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NEW TOOLS IN THE MATRIX FRACTION DESCRIPTION OF LINEAR SYSTEMS

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ABSTRACT

The paper reviews the definition of the generalized Bezoutian matrix, which is defined using the coefficients of the matrix polynomials in prescribed left and right matrix fraction decompositions of a rational transfer function matrix $W(s)$. Like its classical counterpart, the generalized Bezoutian matrix is relevant for studying a wide variety of problems. For example, its rank gives information about the minimality or otherwise of the matrix fraction decompositions and the degree of the rational transfer function matrix. In case $W(s)$ is symmetric, the generalized Bezoutian matrix can be used to define a matrix Cauchy index. In case $W(s)$ is lossless, RC or RL positive real, the generalized Bezoutian matrix possesses certain properties. Another use of the Bezoutian is for the characterization of square nonsingular polynomial matrices which have stable determinants.

1. INTRODUCTION

In this paper, we begin by recalling the definition of the classical Bezoutian matrix. Following this, we define the generalized Bezoutian matrix and develop a number of its properties, indicating relations with classical results where appropriate.

2. THE CLASSICAL BEZOUTIAN MATRIX

Let $p(s)$, $q(s)$ be two polynomials, $p(s)$ of degree n, $q(s)$ of degree $\le n$. The Bezoutian form associated with $p(\cdot)$, $q(\cdot)$ is defined by

$$\Gamma(x,y) = \frac{p(x)q(y) - p(y)q(x)}{x-y} \tag{1}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_{ij} x^{i-1} y^{j-1}$$

and the Bezoutian matrix is $\Delta = (\Delta_{ij})$. It is not hard to verify that Δ is symmetric

3. THE GENERALIZED BEZOUTIAN MATRIX

Many of the useful properties of Δ stem from the fact that it contains information concerning the transfer function $w(s) = q(s)/p(s)$. It follows that we might well seek a generalization which would contain information about a transfer function matrix $W(s)$. The analogy of the representation of $w(s)$ as a ratio of polynomials is the representation of $w(s)$ as a fraction of two polynomial matrices, or matrix fraction decomposition (MFD) [1,2]. Because of the noncommutativity of matrices, we obtain two styles of MFD, thus $W(s) = A^{-1}(s)B(s) = D(s)C^{-1}(s)$, where A, B, C and D are polynomial matrices. Note that in general, it is not possible to have $A = C$ and $B = D$, although in some special cases (e.g. $A = a(s)I$ where $a(\cdot)$ is a scalar polynomial), this is possible.

Partly with the aid of hindsight, one can write down the appropriate generalization of (1), [3], which is

$$\Gamma(x,y) = \frac{A(x)D(y) - B(x)C(y)}{x-y} \tag{2}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta_{ij} x^{i-1} y^{j-1} \tag{3}$$

Notice that Γ is integral, as opposed to rational in x and y; from $A^{-1}B = DC^{-1}$ comes $AD-BC = 0$, which shows that $x-y$ divides the numerator matrix in (2). The integers n,m in (3) are defined as $\max[\deg A,B]$, $\max[\deg C,D]$. From the Δ_{ij} one can construct a matrix Δ with block i-j entry equal to Δ_{ij}. This is the generalized Bezoutian matrix. We note that Δ_{ij} is also given by

$$\Delta_{ij} = \sum_{k\ge0} (A_{n-i-k}D_{m-j+1+k} - B_{n-i-k}C_{m-j+1+k})$$

$$= \sum_{k\ge0} (B_{n-i+k+1}C_{m-j-k} - A_{n-i+k+1}D_{m-j-k})$$

where $A(s) = \sum_{i=0}^{n} A_i s^{n-i}$, etc. These two formulas trivially give expressions for Δ involving triangular "striped" matrices, the first of which is

$$\Delta = \begin{bmatrix} A_{n-1} & A_{n-2} & A_{n-3} & \cdot \\ A_{n-2} & A_{n-3} & \cdot & \\ A_{n-3} & \cdot & & \\ \cdot & & & \\ \cdot & & & \end{bmatrix} \begin{bmatrix} D_m & D_{m-1} & D_{m-2} & \cdot & \cdot \\ 0 & D_m & D_{m-1} & & \\ 0 & 0 & D_m & & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix}$$

$$
-\begin{bmatrix} B_{n-1} & B_{n-2} & B_{n-3} & \cdot & \cdot \\ B_{n-2} & B_{n-3} & \cdot & & \\ B_{n-3} & \cdot & & & \\ \cdot & & & & \\ \cdot & & & & \end{bmatrix} \begin{bmatrix} C_m & C_{m-1} & C_{m-2} & \cdot & \cdot \\ 0 & C_m & C_{m-1} & & \\ 0 & 0 & C_m & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \end{bmatrix}, \quad (4)
$$

The second is obtained in a similar way.

