FORWARDS, BACKWARDS AND DYNAMICALLY REVERSIBLE MARKOVIAN MODELS OF SECOND-ORDER PROCESSES *

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1. Introduction

It is now commonplace to study finite-dimensional models of second-order processes. More explicitly, we often model processes with a given covariance as the output of a finite-dimensional linear system in state-space form with initial state $x(t_0)$ uncorrelated with its white noise input for $t \geq t_0$. These are often known as forwards Markovian models of the process. In several applications e.g., in smoothing problems, it is useful to consider backwards models in which the excitation starts at some time t_1 and runs "backwards" in time. Just reversing time in the forwards model will destroy the Markovianness and therefore a different construction has to be used. The appropriate backwards Markovian model was first described in [1] (see also [2]-[3]).

In this paper, we make a further clarification of the relations betweeen the forward and backward models and we study more fully the case of stationary processes corresponding to time-invariant models that are stable and commence at $t_0 = -\infty$, running forward, or at $t_1 = +\infty$, running backward. This specialization both lends greater intuitive content to some of the results, and allows more results to be obtained. [Moreover, time-varying versions of a number of the results could also be derived.]

Let us now give a preview of the main ideas of the paper. In Section 2, we analyse the scheme in [1-3] for passing from a forward model to a backward model in transfer function terms. It turns out that for single-input single-output systems, the backward model has the same zeros as the forward model and poles which are reflections through the $j\omega$-axis of the forward model poles. In Section 3, we describe the appropriate generalization of this notion with the aid of matrix fraction descriptions [4] of the forward and backward models.

In Section 4 and 5 we study situations in which the forward and associated backward models have the same transfer function matrix, to within sign change of the independent variable. As explained later, certain physical situations may lead to this happening. It turns out that such "self-dual" situations imply that the model output process has a statistical <u>time-reversibility</u> property, which is equivalent to the power spectrum being a symmetric matrix. Furthermore, we show that in the right coordinate basis, processes internal to the model (e.g., the state process when the model is given in matrix-fraction form) also have a generalized (so-called <u>dynamic</u>) reversibility property when considered along with the output process.

In Section 6 we discuss briefly the determination of self-dual models. Appendix I describes a different approach from that in [1] to the construction of the backwards Markovian model; this new approach due to G. Verghese, shows that the state and output processes of the forwards and the backwards models can be taken to be the same.

Finally, we should note that the results in Sec. 4-6 are closely related to ideas in the theory of passive networks, especially as regards the relations between external and internal reciprocity (see [5]-[8]). These connections will be made more explicit in the paper; we might mention that they have also led us to some new views of the network synthesis problem [9].

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2. Forward and Backward Factors

Let $\Phi(s)$ be an $m\times m$ square matrix of real rational functions of s satisfying the following conditions:

$$\Phi(s) \text{ is analytic for } s = j\omega,\ \omega \text{ real, including } \omega = \infty, \tag{2.1}$$

$$\Phi(s) = \Phi'(-s), \tag{2.2}$$

$$\Phi(j\omega) \text{ is nonnegative definite Hermitian for all real } \omega. \tag{2.3}$$

Then there exists a second-order process $y(\cdot)$ whose power spectral density matrix is $\Phi(j\omega)$ (or equivalently such that $\Phi(j\omega)$ is the Fourier transform of the covariance function of $y(\cdot)$) [7]. It is also well known that there are many such processes $y(\cdot)$. One class of such processes can be obtained by passing white-noise $u(\cdot)$ through a linear time-invariant filter with transfer function $W(s)$, where

$$\Phi(s) = W(s)W'(-s) \tag{2.4}$$

and

$W(s)$ is analytic for $\text{Re } s \geq 0$ (including $s = \infty$).

