(\sigma_j) - \nu(\sigma_i)$ (for $j = 1, 2, \ldots, s-1$) and the entries of D_j have zero degree. So the entries of G_j have degrees less than or equal to $\nu(\sigma_j)$. Under this condition G must be degree ordered and minimal otherwise a basis of lower order than the minimal basis would exist.

To demonstrate the converse, use must be made of Forney's echelon form of a minimal basis (Forney, 1975). Denote the (unique) Forney echelon form minimal basis for V by G_E. From Lemma 2, for any minimal basis G of V there exists a ν-form matrix $\tilde{N}$ such that $G = G_E\tilde{N}$. Now G_E is unique (Forney, 1975) so for a given basis $\bar{G}$ there is a ν-form matrix $\hat{N}$ such that $\bar{G} = G_E\hat{N}$. Hence $G = \bar{G}\hat{N}^{-1}\tilde{N}$ and $N = \hat{N}^{-1}\tilde{N}$ is ν-form matrix by Lemma 1.

Theorem 1 tells us that for a vector space V, all minimal bases can be expressed as $\bar{G}N$ where $\bar{G}$ is any given minimal basis and N is a ν-form matrix of parameters. A $\bar{G}$ can be computed as outlined in (Forney, 1975). (Wolovich and co-workers (1975, 1976) mention a computer program for performing this computation). So the collection of minimal bases can thus be expressed using a finite number of parameters.

3. THE MINIMAL DESIGN PROBLEM

Forney (1975) has shown how to proceed from a minimal basis of a particular vector space to a solution of the MDP. Having seen in the preceding section how to obtain all minimal bases of a vector space we can obtain all solutions of the MDP. The relevant theorem (Wolovich and co-workers, 1975, 1976; Forney, 1975) is as follows.

Theorem 2: Let

$$K(s) = \begin{bmatrix} K_m(s) \\ \hline K_q(s) \end{bmatrix} \quad (3.1)$$

be a column degree ordered $(m+q) \times (m+q-p)$ matrix[2] whose columns are a minimal basis of the vector space kernel $\left[T_1(s) \vdots -T_2(s)\right]$. $K_m(s)$ is the first m rows and $K_q(s)$ the last q rows of K(s). $T_1(s)$ and $T_2(s)$ are as described in equation (1.1). Write the high order coefficient matrix of K(s) as

$$\left[K(s)\right]_h = \begin{bmatrix} K_{m\gamma} \\ \hline K_{q\gamma} \end{bmatrix} \quad (3.2)$$

The MDP has a solution if and only if

$$\text{rank}\left[K_{q\gamma}\right] = q \quad (3.3)$$

If condition (3.3) holds, consider the first q columns of K(s) for which the corresponding columns of $K_{q\gamma}$ are linearly independent. Denote these columns by

$$S(s) = \begin{bmatrix} Q(s) \\ \hline P(s) \end{bmatrix} \quad (3.4)$$

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[2] The dimensions follow from those of $T_1(s)$ and $T_2(s)$, as well as the fact that rank $T_1 = p$, see the introduction.

$\{\nu(i)\}$ with $\nu^*(j) \le \nu^*(j+1)$ and let $\bar{V}$ be any vector space with $\{\nu^*(j)\}$ as indices. If $\bar{N}_A(s)$ is a pseudo ν-form matrix corresponding to $\{\nu(i)\}$ and $\{\nu^*(j)\}$, if L is a ν-form matrix corresponding to $\{\nu(i)\}$ and M a ν-form matrix corresponding to $\{\nu^*(j)\}$, then $N_A = L\bar{N}_A M$ is a pseudo ν-form matrix corresponding to $\{\nu(i)\}$ and $\{\nu^*(j)\}$.

Proof: The proof is straightforward and will be omitted.

The same solution (3.5) to the MDP is obtained if the q columns of S(s) in equation (3.4) are replaced by S(s)M(s) where M(s) is an arbitrary q×q ν-form matrix of the vector space $\bar{V}$, the space spanned by the columns of S(s):

$$S(s)M(s) = \begin{bmatrix} Q(s) \\ \hdashline P(s) \end{bmatrix} M(s) \tag{4.2}$$

$$= K(s)\bar{N}_A(s)M(s) \tag{4.3}$$

where K(s) is the known minimal basis of kernel $[T_1(s) \; \vdots \; -T_2(s)]$ and $\bar{N}_A(s)$ is an appropriate pseudo ν-form parameter matrix. Hence all solutions (3.5) could be obtained using $N_A(s) = \bar{N}_A(s)M(s)$ instead of $\bar{N}_A(s)$. We may then choose M(s) to reduce the number of parameters required to represent all solutions of the MDP.

