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DECISION ALGEBRA AND ITS APPLICATION TO LINEAR SYSTEMS

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ABSTRACT

After presentation of the output feedback stabilization problem as a motivating example, the main result of decision algebra is reviewed. Its application to the output feedback stabilization problem and to problems of model matching are described. In both cases, decision algebra is seen to be a tool which allows the handling of problems in which dimensionality and stability constraints are simultaneously in force.

INTRODUCTION

The purpose of this paper is to review what the tool of decision algebra is, and to observe several areas of its applicability to problems of linear system theory. By and large, any systems problem in which there are simultaneously constraints on stability and dimensionality is one for which decision algebra is prima facie suited.

Decision algebra is a tool that was first developed some 40 years ago, although it has only been in the open literature for approximately 25 years. It was first developed by the logician A. Tarski, and described in a brief book authored by him [1]. An algebraic geometer, Seidenberg, obtained the same results by different methods and these are described in [2]. A more extended treatment of Seidenberg's ideas was first available in text book form in [3,4] and a more extended version again, is that contained in a much more recent book, by the author of [3], namely [5]. Below, we indicate what the main results of decision algebra are. However, we shall first indicate the sorts of control problems which led to its first application in control theory several years ago [6]. Further applications to control problems are described after our summary of the main results of decision algebra.

A SIMPLE CONTROL PROBLEM

Consider the system depicted in Figure 1.

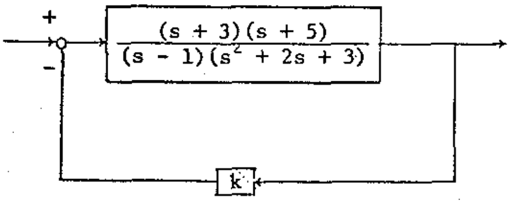

Figure 1.

The plant is open loop unstable. A common question, generally examined in elementary courses on classical control, is whether or not there exists a constant feedback gain which will stabilize the plant. More particularly, one might be interested in describing what the range of feedback gains are which will stabilize the plant, assuming such gains exist.

Answering these questions is fairly easy using root locus diagrams or Nyquist diagrams. If one attempts to translate the content of these diagrams into equations, what one finds is that one must solve polynomial equations, determine certain intervals in terms of the real solutions of these polynomial equations, and check, via a Hurwitz or similar criterion, for the stability of a closed loop system with gains determined in each of the intervals determined by the polynomial equation solutions. The end result is that if there are any stabilizing feedback gains, then they are in certain computable intervals.

Actually, and this is by no means obvious, the question of whether there are any feedback gains stabilizing the closed-loop as distinct from the question of what these gains might be, need not be answered with polynomial factorization.

What this problem is really all about is this: one is seeking to do a controller design, with, simultaneously a stability constraint on the closed loop, and a dimensionality constraint on a controller. Such problems arise frequently in control systems, and it is for tackling more complicated versions of such problems that decision algebra is the obvious tool.

That more complicated versions are indeed difficult to cope with is readily seen by examining the multivariable generalization of the above example. Thus, in state space form, one has the following set of equations, with vector u and y:

$$\dot{x} = Fx + Gu \qquad y = H'x \qquad (1)$$

One then asks whether or not there is a feedback law u = Ky that will stabilize the system, i.e., a K such that F + GKH' has all its eigenvalues in the left half plane; further, if such a stabilizing law exists, one asks how it might be calculated.

For a few special cases, there is a simple answer to these questions. For example, if $H = I$, or if H is an invertible matrix, such a K exists if and only if the pair $[F,G]$ is completely stabilizable. In general however, there is no simple way to answer this seemingly simple question.

Methods of answering the question have been obtained by decision algebra, and actually more recently by algebraic geometry [6,7]. It is a decision algebra approach which we shall consider here, and accordingly, we move on to consider the nature of decision algebra problems and results available in decision algebra.