For rational $\Phi(s)$, $W(s)$ can also be chosen to be rational and therefore can also be described in state-space terms. For example, we can find matrices $\{F,G,H\}$ such that

$$W(s) = J + H(sI-F)^{-1}G \tag{2.5}$$

where F is a stable matrix (i.e., its eigenvalues have negative real parts). Now suppose the corresponding state-space realization $\{F,G,H,J\}$ is driven by a white-noise process $u(\cdot)$, and by a random initial condition $x(t_o) = x_o$ that is uncorrelated with $u(\cdot)$, i.e.,

$$E[u(t)u'(s)] = I\delta(t-s),\ Eu(t)x_o' \equiv 0,\ t > t_o \quad (2.6a)$$

Then provided the initial variance[†]

$$E\, x_o x_o' \overset{\Delta}{=} \Pi \quad (2.6b)$$

is the unique nonnegative definite solution of the (Lyapunov) equation

$$F\Pi + \Pi F' + GG' = 0 \quad (2.6c)$$

it can be checked that the process $y(\cdot)$ described by

$$y(t) = Hx(t) + Ju(t), \quad t \geq t_o \quad (2.7)$$
$$\dot{x}(t) = Fx(t) + Gu(t), \quad x(t_o) = x_o$$

has power spectral density matrix

$$\Phi(j\omega) = W(j\omega)W'(-j\omega),\ W(j\omega) = J + H(sI-F)^{-1}G. \quad (2.8)$$

It can also be checked that (as expected by stationarity)

$$Ex(t)x'(t) = \Pi \quad \text{for all} \quad t \geq t_o \geq -\infty .$$

Some calculation also shows that $\Phi(\cdot)$ itself can also be written in "rational form" as

$$\Phi(s) = JJ' + H(sI-F)^{-1}K + K'(-sI-F')^{-1}H' \quad (2.9a)$$

where

$$K = \Pi H' + GJ' \quad (2.9b)$$

It will simplify various calculations if we assume that the state-process $x(\cdot)$ is nondegenerate (purely nondeterministic) in the sense that the state-variance

$$\Pi \text{ is positive definite .} \quad (2.10a)$$

It is known (see, e.g., [7, p.131]) that this is ensured by the assumption that

$$\{F,G\} \text{ is controllable .} \quad (2.10b)$$

Now by reversing the direction of time in the above equation (2.7), we shall obviously get a "backwards" model for the process $y(\cdot)$ (and, by an abuse of notation, for the spectral matrix $\Phi(\cdot)$). However an important property of the "forwards" model (2.6)-(2.7) will be lost. Thus note that, because $u(\cdot)$ is white (Gaussian) and uncorrelated with x_o, (2.6)-(2.7) is termed a Markovian model for $y(\cdot)$. Now if we just reverse time in (2.7), the "final" value will be correlated with the (backwards) white noise $u(\cdot)$ and this Markovian property will be lost.

However with a little effort, a backwards Markovian model can indeed be obtained. This was shown in [1] (see also [2-3]) for processes with possibly time- variant state-space models. For the case of interest, the results specialize as follows.

Proposition 1: Given a forwards Markovian model obeying (2.6)-(2.7), a backwards Markovian model

$$-\dot{x}_b(t) = F_b x_b(t) + G_b\tilde{u}(t), \quad x(t_1) = x_1$$

$$y(t) = H_b x_b(t) + J_b\tilde{u}(t), \quad t \leq t_1 \leq \infty \quad (2.11)$$

with

$$E\tilde{u}(t)\tilde{u}'(s) = I\delta(t-s),\ E\tilde{u}(t)x_1 \equiv 0,\ Ex_1x_1' = \Pi$$

can be obtained by choosing $\{F_b, G_b, H_b, J_b\}$ such that

$$F_b = -(F+GG'\Pi^{-1}) = \Pi F'\Pi^{-1}$$

$$G_b = G,\ \Pi H_b' = -\Pi H' - GJ',\ J_b = J \quad . \quad (2.12)$$

Such choice will ensure that $x_b(\cdot)$ and $x(\cdot)$ have the same covariance

$$E[x_b(\sigma)x_b'(\rho)] = E[x(\sigma)x'(\rho)] = \Pi \quad (2.13)$$

and of course also $y_b(\cdot)$ and $y(\cdot)$

$$E[y_b(\sigma)y_b'(\rho)] = E[y(\sigma)y'(\rho)] \ . \quad (2.14)$$

The transfer function of this backwards model is

$$W_b(s) = J - H_b(sI+F_b)^{-1}G_b = J + H_b(-sI-F_b)^{-1}G_b \quad (2.15)$$

and of course it must obey the relation

$$\Phi(s) = W(s)W'(-s) = W_b(s)W_b'(-s) \ . \quad (2.16)$$

Remark 1. One of our students, G. Verghese, has noted that in fact one can identify the trajectories of $x(\cdot)$ and $x_b(\cdot)$ (and hence also $y(\cdot)$ and $y_b(\cdot)$). His proof will be described in Appendix I.

