A SOLUTION TO THE LINEAR-QUADRATIC SINGULAR CONTROL PROBLEM

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Abstract

General linear-quadratic singular variational problems with free end-point are examined. Procedures are given for establishing whether a prescribed problem has a solution, and for computing the optimal performance index and optimal controls when these exist.

1. Introduction

In this paper we are concerned with the existence and computation of optimal controls in the singular, linear-quadratic control problem with no end-point constraints. In conjunction with this we are also interested in the problem of finding necessary and sufficient conditions for a quadratic cost functional to be bounded below independently of the control function, subject to linear differential equation constraints.

In particular, on the finite interval $[t_0, T]$ define the cost functional

$$V[x_0, u(\cdot)] = x'(T)Sx(T) + \int_{t_0}^{T} \{x'Qx + 2x'Hu + u'Ru\}dt \qquad (1.1)$$

for each admissible control function $u(\cdot)$ (to be specified subsequently), and let the linear differential constraints be

$$\dot{x} = Fx + Gu \qquad x(t_0) = x_0. \qquad (1.2)$$

In (1.1) and (1.2), u is an m-dimensional control vector and x is an n-dimensional state vector. The matrices Q, H, R, F and G have dimensions consistent with x and u, and are time-varying; S is a constant matrix. Without loss of generality we assume S, Q and R are symmetric. We further assume that the various matrices F, G, etc., all have differentiability properties sufficient to allow the carrying out of certain transformations (involving differentiation) which are explained subsequently. The number of such transformations can vary from problem to problem and so consequently does the required degree of differentiability of the coefficient matrices. At each stage of the development of the algorithm described in this paper we will state the degree of differentiability sufficient for the carrying out of that stage. (In case only one complete cycle of the algorithm is required, continuous differentiability of Q, R, F and G, and H twice continuously differentiable are sufficient.) Besides this class of assumptions, we shall also on occasions need to assume the constancy of rank on $[t_0, T]$ of certain matrices constructed from F, G, etc.

Define the set of admissible controls U as the set of m-vector functions $u(\cdot)$ such that each component of $u(\cdot)$ is piecewise continuous on $[t_0, T]$.

Let us now define two problems of interest, the first problem being more general than the second.

Problem 1: The Nonnegativity Problem

Find necessary and sufficient conditions for $V[0, u(\cdot)]$ to be nonnegative for all $u(\cdot) \in U$. Equivalently, find necessary and sufficient conditions for $V^*[0] = 0$ where $V^*[x_0] = \inf_{u(\cdot)\in U} V[x_0, u(\cdot)]$.

Problem 2: The Finiteness Problem

Find necessary and sufficient conditions for $V[x_0, u(\cdot)]$ to be bounded below independently of $u(\cdot)\in U$ for each x_0 i.e. for $V^*[x_0]$ to be finite for each x_0.

A well-known necessary condition for the nonnegativity problem (and therefore for the finiteness problem) is $R(t) \geq 0$ for all $t\in[t_0, T]$, this being the classical Legendre-Clebsch condition for the second variation [1]. In case $R(t) > 0$ for all $t\in[t_0, T]$ the problem is termed nonsingular and is easily solved by Riccati equation methods. The interesting cases are those when $R(t) = 0$ (the totally singular case), and $R(t)$ is nonzero and singular (the partially singular case).

These problems are of importance in several areas. The nonnegativity problem is just the second variation problem of optimal control [2] and also appears in the dual control problem of covariance factorization [5], [6], [7], while the finiteness problem is closely connected with linear-quadratic control [3], [4] and also with the synthesis of passive networks [8].