DECISION ALGEBRA

Consider the following very simple problem. Given the polynomial $x^n + a_1x^{n-1} + \dots a_n$, is there a root? The answer to this question is so well known that it is a slight surprise to realize that a slight variation on this question is a lot more difficult to answer. Suppose we ask: is there a real root? Obviously, if n is odd, there is, but what if n is even? Two approaches are possible. First, one can factor the polynomial. For high order polynomials of course this requires computer facilities. Second, however, there exists a set of rational inequalities on the coefficients a_i which if satisfied guarantee that there is a real root. This result, known as Sturm's Theorem, is 140 or so years old. The derivation of these conditions is part of the content of the theorem.

The question of when a prescribed polynomial has a real root is the most elementary question of decision algebra, and we note that it is answerable with rational calculations, i.e., addition, subtraction, multiplication, division and sign checking, but not rootfinding.

Now we consider how this problem could be generalized. Different people at different times have made generalizations in different directions, but we can sum up all these generalizations in the diagram of Figure 2.

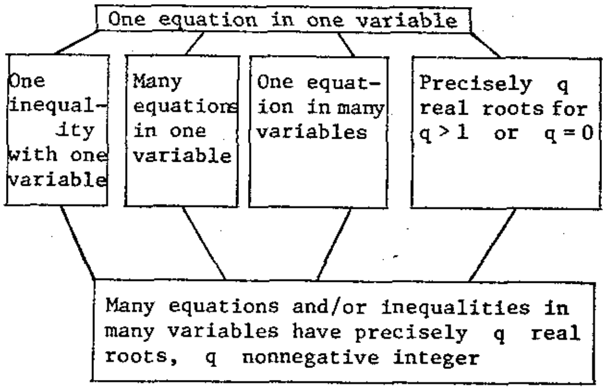

Figure 1

The general question that decision algebra tackles is therefore as follows: given a set of polynomial equalities and inequalities with real coefficients and an arbitrary number of variables and a nonnegative integer q, state whether or not there are precisely q real solutions to the set of polynomial equalities and inequalities. Something which is also of interest, but which is not normally discussed in decision algebra is this. Assuming there are q real solutions to the equality/inequality set, how may one calculate those solutions?

For example, one might ask whether or not the following set has precisely four real solution triples:

$$\begin{aligned} x^2 + y^2 - z &> 0 \\ x^3 - y^4 + 2z &= 0 \\ yz &\le 0 \\ x^2y^2z^2 &= 1 \end{aligned} \tag{2}$$

The main result of decision algebra is that any decision algebra problem is answerable by a gigantic extension of Sturm's Theorem, which it will be recalled answers the simplest possible decision algebra problem, that associated with a single polynomial equation in a single variable. The calculations tend to grow exponentially with the number of equations involved, and procedures for carrying out the calculations have been heavily studied in recent times by computer scientists (see e.g. [8]). All the calculations in checking existence are rational, i.e. involve nothing more than addition, subtraction, multiplication, division and sign checking.

Assuming that the answer to the existence question is affirmative, it turns out that the solutions can be calculated by using methods which ultimately depend on the ability to factor a single polynomial in a single variable. Therefore, polynomial factorization is required in solving the construction problem, in general. A discussion of this extension to the basic theory can be found in Reference [6].

THE OUTPUT FEEDBACK STABILIZATION PROBLEM AS A DECISION ALGEBRA PROBLEM

Let us now consider how the multivariable output feedback stabilization feedback problem can be regarded as a problem of decision algebra. Recall that we are given three matrices, F, G, and H, and we seek a matrix K such that the eigenvalues of $F + GKH'$ have negative real parts.