Remark 2. The formula $F_b = \Pi F \Pi^{-1}$ shows that F and F_b will have the same eigenvalues (all in the left-half plane). Therefore $W_b(s)$ will have its poles in the right half-plane, corresponding to an anti-causal impulse response (zero for $t > 0$).

Proposition 2: With quantities as defined above, $W(s)$ and $W_b(s)$ can be related as

$$W_b(s)U(s) = W(s) \quad (2.17)$$

where

$$U(s) \overset{\Delta}{=} I - G'\Pi^{-1}(sI-F)^{-1}G \quad (2.18a)$$

and $U(s)$ is 'para-unitary' or 'lossless' in the sense that

$$U(s)U'(-s) = I \quad (2.18b)$$

Proof: By verification. ∎

In contrast to the $W_b(s)$ of the backwards Markovian model, we may note that the backwards model obtained by just reversing the direction of time in the forwards model will have transfer function

$$W(-s) = H(-sI-F)^{-1}G + J \overset{\Delta}{=} W_B(s) \text{ , say} \quad (2.19)$$

It is interesting to compare $W_B(s)$ and $W_b(s)$ in the scalar case.

It is at once clear from (2.19) that if $W(s) = n(s)/d(s)$ for polynomials $\{n(s), d(s)\}$, then

$$W_B(s) = n(-s)/d(-s)$$

† All random variables will be assumed to have zero mean.

and is obtained from $W(s)$ by reflecting its poles and zeros across the $j\omega$-axis. The same $W_B(s)$ results from minimal or nonminimal realizations of $W(s)$. In contrast, use of (2.18a) will show that *with* $d(s) \underline{\Delta} |sI-F|$, $U(s)$ *will be* $d(-s)/d(s)$, *so* that

$$W_b(s) = n(s)/d(-s) \ .$$

Thus the poles, including those corresponding to unobservable modes *of the state-variable realization* of $W(s)$, are reflected to yield $W_b(s)$, while the zeros are not affected. In general, different $W_b(s)$, will result from minimal and nonminimal realizations of $W(s)$. As it turns out, this point is not just of academic interest in the network theoretic applications of these ideas [9].

The poles have to be reflected for all backward models in order to have them purely noncausal (anti-causal), but there is no such constraint on the zeros. Different anti-causal backwards models can be obtained by flipping various combinations of zeros across the $j\omega$-axis and the backwards Markovian model is the one corresponding to no zeros being flipped.

That the zeros will remain invariant can also be seen from the fact that the expression $F_b = -(F + GG'\Pi^{-1})$ corresponds to using state-feedback,

$$u(\cdot) = -\tilde{u}(\cdot) + G'\Pi^{-1}x(\cdot)$$

in the original forwards realization (2.7). For controllable scalar systems it is well known and easy to prove that state-feedback only moves the poles and does not directly affect the zeros (see, e.g., [4]).

To obtain the corresponding results for multivariable systems, we should use the so-called matrix-fraction descriptions (MFDs), as will be explained in the next section.

3. Matrix-Fraction Description of Backwards Markovian Models

It is a standard result [4] that one may associate with a completely controllable pair $\{F,G\}$ a square-polynomial matrix $D(s)$ such that

$$\det D(s) = \det(sI-F) \tag{3.1}$$

and

$$W(s) = J + H(sI-F)^{-1}G = N(s)D^{-1}(s) \tag{3.2}$$

for some polynomial matrix $N(s)$. The pair $\{N(s),D(s)\}$, or more loosely the rational function $N(s)D^{-1}(s)$, is called a matrix-fraction description of the matrix transfer function $W(s)$.