Using the formula (2), several trivial identities can be established between the Bezoutian matrices of related transfer function matrices, [3]

1. Let $V(z) = W(z) + J$, so that $(A,B+AJ)$ and $(C,D+JC)$ are MFDs of $V(z)$. Then $\Gamma_V = \Gamma_W$.

2. Let $U(z) = W(z)[I+FW(z)]^{-1}$ for some constant F; so that $(A+BF,B)$ and $(C+FD,D)$ are MFDs of $U(z)$. Then $\Gamma_U = \Gamma_W$

3. Let W be invertible, so that (B,A) and (D,C) are MFDs of $W^{-1}(z)$. Then $\Gamma_{W^{-1}} = -\Gamma_W$.

When $W(s)$ is a scalar, the matrix Δ can be viewed as a coordinate basis transformation matrix between two canonical state-space representations of $W(s)$, see [4]. It appears as if an analogous result holds for matrix $W(s)$, but the details have yet to be worked out in full.

4. RANK INFORMATION IN Δ

An old result associated with scalar Bezoutians is that the nullity of Δ is the degree of the greatest common divisor of $p(s)$ and $q(s)$, see [5]. A translation of that result to the matrix case turns out to be the following, [3].

<u>Theorem</u> Let Δ be formed from two MFDs of a transfer function matrix $W(s)$, as described above. Then rank $\Delta = \delta[W(s)]$, the McMillan degree of $W(s)$, or, in case $W(s)$ is proper, the dimension of a minimal state-variable realization of $W(s)$.

Several points should be noted.

1. The structure in the formula (4) may allow the development of fast computational algorithms for evaluating the rank of Δ.

2. Δ is not uniquely defined by $W(s)$, but will vary with the MFDs chosen. However, as the theorem shows, all Δ have the same rank. In fact, it is possible to show, see [3], that there is a particular pair of MFDs with an associated Bezoutian matrix, call it $\bar{\Delta}$, such that any other Δ is of the form $\Delta = E\bar{\Delta}F$ where E and F are matrices of full column and row rank respectively.

3. One way of obtaining the result of the theorem is to use an important formula linking Δ with a truncated Hankel matrix associated with $W(s)$, at least when $W(s)$ is proper. Suppose that $W(s) = W_0 s^{-1} + W_1 s^{-2} + \ldots$ and let H_{nm} denote the matrix with n block rows and m block columns with block i-j element W_{i+j-2}. Then

$$
\Delta = \begin{bmatrix} A_{n-1} & A_{n-2} & \cdots & A_0 \\ A_{n-2} & A_{n-3} & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ A_0 & 0 & \cdots & 0 \end{bmatrix} H_{nm} \begin{bmatrix} C_{m-1} & C_{m-2} & \cdots & C_0 \\ C_{m-2} & C_{m-3} & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ C_0 & 0 & \cdots & 0 \end{bmatrix} \quad (5)
$$

Linear systems theory [6] tells us that rank $H_{nm} = \delta[W(s)]$, and if A_0, C_0 are nonsingular, one immediately has rank Δ = rank H_{nm}. (The difficult part in proving the theorem is handling singular A_0 or C_0)

4. The theorem provides a test for relative primeness of (A,B) and (C,D). For if (A,B) are relatively prime, then $\delta[W(s)] = \deg[\det A(s)]$, by a result of linear system theory [2]. Thus if rank $\Delta = \deg[\det A(s)]$, (A,B) are relatively prime. Similarly for (C,D).