Let $\bar{N}_A(s)$ be as described in (4.1) and choose $M(s) = \hat{M}^{-1}(s)$ where $\hat{M}(s)$ is a q×q ν-form matrix of $\bar{V}$, determined as follows. For the first t_1 rows of $\hat{M}(s)$ select, from the first r_1 rows of $\bar{N}_A(s)$, the last t_1 rows for which the corresponding rows of $\bar{D}_1$ are linearly independent. For the next t_2 rows of $\hat{M}(s)$ select, from the next r_2 rows of $\bar{N}_A(s)$, the last t_2 rows for which the corresponding rows of $\bar{D}_2$ are linearly independent. Continue in this way until q rows have been selected for $\hat{M}(s)$. Then $\hat{M}(s)$ is a q×q ν-form matrix corresponding to the indices of $\bar{V}$ with $t_i \times t_i$ diagonal blocks. Then by Lemma 1, the inverse of $\hat{M}(s)$ is a permissible choice of M(s). So

$$N_A(s) = \bar{N}_A(s)M(s) = \bar{N}_A(s)\hat{M}^{-1}(s)$$

$$= \begin{bmatrix} W_1 & V_{12} & \cdots & V_{1s} \\ 0 & W_2 & \cdots & V_{2s} \\ \vdots & & & \\ 0 & 0 & \cdots & W_s \end{bmatrix} \tag{4.4}$$

where W_i is a real $r_i \times t_i$ block (of free parameters) in column reduced echelon form, a standard canonical form for real matrices (Loomis and Sternberg, 1968). The $r_i \times t_j$ blocks V_{ij} have zero rows corresponding to the rows in W_i which have the 1's characteristic of the echelon form. The remaining entries of V_{ij} are polynomials satisfying the usual bounds on degrees and whose coefficients are free parameters. Clearly $N_A(s)$ has fewer parameters than $\bar{N}_A(s)$; the reduction is achieved by eliminating overlap of solutions (that is the same solution (3.5) occurring when different sets of numbers are substituted for the parameters in $\bar{N}_A(s)$.

When we consider the construction of the original pseudo ν-form matrix $\bar{N}_A(s)$ it is clear that every entry in and above the diagonal blocks $\bar{D}_i$ is non zero (i.e. not identically zero). Consequently the very last t_i rows of all the above sets of r_i rows of $\bar{N}_A$ will, generically, satisfy the condition of linear independence of the corresponding rows of $\bar{D}_i$. So W_i and V_{ij} will generically have the form

$$W_i = \begin{bmatrix} Y_i \\ \hdashline I_{ti} \end{bmatrix} \quad ; \quad V_{ij} = \begin{bmatrix} Z_{ij} \\ \hdashline 0_{ij} \end{bmatrix} \tag{4.5}$$

where I_{ti} is the $t_i \times t_i$ unit matrix, 0_{ij} is the $t_i \times t_j$ zero matrix, Y_i is a $(r_i - t_i) \times t_i$ real matrix, all of whose entries are free parameters, and Z_{ij} is a $(r-t_i) \times t_j$ polynomial matrix whose entries have degree $\nu^*(\sigma_j) - \nu(\sigma_i)$ and whose real coefficients are all free parameters.

However, one can readily conceive of cases where, upon assigning numerical values to the free parameters in $\bar{N}_A$, the very last t_i rows of $\bar{D}_i$ will not be linearly independent and (4.5) will not be the correct echelon form. Nevertheless if we allow the free parameters to be real or tend to infinity (in a sense made clear below in an example) then the pathological cases are covered by the structure of (4.5).

For example, if $r_1 = 3$ and $t_1 = 2$, a particular $N_A(s)$ could have any of the following three echelon forms with r_{ij} being real and p_{ij} real polynomials.

$$\left[\begin{array}{cc|ccc} r_{11} & r_{12} & p_{13} & p_{14} & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \hline & & & & \\ 0 & & & & \end{array}\right] \tag{4.6a}$$

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & r_{22} & p_{23} & p_{24} & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \hline & & & & \\ 0 & & & & \end{array}\right] \tag{4.6b}$$

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & p_{33} & p_{34} & \cdots \\ \hline & 0 & & & \end{array}\right] \qquad (4.6c)$$

Further column manipulation of (4.6a) by means of a slightly altered ν-form $M(s)$ yields the following two equivalent matrices