Now it can be shown that the effect of state-variable feedback on a *realization* $\{F,G,H,J\}$ *is* mirrored in the associated MFD by the fact that the new transfer function will have the same "numerator polynomial," but a different denominator polynomial. In our problem, the state-feedback interpretation of (2.12) will show that (see, e.g., [4,p.229])

$$W_b(s) = N(s)D_b^{-1}(s) \tag{3.3}$$

where $D_b(s)$ is a polynomial matrix such that

$$\begin{aligned} D_b(s) &= (I-G'\Pi^{-1}(sI-F)^{-1}G)D(s) \\ &= U(s)D(s) \end{aligned} \tag{3.4}$$

where $U(s)$ was defined in (2.18b). Note that

$$D_b^{-1}(s) = D^{-1}(s)U^{-1}(s)$$

so that

$$N(s)D_b^{-1}(s)U(s) = N(s)D^{-1}(s)$$

consistent with (2.18a). We have also that

$$\det D_b(-s) = \pm \det D(s) \ . \tag{3.5}$$

In the scalar case this uniquely determines $D(s)$ (upto sign), but there is no such simple relation in the matrix case (unless $D(s)$ is diagonal). From (3.4) and (2.19) we can say that

$$D'(-s)D(s) = D_b'(s)D_b(-s) \ . \tag{3.6}$$

Note that in our problem

$$\det D(s) \quad \text{has its roots in the LHP} \tag{3.7a}$$

and (consequently)

$$\det D_b(s) \quad \text{has its roots in the RHP.} \tag{3.7b}$$

Matrix pairs obeying (3.5)-(3.7) arise as the forward and backward predictors in the so-called LWR (Leirnson-Whittle-Wiggins-Robinson) algorithm for predicting stationary sequences, and in the closely related theory of matrix orthogonal polynomials (cf., [10,Sec.] and the references therein). Such pairs, without the restriction (3.7), also arise in a stability test for matrix polynomials [11].

We shall say that two matrix polynomials $\{D(s),D_b(s)\}$ are dual if they are related as in (3.5)-(3.6).

The following theorem, proved in Appendix II, guarantees the existence of $D_b(s)$. A constructive procedure may be found in the proof of the theorem, and it is closely tied to the construction in Proposition 1.

Theorem 1: Let $D(s)$ be an $n \times n$ nonsingular matrix polynomial. Then there exists a "dual" matrix polynomial $D_b(s)$ obeying (3.5) and (3.6). Further, if $\det D(s)$ and $\det D(-s)$ are coprime, $D_b(s)$ is unique to within left multiplication by a constant orthogonal matrix, and there is one and only one $D_b(s)$ with the property that

$$\lim_{s\to\infty} D(s)D_b^{-1}(-s) = I \tag{3.8}$$

Several remarks can be made:

1. The theorem is more general than needed for this paper, where for stationarity of the process $y(\cdot)$ we shall require that $\det D(s)$ have all its zeros in $\mathrm{Re}[s] < 0$. Then the condition guaranteeing uniqueness of a $D_b(s)$ satisfying (3.8) *is automatically fulfilled.*

2. If $D(s)$ is diagonal, $D_b(s) = D(s)$ satisfies (3.5) and (3.6) and, to within sign change of the diagonal entries, (3.8). Generally,

though, construction of $D_b(s)$ from $D(s)$ is not straightforward.

3. If $D_b(s)$ is the dual of $D(s)$, then $D(s)$ is the dual of $D_b(s)$. An absence of a transpose caused a very similar definition in [11] to lack this property.

4. Note that by our earlier discussions, we have an explicit state-space construction, at least when (3.7a) holds, for going from a polynomial matrix to its dual.

Network Interpretations

Several physical systems provide examples of the simultaneous occurrence of spectral factors and their duals. The following ideas are developed at greater length elsewhere, [9]. Consider an n-port network N comprised of a finite number of passive capacitors, inductors, transformers, gyrators and resistors and suppose it is drawn as in Figure 1, as a lossless $(n+m)$ - port terminated at m ports in unit resistors. (Transformer normalization will take care of unit resistors). Suppose the network possesses an impedance matrix $Z(s)$. Let $W(s)$ be the transfer function matrix linking current sources in parallel with the resistors to the voltage vector at the n input ports of the network. Then it can be shown that $W(s)$ is a (forward) spectral factor for $\Phi(s) = Z(s) + Z'(-s)$ (which is a power spectrum matrix); in fact, by Nyquist's theorem (see, e.g., [12]) the thermal noise in the resistors of the network induces a random process at the ports of N which has power spectrum $\Phi(s)$ to within a scaling constant. In a physical sense, the dual of $W(s)$ can be obtained in two ways. First it can be shown that $W_b'(-s)$ is the transfer function matrix linking current generators at the n ports of N to the voltages across the resistors. Second, if the direction of all gyrator polarities is reversed, it can be shown that $W_b(-s)$ † is the transfer function matrix linking current sources in parallel with the resistors to the voltage vector at the n ports. The state space description or matrix fraction description used in forming the dual is one induced by a corresponding description for the lossless $(m+n)$ - port embedded in N.