Consider Figure 3.

$$\begin{tikzcd}
\text{Entries of } K \arrow[d, "\text{Polynomial Map}"] \arrow[dd, bend left=60, "\text{Polynomial Map}"] \\
\text{Coefficients of } \det[sI-F-GKH'] \arrow[d, "\text{Polynomial Map}"] \\
\text{Hurwitz Stability Conditions (Inequalities)}
\end{tikzcd}$$

Figure 3

This figure attempts to illustrate the fact that the problem of finding K such that the closed loop *matrix is stable is equivalent to the problem of* deciding whether or not there are real solutions to a set of polynomial inequalities. (Finding whether there is any set of real solutions to a set of polynomial inequalities is equivalent to checking whether or not there are no solutions to the inequalities, which is a problem of decision algebra. If it is false that there are no real solutions to the polynomial inequalities, then of course it follows that there are real solutions, and obviously an infinity of them, to the inequalities.) The construction of the inequalities proceed in two steps. First, one can form the characteristic polynomial of the matrix $F + GKH'$. The coefficients of this characteristic polynomial will be expressible in terms of the entries of K, and in fact will be multivariable polynomials in the entries of K. Thus the first map in the left side of Figure 3 is a polynomial map. Next, we recall that given an arbitrary polynomial with real coefficients, one may test for the property of this polynomial possessing all its zeros in the left half plane by the Hurwitz stability conditions. This involves taking the coefficients of the polynomial and forming certain determinants using these coefficients, which are positive if and only if the polynomial is stable. These determinants are in effect polynomial functions of the coefficients. Accordingly, the composition of two polynomial maps allows the expressing of the stability conditions as polynomial inequalities in the entries of K.

For example, consider the system of Equation 3.

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$K = [v \quad w] \qquad u = Ky \tag{3}$$

The closed-loop characteristic polynomial is $s^3 + vs^2 + (w - 5v - 13)s + w$. Observe the coefficients of this polynomial are themselves polynomial in the unknown entries of K. Using the Hurwitz stability conditions, we see that this characteristic polynomial has all its zeros in the left half plane if and only if

$$v > 0 \quad w > 0 \quad w(v-1) - (5v^2 + 13v) > 0 \tag{4}$$

Thus the plant is stabilizable if and only if the inequalities (4) have a real solution. The checking of this is a standard problem of decision algebra, involving only rational calculations, and the construction of a solution assuming that Equations 4 do have a real solution proceeds by methods as outlined in [6], and in general requires polynomial factorization.

As commented earlier, the output feedback stabilization problem is an example of a problem in which one is required to design a controller with a dimensionality constraint on the controller and a stability constraint on the closed loop system. The whole class of such problems can in fact be tackled by decision algebra methods. Thus, for example, if one postulates a controller with state variable dimension two, one can seek to determine whether *there exists such a controller which at the same* time stabilizes the closed loop system, and in this case, again, multivariable polynomial inequalities eventuate. It does seem that it is a combination of the dimensionality constraint and the stability constraint which causes the complexity involved in any decision algebra problem. Thus, if one drops the dimensionality constraint and insists merely on stability, it is easy to give an answer to the question of whether or not a stable closed-loop system can be found, and if one drops the stability constraint, naturally there is a controller of any dimensionality. Reference [6] should be consulted for further details on controller, and in fact, observer, design with dimensionality and stability restrictions.

MODEL MATCHING

There is another group of system theory problems which seem suited to an examination involving some decision algebra ideas. The model matching problem is as follows. One is given rational matrices $T_1(s)$, $T_2(s)$ and one seeks a $T(s)$ such that $T_1T = T_2$. The dimensionalities of T_1 and T_2 are usually such that there is normally an infinity of T satisfying this particular equation. One is usually interested in restricting the solution in some way, and the three types of restriction are those on the properness of T, i.e. ensuring $T(\infty) < \infty$, on the degree of T, i.e., the dimension of a minimal state variable realization, and on the stability of T.

Papers dealing with the model matching problem include [9-15]. The model matching problem captures the situation of searching for an inverse, e.g., take $T_2 = I$, and (less obviously) of searching for reduced dimension observers, see [9].

If one simply insists on the properness of T, there is a very simple test to see if such a T exists, [9,10]. If one avoids consideration of stability, it is in fact easy to calculate a solution T, assuming one exists, with degree equal to the least degree of all solutions of $T_1T = T_2$. It turns out however that if one imposes simultaneously a dimensionality and a stability constraint, it is much harder to answer both the existence and construction questions.