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{r_{11}} & \frac{-r_{12}}{r_{11}} & \frac{-p_{13}}{r_{11}} & \frac{-p_{14}}{r_{11}} & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \hline & 0 & & & \end{array}\right] \qquad (4.7a)$$

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \frac{1}{r_{12}} & \frac{-r_{11}}{r_{12}} & \frac{-p_{13}}{r_{12}} & \frac{-p_{14}}{r_{12}} & \cdots \\ \hline & 0 & & & \end{array}\right] \qquad (4.7b)$$

If r_{11}, r_{12} and certain coefficients of the polynomials p_{1j} approach infinity at related rates, then (4.7a) approaches the same form as (4.6b) with

$$r_{22} = -\frac{r_{12}}{r_{11}}, \quad p_{23} = -\frac{p_{13}}{r_{11}}, \quad \text{etc.}$$

Likewise if in (4.7b), r_{12} and certain coefficients of p_{1j} approach infinity then (4.7b) can tend to the same form as (4.6c) with

$$p_{33} = -\frac{p_{13}}{r_{12}}, \quad \text{etc.}$$

So by allowing parameters to be real and to tend to infinity the structure (4.6a) covers all cases.

Let us summarize:

MDP CONSTRUCTION PROCEDURE. The parametrically expressed set of all solutions to the MDP (1.1) is constructed as follows. First compute a minimal basis of kernel $[T_1(s) \vdots -T_2(s)]$. Then multiply this minimal basis by an appropriate pseudo ν-form matrix $N_A(s)$ as described above, see (4.4) and (4.5). Partition the resulting matrix, $K(s)N_A(s)$, as $[Q(s)^T \vdots P(s)^T]^T$ and obtain the solutions $T(s) = Q(s)P^{-1}(s)$. All solutions of the MDP are obtained, provided parameters are allowed to tend to infinity in an appropriate manner.

The question arises as to whether these solutions will be proper. The answer is that they will be at least generically proper. There is a choice for $N_A(s)$ which simply picks out the first q columns of $K(s)$ for which the corresponding columns of $K_{q\gamma}$ are linearly independent. Because one $N_A(s)$ has this property, it is not hard to show that almost all have it. On the other hand, if there are choices of q columns of $K(s)$ with the correct controllability indices for which the corresponding matrix $K_{q\gamma}$ does not have linearly independent columns, as well there may be in a specific problem, this means that there exist specific choices of $N_A(s)$ which would result in an improper $T(s)$.

Finally in this section, we comment on the question of stability. The condition for stability is that the determinant of $P(s)$ be Hurwitz (i.e. have all its roots in the open left half plane). This defines a set of polynomial inequalities in the parameters, the solution of which defines that set of real numbers which can be inserted into the parameters of $T_i(s)$ to give the set of stable solutions of the MDP. Solution of a set of polynomial inequalities has been dealt with in (Anderson, 1975, 1976b). Of course the evaluation of the determinant of a matrix with literal entries, while in principle possible, may not be an easy task. A discussion of the problem can be found in (Collins, 1973).

5. NONMINIMAL SOLUTIONS TO THE MODEL MATCHING PROBLEM AND CONSTRUCTION OF A LEAST ORDER STABLE SOLUTION

If there are no stable solutions to the MDP, it may well be of interest to examine nonminimal solutions to obtain a stable solution. (Recall that the existence of a stable solution of some suitably high order can always be checked moderately easily, as noted in Section 1). Accordingly, we devote most of this section to describing a parametrisation of nonminimal solutions to the model matching problem. Incorporation of the stability constraint, at least in formal terms, is then easily achieved.

We shall first see how, with an excess number of parameters, a solution to the model-matching problem with prescribed column degrees can be achieved.

Lemma 4: With notation as in earlier sections, let $\nu^*(1) \le \nu^*(2) \ldots \le \nu^*(q)$ be a set of column degrees with $\nu(i) \le \nu^*(i)$. Let $\bar{N}(s)$ be a $(m+q-p) \times q$ polynomial matrix with $\partial[\bar{n}_{ij}(s)] = \nu^*(j) - \nu(i)$. (If $\nu^*(j) - \nu(i) < 0$, we take $\bar{n}_{ij}(s) = 0$). Then, generically the column degrees of $K(s)\bar{N}(s)$ are $\nu^*(1), \ldots, \nu^*(q)$. Conversely, suppose $\bar{N}(s)$ is a polynomial matrix such that $K(s)\bar{N}(s)$ has column degrees $\nu^*(1), \ldots, \nu^*(q)$. Then $\partial[n_{ij}(s)] \le \nu^*(j) - \nu(i)$ with, for each j, equality for at least one i.

identity matrix as a coefficient of highest degree terms in the first nonzero block and obtaining terms of lower degree in appropriate rows of blocks to the right of this nonzero block.