In case N is reciprocal, i.e., contains no gyrators, we shall have $W(s) = W_b(-s)$, a situation which we shall describe as self-dual. In the next section, we delve more deeply into the self-duality property.

4. Self-Dual Models and State Reversibility

In this and the next section, we shall consider the question of when a forward model and associated backward model are related by the self-duality equation $W(s) = W_b(-s)$. The main conclusions are that the process of which $W(s)W'(-s)$ is the power spectrum matrix must have a reversibility property, and that in the right coordinate basis, the state (or partial state) of the realization of $W(s)$ will have a generalized (so-called dynamic) reversibility property. This section considers state reversibility, while the next section considers partial state reversibility.

A necessary condition for self-duality is easily obtained:

Theorem 2: Let $W(s)$ be a spectral factor of a prescribed rational power spectrum matrix, and let $W_b(s)$ be the dual spectral factor, obtained from a state-variable or matrix fraction description of $W(s)$ via the procedure of Section 2 or 3. If $W_b(-s)Q = W(s)$ for some orthogonal Q, then $\Phi(s) = \Phi'(s)$.

Note that symmetry of $\Phi(s)$ is not sufficient for self duality. Thus all scalar $\Phi(s)$ are symmetric, but the self duality property could only follow for a scalar transfer function $W(s)$ in case the zero pattern of $W(s)$ was symmetric with respect to the $j\omega$-axis.

Note too that, as suggested by the statement of the above theorem, it proves convenient to expand the definition of self-duality to permit an orthogonal "normalizing" matrix. In the scalar case, this is equivalent to allowing a ± 1 multiplier; recall also the possibility of the arbitrary orthogonal matrix in the duality result of Theorem 1.

Some elementary concepts from statistical thermodynamics [12] provide helpful insights into the symmetry of $\Phi(\cdot)$ and possible self-duality of a spectral factor. Let $y(\cdot)$ be a stationary vector random process. Then $y(\cdot)$ has a second order reversibility property if statistics computed running forwards in time are the same as those running backwards in time. [Thus a tape recording of $y(\cdot)$ would be indistinguishable as far as statistics from the same tape recording played backwards.] What is the condition for $y(\cdot)$ to have the reversibility property? Clearly, for all t and τ,

$$E[y(t)y'(t+\tau)] = E[y(t)y'(t-\tau)] \quad . \tag{4.1}$$

Using stationarity, we have

$$E[y(t)y'(t-\tau)] = E[y(t+\tau)y'(t)]$$

Thus (4.1) implies that the covariance of $y(\cdot)$ is symmetric, and thus so is its power spectrum matrix. The converse is easy. In summary:

Proposition 3: A stationary process $y(\cdot)$ has the second order reversibility property if and only if its power spectrum matrix is symmetric.

Note that every scalar process is reversible. Note also that the content of Theorem 2 is that reversible models always yield reversible processes, (one can also show that reversible processes have some models which are reversible.)

We shall now push the reversible process idea further, in order to show that when a state-variable model defined by $W(s)$ is reversible then the state process of the model has a reversibility type of property (as a process) (at least in an appropriate coordinate basis and when the model order is minimal in a sense described below). Thus self-dual or reversible models can be thought of as having an internal kind of reversibility property.

† We have $-s$ here because in the network all transfer functions are causal.