Initially, it was as the second variation problem of optimal control that the nonnegativity problem was studied. Stronger necessary conditions than the classical Legendre-Clebsch condition were needed to eliminate singular extremals from consideration as minimizing arcs for problems which arose in aerospace trajectory optimization. For more detailed information of the history of this problem see the surveys [9], [10] and the references therein. Arising from these studies were the generalized Legendre-Clebsch conditions which in the totally singular case can be written as

$$\frac{\partial}{\partial u}\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right)' = 0 \quad \text{on } [t_0, T] \tag{1.4}$$

$$-\frac{\partial}{\partial u}\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial u}\right)' \geq 0 \quad \text{on } [t_0, T] \tag{1.5}$$

where H is the Hamiltonian associated with (1.1) and (1.2), i.e.

$$H = x'Qx + 2x'Hu + \lambda'(Fx + Gu) \tag{1.6}$$

$$-\dot{\lambda} = \frac{\partial H}{\partial x} \tag{1.7}$$

and λ is the costate vector. If (1.5) is met with equality, the procedure leading to (1.4) and (1.5) can be extended to give further necessary conditions. In general, the necessary conditions become

$$\frac{\partial}{\partial u}\frac{d^q}{dt^q}\left(\frac{\partial H}{\partial u}\right)' = 0 \quad \text{on } [t_0, T], \quad q < 2p \tag{1.8}$$

$$(-1)^p\frac{\partial}{\partial u}\frac{d^{2p}}{dt^{2p}}\left(\frac{\partial H}{\partial u}\right)' \geq 0 \quad \text{but nonzero on } [t_0, T] \tag{1.9}$$

where $\frac{d^{2p}}{dt^{2p}}\left(\frac{\partial H}{\partial u}\right)'$ is the lowest order time derivative of $\left(\frac{\partial H}{\partial u}\right)'$ in which some component of the control u appears explicity with a nonzero coefficient. The integer p is called the order of the singular arc for scalar u; for vector u an extension of this definition is needed [15].

These conditions (1.8), (1.9) were initially derived by Kelley [11], [12] for scalar controls only, in which case it is not difficult to show that, for odd q, (1.8) is automatically satisfied. The original derivation [11] used the classical method of constructing special variations and considering terms of comparable orders. In [12] and [13], a transformation technique for deriving (1.9) is described and it is this transformation which will be studied in this paper for the general case of vector controls.

It should be noted that passing from the scalar case to the vector case is generally far from easy. To suggest why the extension is nontrivial, (1.5) can be examined. In the scalar case, two possibilities arise, of equality (leading on to (1.8) and (1.9)) or inequality. In the vector control case, there are really three possibilities; (1.5) can hold with equality (leading as before to (1.8) and (1.9), or at least some of these equations) or it can hold with strict inequality (as with the scalar case), or it can hold with a loose inequality, the matrix on the left side of (1.5) being singular and nonzero. Some modification of (1.8) and (1.9) is called for to cope with this case. In the dual problem of spectral factorization, the vector problem though now solved, took much longer to solve than the scalar problem; this fact also suggests the nontriviality of the scalar-to-vector extension. Nevertheless results have been obtained for the vector control problem; a general form of the generalized Legendre-Clebsch conditions [(1.8) and (1.9) being inadequate to cover all possibilities as just noted] has been derived by Goh [14] and Robbins [15]. Robbins' method was essentially variational, whereas Goh used a transformation on the states and controls in a treatment which represents an application of work he had done on the singular Bolza problem in the calculus of variations [16]. Though both Kelley and Goh use transformation methods, there is a major difference in the style of the transformations. Kelley's transformation procedure replaces the original performance index and linear system equation by one involving a state variable of lower dimension than the original. Goh retains the full state-space dimension and there arise as a result a number of extra constraint conditions over and above those which might fairly be termed generalized Legendre-Clebsch conditions. These extra constraint conditions have been examined at length in [22].

Subsequent work on the nonnegativity problem has culminated in the following theorem.