5. When (A,B) or (C,D) are not relatively prime it is possible to find their greatest common divisor by constructing an echelon form. Suppose the g.c.d. of (C,D) is desired. Let $\tilde{\Delta}$ denote Δ with its block columns reversed. Form the row echelon form of

$$
\left[\begin{array}{c|c} C_0 & C_1 \; \cdots \; C_m \\ \hline 0 & \\ \cdot & \tilde{\Delta} \\ \cdot & \\ \cdot & \end{array}\right]
$$

where $C(s) = C_0 s^m + C_1 s^{m-1} + \ldots + C_m$. Select rows according to the following scheme. Choose the last row. If its leading element is in column ℓ_1, delete all rows whose leading element is in column $\ell_1 - iq$ for any integer i, where $C(s)$ is $q \times q$. Choose the second last row remaining. If its leading element is in column ℓ_2, delete all rows with leading element in column $\ell_2 - iq$. Continue until all rows have been deleted or included. Post multiply $[s^m I \; s^{m-1} \; \ldots \; I]'$ to obtain the g.c.d. A similar procedure based on the column echelon form recovers the g.c.d. of (A,B).

6. The row echelon form of the same matrix mentioned in 5 also contains information as to the observability and controllability indices of $W(s)$. This will be reported in future work.

7. Specialized versions of these results [usually requiring that $A(s)$ be of the form $a(s)I$ for a scalar polynomial $a(s)$] have been known for some little time, see e.g. [7].

5. SYMMETRIC TRANSFER FUNCTION MATRICES

Reference to the formula (1) for the classical Bezoutian form and comparison with (2) shows that (1) is the form associated with a left MFD (p,q) and a right MFD (p,q); the resulting Δ is symmetric. The evident self-duality of the left and right MFDs and resulting symmetry in Δ is not inherited in the transfer function matrix case, at least in general. However if we demand that

$$W(s) = W'(s) \tag{6}$$

we can make some progress. For then if (A,B) is a left MFD, (A', B') is a naturally associated right MFD, and one can verify that Δ is then symmetric.

One is then led to ask whether the signature of Δ has any significance. In the case of scalar $w(s)$, this is certainly so: one has, [4],

$$\text{signature } \Delta = \operatorname*{I}_{-\infty}^{+\infty} w(s) \tag{7}$$

Here, $\operatorname*{I}_{-\infty}^{+\infty} w(s)$, denotes the Cauchy index of $w(s)$ evaluated over $(-\infty,\infty)$, which is the number of jumps of $w(s)$ from $-\infty$ to $+\infty$ less the number from $+\infty$ to $-\infty$ as s moves from $-\infty$ to $+\infty$, [8]. The Cauchy index is a useful tool for counting the zeros of a polynomial along the real axis, imaginary axis, or in a half plane. For example, if $p(s)$ is a polynomial, $\operatorname*{I}_{-\infty}^{+\infty} p'(s)/p(s)$ is the number of distinct real zeros of $p(s)$ in $(-\infty,\infty)$, [8].

When $W(s)$ is a real rational matrix, we can proceed as follows.

<u>Definition</u> The Cauchy index of $W(s) = W'(s)$ over (a,b), written $\operatorname*{I}_{-\infty}^{+\infty} W(s)$, is the number of jumps from $-\infty$ to $+\infty$ less the number of jumps from ∞ to $-\infty$ of eigenvalues of $W(s)$ as s moves from (a,b), jumps at a and b not being counted.

One can then prove the following, [9]

<u>Theorem</u> Let $W(s) = W'(s)$, let (A,B) be a left MFD of $W(s)$. Let Δ be defined using (A',B') as a right MFD. Then

$$\text{signature } \Delta = \operatorname*{I}_{-\infty}^{+\infty} W(s) \tag{8}$$

There are a number of things one can do with the above definition and theorem. One is to obtain various properties of the matrix Cauchy index definition which are analogous to properties involving the scalar Cauchy index as set out in e.g. [8]. This is done in [9], with the most important properties being those showing how the Cauchy index can be defined using a matrix Sturm Theorem, and using the Hankel matrix defined by $W(s)$. The second thing that one can do with the definition is to try to obtain results of a specifically system theoretic nature. Such results can be stated using simply Bezoutian matrix properties if desired but it should be recalled that the statement of the corresponding scalar results have traditionally involved Cauchy index ideas.

In this paper, we examine two system theoretic results, associated with lossless positive real matrices and stability of matrix polynomials.

6. LOSSLESS POSITIVE REAL MATRICES

Let $Z(s)$ be an $n\times n$ rational transfer function matrix. We say that $Z(s)$ is lossless positive real [10] if and only if $Z(s) + Z'(-s) = 0$ and all poles of $Z(s)$ are pure imaginary, simple, and have positive residue matrix. Lossless positive real matrices correspond to impedances of lossless networks.