For this analysis, we develop state-variable interpretations in Proposition 4 below of the symmetry of $\Phi(s)$ and the self-duality or model reversibility of a spectral factor in state-variable form. The calculations are very similar to some that have been used in studying reciprocal networks, [7,8]. Thus let us suppose that the model $W(s)$ is defined by a quadruple $\{F,G,H,J\}$ and that the associated power spectrum is as given in (2.9), which we repeat as

$$\Phi(s) = JJ'+H(sI-F)^{-1}K+K'(sI-F')^{-1}H' \quad . \qquad (4.2)$$

The next proposition which slightly extends a result of Youla and Tissi [6] sets up the existence of a certain matrix which is used in Theorem 3 to define a new coordinate basis displaying internal reversibility.

[For reasons of space, all proofs will be omitted in the rest of the paper.]

Proposition 4: Let $W(s)$ with minimal state variable realization $\{F,G,H,J\}$ be a spectral factor of $\Phi(s)$ *such that* $Re\lambda_i(F) < 0$. *Suppose that*

$$W(s) = W_b(-s)Q, \text{ some orthogonal } Q \qquad (4.3a)$$

i.e.,

$$J+H'(sI-F)^{-1}G=JQ - (H+\Pi^{-1}GJ')'(sI-\Pi F'\Pi^{-1})^{-1}GQ, \qquad (4.3b)$$

where Π satisfies $\Pi F'+F\Pi=-GG'$. Then there exists *a unique nonsingular symmetric* T *such that*

$$K'T=H' \quad T^{-1}F'T=F \quad -T^{-1}\Pi^{-1}GQ=G \quad T\Pi T=\Pi^{-1} \qquad (4.4)$$

where $K \triangleq \Pi H'+GJ'$.

Our task is now to exhibit a reversibility property of the state process of a reversible or self-dual model. In point of fact, we shall show that in a suitable coordinate basis, the state vector, call it $\bar{x}$, has the property

$$\Sigma E\{\bar{x}(t)\bar{x}'(t-\tau)\} = E\{\bar{x}(t)\bar{x}'(t+\tau)\}\Sigma \qquad (4.5)$$

where Σ is a diagonal (so-called signature) matrix with +1, -1 elements on the diagonal. By arranging the vector entries so that $\Sigma = I_p \dotplus (-I_q)$, (4.5) says that

$$E[\bar{x}_p(t)\bar{x}_p'(t-\tau)] = E[\bar{x}_p(t)\bar{x}_p'(t+\tau)]$$

$$E[\bar{x}_q(t)\bar{x}_q'(t-\tau)] = E[\bar{x}_q(t)\bar{x}_q'(t+\tau)]$$

$$E[\bar{x}_p(t)\bar{x}_q'(t-\tau)] = -E[\bar{x}_p(t)\bar{x}_q'(t+\tau)]$$

Thus, $\bar{x}_p(\cdot)$ and $\bar{x}_q(\cdot)$ are separately reversible, but jointly are not. However, modulo a sign change there is a reversibility. This type of generalized reversibility of a process is a standard concept in the thermodynamics of irreversible processes, see [12, Chapter XI] and also arises in network theory (see, e.g., [7,8]). The following theorem is related to one in network theory (cf., [7,p.324]).

Theorem 3: Let $W(s)$ with minimal state variable realization $\{F,G,H,J\}$ be a spectral factor of a symmetric rational power spectrum matrix $\Phi(s)$ such that $Re\lambda_i(F) < 0$. With notation as earlier, suppose that $W(s) = W_b(-s)Q$, i.e., (4.3) holds where Π solves $\Pi F'+F\Pi=-GG'$. Let T be the unique symmetric nonsingular matrix satisfying (4.4), with $K=\Pi H'+GJ'$. Then there exists an orthogonal matrix V such that

$$\Pi^{\frac{1}{2}}T\Pi^{\frac{1}{2}} = V'\Sigma V \qquad (4.6)$$

where $\Pi^{\frac{1}{2}}$ is the positive definite square root of Π and Σ is a diagonal matrix with +1 and -1 elements on the diagonal. Define a state space transformation

$$\bar{x} = V\Pi^{-\frac{1}{2}}x \qquad (4.7)$$

where x is the state-variable in the noise model (2.9) defined by $W(s)$. Then $\bar{x}$ has the generalized reversibility property of (4.5).