Some partial answers to these questions can be found in [12,13]. One interesting result of [13] is that there is a set of poles, possibly empty, which are common poles of all solutions to $T_1T = 2$. If one or more of these poles lies in $Re[s] \geq 0$, then naturally no stable solutions exist. If on the other hand, all lie in $Re[s] < 0$, then, provided one takes T of sufficiently high degree, one can find a stable T. Thus if the dimensionality constraint is absent and the stability constraint present, under easily checked conditions the problem is solvable.

To proceed further, one does however need decision algebra. There are in fact two broad thrusts, which are as follows.

First, assuming there are any stable solutions of $T_1T = T_2$, one can obtain a parametrization of all stable solutions of the problem. Thus, it can be shown (see [14]) that if there are stable proper solutions, then they can all be found from the following formula

$$T = B_1 + B_2K \tag{5}$$

where B_1 and B_2 are certain fixed stable transfer function matrices, computable from T_1 and T_2, and K is a stable proper transfer function matrix which is otherwise arbitrary. (Parenthetically, we remark that the mathematics required for constructing (5) involves an interesting variation on a construction - the Smith-McMillan form - well known in system theory; the variation involves recognition that the Smith form is really a construction of matrices with elements in any principal ideal domain, not just polynomial matrices, and the class of stable, proper rational transfer functions is a principal ideal domain).

To express the stability of K in parametric form, it is easiest to represent K as a sum of terms of the form $(C_is + D_i)(s^2 + 2\alpha_i^2s + \alpha_i^4 + \beta_i^2)^{-1}$ and $E_i(s + \gamma_i^2)^{-1}$. Then, having obtained the parametrisation (5), i.e. having imposed a stability constraint and a constraint of properness, the next task is to impose a dimensionality constraint. In principle, one can see how to do this. A dimensionality constraint on T corresponds to the existence of certain constraints, certainly expressible rationally and ultimately via polynomial equalities, on the coefficients of the denominator and numerator polynomials in B_1, B_2 and K. Thus in this case, a decision algebra problem evolves which involves polynomial equalities for the real parameters appearing in K (The matrices B_1 and B_2 are fixed). If and only if these equations have a real solution, there will be a T which is stable, and proper and of prescribed degree.

The second way of tackling the problem [15] proceeds by seeking a parametrisation of all proper solutions of $T_1T = T_2$ of given degree, without regard for the stability constraint, and then imposing the stability constraint. More specifically, we think of any $T(s)$ as being defined by a matrix fraction description $Q(s)P^{-1}(s)$, and find one can construct equations for the polynomial matrices $P(s)$ and $Q(s)$ of the form

$$P(s) = X(s)N(s) \qquad Q(s) = Y(s)N(s)$$

in which $X(s)$ and $Y(s)$ are known polynomial matrices constructible from $T_1(s)$ and $T_2(s)$, and $N(s)$ is a polynomial matrix containing a number of free parameters, but with some structure; $T(s)$ of different degree are obtained by varying the structure. The stability of $T(s)$ corresponds to having $\det Q(s)$ a Hurwitz polynomial. The coefficients of $\det Q(s)$ are polynomial in the parameters of $N(s)$, and thus the stability constraint translates to a set of polynomial inequalities in the free parameters of $N(s)$. Again then, decision algebra is brought to bear on the problem.

CONCLUDING REMARKS

At this stage it is fair to say that decision algebra provides a useful conceptual framework in which to locate certain problems of system theory and control system design, particularly problems which involve simultaneously a stability constraint and some sort of dimensionality or similar constraint. The numerical techniques presently available for answering decision algebra questions of existence, and then going on to solving the construction problem assuming an affirmative answer to the existence question, are however very consuming of computer time. It is therefore not clear whether or not decision algebra ideas will find great application for design in specific instances. The reliance of these numerical techniques on rational operations, and a comparative avoidance of root finding, suggests that alternative techniques might well be available which are both faster and which use more of the root finding capability of a computer. If this were indeed the case, this would make the practical application of decision algebra techniques much more attractive. It is conceivable that ideas of algebraic geometry [7] could prove helpful in developing the needed techniques, but much remains to be done.

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