The next theorem is established in [9] and [11]. It has a Cauchy index interpretation: if one sets $\omega = s/j$, and $W(\omega) = jZ(j\omega)$, then $W(\omega) = W'^*(\omega)$, and $\operatorname*{I}_{-\infty}^{+\infty} W(\omega)$ can be well defined. The pole constraints on $Z(s)$ translate to constraints on the real poles of $W(\omega)$ which in turn translates to a statement concerning $\operatorname*{I}_{-\infty}^{+\infty} W(\omega)$. For details, see [9]

<u>Theorem</u> Let $W(s) = A^{-1}(s)B(s)$ be such that $W(s)+W'(-s) = 0$. With Δ formed using (A,B) and $(-A'(-s),B'(-s))$, a naturally associated right MFD, and with $\Sigma = \text{diag}\,[I,-I,I,-I,\ldots]$, $W(s)$ is lossless positive real if and only if $\Sigma\Delta$ is nonnegative definite symmetric.

Results are also obtainable for two classes of positive real matrices closely linked to lossless positive real matricies, viz. RC and RL positive real matrices, see [9].

The above theorem can be thought of as a multivariable extension of a result derived in [12], where the role of $\Sigma\Delta$ is filled by the classical Hermite matrix.

7. STABILITY OF MATRIX POLYNOMIALS

Given a MFD $A^{-1}(s)B(s)$, it will often be of interest to know whether the associated transfer function matrix is stable, i.e. $\det A(s)$ has all its zeros in $\text{Re}[s] < 0$. One way to check this condition is clearly to evaluate the determinant and then to apply a classical stability test. But since classical stability tests in effect amount to Cauchy index calculations, it is clearly reasonable to attempt to find a more direct test, not involving the computation of $\det A(s)$, but using a matrix Cauchy index concept.

To obtain a guide as to what to do, we recall some results for scalar $a(s)$. Let $a(s)$ be monic and of even degree. One can show that $a(s)$ is stable if and only if $z(s) = \dfrac{a(s) - a(-s)}{a(s) + a(-s)}$ is lossless positive real, with one proof of this result involving use of the Hermite matrix [12]. This suggests that one should, in the matrix case, attempt to pass from $A(s)$ to a matrix $Z(s)$, perhaps $[A(s) + A(-s)]^{-1}[A(s)-A(-s)]$, with the property that $Z(s)$ is lossless positive real if and only if $\det A(s)$ is stable. The lossless positive real property can be checked via the Bezoutian Δ, and in this way straightforward manipulation of the coefficients of $A(s)$ is involved. Unfortunately, the connection between $A(s)$ and $Z(s)$ as stated above is not satisfactory. Instead, we have the following

Theorem Let $A(s)$ be an $n \times n$ matrix polynomial and let $B(s)$ be another $n \times n$ matrix polynomial such that

$$B(s)B'(-s) = A'(-s)A(s) \quad (9a)$$
$$\det A(s) = \det B(s) \quad (9b)$$

Then $\det A(s)$ is stable if and only if

$$Z(s) = (A+B_*)^{-1}(A-B_*) \quad (10)$$

with $B_*(s) = B'(-s)$ is lossless positive real.

The proof will be presented elsewhere. Several points should be noted. In the scalar case, because $A(s)$ commutes with $A'(-s)$, (9) implies that $A(s) = B(s)$ and (10) reduces to the expression given earlier. Thus there is a self duality present in the scalar case which is not in general present in the matrix case. Second, given $A(s)$ alone, the theorem will only define a computationally attractive method for checking stability if $B(s)$ can be easily found. Techniques for computing $B(s)$ are available, but do not seem straightforward, and may not necessarily involve less work then that involved in computing $|A(s)|$. Third, the self duality of the scalar case and difficulties of computing the dual polynomial in the matrix case are also encountered in Levinson filtering [13]

8. FUTURE WORK

Let us note three directions in which work is at present going forward

1. The generalized Bezoutian is closely connected with a generalized Sylvester matrix [3]. The development and use of these connections is being investigated.

2. A square nonsingular Toeplitz matrix has an inverse which is a (classical) Bezoutian matrix. Multivariable generalizations are being examined.

3. The classical Bezoutian figures in a number of formulas connecting coordinate basis transformations between different canonical state-space representations of the one scalar transfer function. Generalizations are being sought, which would give added insight into formulas like, for example, (5).

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