It is important to note that the joint process $[\bar{x}'(t) \;\; y'(t)]'$ possesses a dynamic reversibility property. One can check that

$$\begin{bmatrix}\Sigma & 0\\ 0 & I\end{bmatrix} E\left\{\begin{bmatrix}\bar{x}(t)\\ y(t)\end{bmatrix}[\bar{x}'(t-\tau) \;\; y'(t-\tau)]\right\} =$$

$$= E\left\{\begin{bmatrix}\bar{x}(t)\\ y(t)\end{bmatrix}[\bar{x}'(t+\tau) \;\; y'(t+\tau)]\right\}\begin{bmatrix}\Sigma & 0\\ 0 & I\end{bmatrix} \qquad (4.8)$$

Given any spectral factor in state-variable form with state process $x(t)$ and dynamic reversibility of $[x'(t) \;\; y'(t)]'$, it does not follow that $E[x(t)x'(t)] = I$, but it does follow that the self-duality property (4.3) holds, and in this sense we have a converse of Theorem 3. To check this, observe that the dynamic reversibility property is equivalent to

$$\begin{bmatrix}\Sigma & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}e^{F\tau}\Pi & e^{F\tau}K\\ He^{F\tau}\Pi & He^{F\tau}K\end{bmatrix} = \begin{bmatrix}\Pi e^{F'\tau} & \Pi e^{F'\tau}H'\\ K'e^{F'\tau} & K'e^{F'\tau}H'\end{bmatrix}\begin{bmatrix}\Sigma & 0\\ 0 & I\end{bmatrix}$$

from which $\Sigma\Pi = \Pi\Sigma$, $\Sigma F\Pi = \Pi F'\Sigma = \Pi F'\Sigma$, $\Sigma K = \Pi H$. These equations imply that

$$J+H(sI-F)^{-1}G = J+K'\Pi^{-1}\Sigma(sI-\Sigma\Pi F'\Pi^{-1}\Sigma)^{-1}G$$
$$= J+K'(sI-\Pi F'\Pi^{-1})\Sigma G$$

Also, pre and post multiplying $\Pi F'+F\Pi = -GG'$ by Σ yields $\Sigma GG'\Sigma = GG'$ or $\Sigma G = GQ$ for some orthogonal Q, i.e., (4.3) holds.

5. Self-Dual Models and Partial State Reversibility

In this section, we consider the situation in which an intermediate variable $z(\cdot)$, the "partial state" associated with a matrix fraction description of a spectral factor matrix, links the input $u(\cdot)$ and output $y(\cdot)$:

$$D(\tfrac{d}{dt})\, z(t) = u(t) \qquad y(t) = N(\tfrac{d}{dt})\, z(t) \qquad (5.1)$$

We aim to discover a reversibility property of $z(\cdot)$ given the self-duality of $W(s) = N(s)D^{-1}(s)$. However, it is preferable to consider $z(t)$ as part of a larger process $[z'(t) \;\; y'(t)]'$, and look for a reversibility property of this process. For otherwise, we could conceive of a non self-dual scalar spectral factor with $\{D(s), N(s)\}$ scalar: Then because $z(\cdot)$ is a scalar process, it automatically is reversible. By considering the larger process, $[z'(t) \;\; y'(t)]'$, we shall be able to draw the conclusion that in the right coordinate basis, this process has dynamic reversibility if and only if $W(s) = N(s)D^{-1}(s)$ has the self-duality property.

To discover a reversibility property for $z(\cdot)$ given the self duality of $W(s) = N(s)D^{-1}(s)$, the general approach is to obtain a unimodular matrix which transforms D and N so that the resulting new $z(t)$, call it $\tilde{z}(t)$, has with $y(t)$ the same sort of generalized reversibility property as did the transformed state variable $\bar{x}(t)$ with $y(t)$ in Therem 3. We begin with a proposition which will allow construction of the transforming unimodular matrix.