4. Then pass to the third last block row, etc.

In this way, $\bar{N}(s)$ is replaced by $N(s)$, a matrix with fewer parameters. No solutions of the model matching problem are thrown away in the process.

We remark that as with the minimal design problem, one can obtain certain solutions by letting parameters tend to infinity in a certain way, and by this means avoid the apparent gap in step 1, in which a certain adjustment is only generically possible.

Proper solutions $T(s)$ will, for a given set of $\nu^*(j)$, be either obtained generically, or not at all. It is easy in a given instance to establish whether proper solutions will be obtained.

It is clear that, as with the parametrised minimal degree solutions, we are in a position to write down polynomial inequalities involving the free parameters in $N(s)$ which, if satisfied by particular values of those parameters, constitute stability conditions for $T(s)$.

Solutions will have order $\sum_{j=1}^{q} \nu^*(j)$ and so by considering all sets $\{\nu^*(j)\}$ with a given sum we can construct all solutions of a given order. Hence, if there is no stable minimal order solution we may systematically test for stable solutions of progressively higher order until the least order stable solutions are found.

In this searching process, certain combinations of $\nu^*(j)$ can be omitted. It is sometimes possible to choose the $\nu^*(j)$ so that $\bar{N}(s)$ has the form

$$\bar{N}(s) = \begin{bmatrix} \tilde{N}(s) \\ 0 \end{bmatrix}$$

where $\tilde{N}(s)$ is a square matrix. (This will happen precisely when $\nu^*(q) < \nu(q+1)$ as may be checked). Then if

$$K(s) = \begin{bmatrix} \tilde{Q}(s) & X \\ \tilde{P}(s) & X \end{bmatrix}$$

(where X denotes entries whose particular form is not important), and if $K(s)\bar{N}(s) = [Q^T(s) \vdots P^T(s)]^T$, it follows that $T(s) = Q(s)P^{-1}(s) = (\tilde{Q}(s)\tilde{N}(s)(\tilde{P}(s)\tilde{N}(s))^{-1} = \tilde{Q}(s)\tilde{P}^{-1}(s)$. Thus no new solutions $T(s)$ to the model matching problem are really obtained; rather, only new matrix fraction decompositions of linear solutions are obtained.

CONCLUSION

In this paper we have shown how to find all solutions of the MDP in parametric form and then how to select stable solutions from among these. If there is no stable solution of the MDP we have shown how to find all solutions, of a given order, of the exact model matching problem. We have further indicated how, by considering orders which are progressively larger than the minimum, the least order stable solutions can be determined.

This approach to finding a least order stable solution can be contrasted with that suggested by Anderson and Scott (1977) : initial parameterization of all stable solutions, followed by the imposition of sets of polynomial equalities, with the property that there exists a real solution to a given set if and only if there exists a stable solution of a certain degree. Put another way, in this paper we first parameterise all solutions of the exact model matching problem of certain dimensional complexity, and then search for a stable solution; in (Anderson and Scott, 1977) we parameterize all stable solutions and then search for a solution of a certain dimensional complexity.

It is conceivable that state variable descriptions and methods could still be used. A substantial extension of (Moore and Silverman, 1972) would be necessary, and it is our conviction that decision algebra would at some stage be required, just as in the schemes of this paper and (Anderson and Scott, 1977) for the stable least order problem.

REFERENCES

B. Anderson et al. IEEE Trans.AC-20, 53.
B. Anderson et al. IEEE Trans.AC-22, 137.
B. Anderson et al. Proc. IEEE, 65, 849
G. Collins, Am.Math.M, 80, 725.
G. Forney, SIAM J.Contr. 13, 493.
L. Loomis et al. Advanced Calculus, Addison Wesley.
B. Moore et al. IEEE Trans.AC-17, 491.
A. Morse, IEEE Trans.AC-18, 346.
A. Morse, IEEE Trans.AC-21, 131.
S. Wang et al. IEEE Trans.AC-18,220.
S. Wang et al. Proc. '73 JACC, 269.
W. Wolovich, Linear Multivariable Systems, Springer.
W. Wolovich, Proc. '74 JACC, 177.
W. Wolovich et al. Proc. '75 JACC, 215.
W. Wolovich et al. IEEE Trans.AC-22, 88.