Proposition 5: Let $W(s)$ with matrix fraction description ND^{-1} where N and D are relatively right prime be a spectral factor of a power spectrum matrix $\Phi(s)$ with all zeros of det D in $\mathrm{Re}[s] < 0$. Let D_b be the unique dual matrix polynomial to D and suppose that $N(-s)D_b^{-1}(s)Q = N(s)D^{-1}(s)$ for some orthogonal Q. Then for some orthogonal T and diagonal signature matrix Σ,

$$N(s)D^{-1}(s) = N(-s)D_b^{-1}(s)T\Sigma T' \tag{5.2}$$

and there exists a unimodular matrix $V(s)$ such that

$$N(s)V(s) = D(-s) \quad D(s)V(s) = T\Sigma T'D_b(s)$$

$$V(s)V(-s) = I \quad . \tag{5.3}$$

The unimodular nature of $V(s)$ and the fact that it satisfies the third equation in (5.3) allows us to obtain a certain decomposition, proved in [13]:

Proposition 5: Let $V(s)$ be a unimodular matrix such that $V(s)V(-s) = I$. Then there exists a unimodular $M(s)$ and a diagonal signature matrix $\tilde{\Sigma}$ such that

$$V(s) = M(s)\ \tilde{\Sigma}M^{-1}(-s) \tag{5.5}$$

The matrix $M(s)$ is now used to set up a new matrix fraction description of $W(s)$, as

$$\tilde{D}(s) = D(s)M(s) \qquad \tilde{N}(s) = N(s)M(s)$$

The dual polynomial of $\tilde{D}(s)$, which we shall denote by $\tilde{D}_b(s)$, is immediately formed as

$$\hat{D}(s) = M'(-s)\bar{C}(s)$$

Corresponding to the first two equations of (5.3), we have

$$\tilde{D}(s)\tilde{\Sigma} = \tilde{D}(-s) \tag{5.6a}$$

$$\tilde{C}(s) = T\Sigma T'\hat{C}'(s) \tag{5.6b}$$

(The third equation is simply $\tilde{\Sigma}^2 = I$). This new matrix fraction description $\tilde{D}C^{-1}$ of $W(s)$ gives us the dynamic reversibility property we are seeking.

Theorem 4: With the hypotheses of Proposition 4, let $C(s)D^{-1}(s) = C(s)N(s)[D(s)N(s)]^{-1}$ be another right matrix fraction decomposition of $W(s)$ with $N(s)$ constructed as described in Proposition 4 and 5, and let $\tilde{z}(\cdot)$ be defined by $\tilde{D}(\frac{d}{dt})\ \tilde{z}(t) = u(t)$, $y(t) = \tilde{C}(\frac{d}{dt})\tilde{z}(t)$ with $u(\cdot)$ a vector of unit intensity independent white noise sources.

$$\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 1 \end{bmatrix} E\{\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} [z'(t-\tau)y'(t-\tau)]\} =$$

$$= E\{\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} [z'(t+\tau)y'(t+\tau)]\}\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & I \end{bmatrix} . \tag{5.7}$$

6. Self-Dual Model Construction

Statistical mechanics suggests that many physically based random processes have an internal dynamic reversibility property [8], [12]. This suggests that we should be concerned with the construction and even identification of self-dual models for symmetric power spectrum matrices. In this section, we refer briefly to some constructions for self-dual models. These constructions are essentially tied to reciprocal network synthesis procedures for passive symmetric impedance or scattering matrices [5], [7].

If ND^{-1} is a spectral factor and D_b is the dual, then

$$W(s) = \frac{1}{\sqrt{2}}[N(s) \quad N(-s)]\begin{bmatrix} D(s) & 0 \\ 0 & D_b(s) \end{bmatrix}^{-1}$$

turns out to be self-dual if $\Phi(s) = \Phi'(s)$. For a state-variable version considered in network-theoretic terms see [7, Section 9.4].

The Darlington network synthesis procedure takes a rational scalar spectral factor $p(s)|q(s)$ and obtains from it a spectral factor for the same spectrum of the form $m(s)|n(s)$ with $m(s)$ possessing a new pattern symmetric with respect to the $j\omega$-axis. A multivariable version--the reciprocal Bayard synthesis--can be found in [5], with a state-space description of the procedure in [7].

Finally, given a state-variable realization of a spectral factor $W(s)$, viz., $\dot{x} = Fx + Gu$, $y = Hx + Ju$, with the McMillan degree of $W(s)$ one half the McMillan degree of $W(s)W'(-s)$ one can find a second spectral factor with the same F, H but different $\bar{G}$, $\bar{J}$ by either of two equivalent procedures described in [7] and [8].

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