Theorem 1.1 [18] Suppose (1.2) is controllable from time t_0 to time τ for all $\tau \in (t_0, T]$; that is,

$$\int_{t_0}^{\tau} \phi(\tau,\sigma)G(\sigma)G'(\sigma)\phi'(\tau,\sigma)d\sigma > 0 \quad \text{for } \tau \in (t_0, T] \tag{1.10}$$

where

$$\frac{\partial}{\partial \tau}\phi(\tau,\sigma) = F(\tau)\phi(\tau,\sigma), \quad \phi(\sigma,\sigma) = I.$$

Then, assuming only continuity of F, G, H, Q and R,

(a) a necessary condition for $V[0, u(\cdot)]$ to be nonnegative for all $u(\cdot) \in U$ subject to (1.2) is that there exists an $n\times n$ matrix $P(t)$, symmetric, and of bounded variation on each interval, $[t_1, t_2] \subset (t_0, T]$ such that

(i) $P(T) = S$ (1.11)

and (ii) the Riemann-Stieltjes integral inequality

$$\int_{t_1}^{t_2} [x' \; u'] \begin{bmatrix} dP + (PF + F'P + Q)dt & (PG+H)dt \\ (PG+H)'dt & R\,dt \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \geq 0 \tag{1.12}$$

holds for all $u(\cdot) \in U$, subject to $\dot{x} = Fx + Gu$ on $[t_1, t_2]$ and $x(t_1)$ arbitrary.

(b) A sufficient condition is that there exists a symmetric $P(t)$ of bounded variation on $[t_0, T]$, with $P(T) \leq S$, and such that (1.2) holds on $[t_0, T]$. (The controllability assumption (1.10) is not required for sufficiency).

Several points should be noted. First this theorem is purely an existence theorem. It gives no insight into how any matrix P satisfying the theorem might be computed. Second, it is not immediately clear what the connection is, if any, to the control problem where one is of course also interested in establishing conditions for the existence of an optimal control and performance index, and in evaluating these quantities. Third, there is evidently a gap between the necessary and sufficient conditions, and there are examples where the necessity conditions holds, but not the sufficiency condition. (Interestingly, in studying the case of arbitrary x_0, it will turn out that the gap between necessity and sufficiency conditions disappears). Fourth, suppose (1.2) is not controllable, i.e. (1.10) fails. A condition for nonnegativity of $V[0, u(\cdot)]$ could be obtained by studying the controllable part of (1.2) alone, since, with $x(t_0) = 0$, it is only this part of (1.2) which can affect the value taken by $V[0, u(\cdot)]$.

The proof of a theorem very similar to Theorem 1.1 appears in [17] as an extension of the totally singular case studied in [19]. The approach in [19] was to regularize the singular problem, replacing it with a nonsingular one obtained by adding the positive quantity $\int_{t_0}^{T} \varepsilon u'u \, dt$ to the cost in the singular problem and allowing ε to approach zero. The nonsingular problem is of course much easier to solve, but one naturally has to prove things concerning the limit as $\varepsilon \to 0$. A cleaner derivation, bypassing the need to obtain conditions for the totally singular case prior to the partially singular case, is to be found in [6] and [8] (modulo minor changes such as time reversal); an important feature of the proof is to use the Helly convergence theorem for sequences of functions of bounded variation. A proof of a quite different character in [18] proceeds by establishing that the performance index is of a quadratic nature irrespective of the singularity or nonsingularity of the problem.

In this paper, we are interested in a set of necessary and sufficient conditions for $V[0, u(\cdot)]$ to be nonnegative for all $u(\cdot) \in U$ of more limited applicability than those of Theorem 1.1. The limitation stems from the need to have certain differentiability and constancy of rank conditions satisfied; the advantage gained is that the conditions are highly pertinent to the problem of computing an optimal control and performance index for the pair (1.1), (1.2).

These conditions fit in with previous work in the following way. First, they are an extension of conditions published in [20] applicable to scalar controls when the singular arcs are of order 1. Second, the conditions are obtained using a vector generalization of the Kelley transformation (which it should be recalled, is limited to the scalar control case). Third, the various steps required to obtain the conditions are in large measure the dual of those arising in an algorithm of Anderson and Moylan, which we now discuss further.

The problem of constructing the P matrix in (1.12) is central to the problem of covariance factorization and time-varying passive network synthesis. It was in the latter context that an algorithm suitable for the stationary case was developed [21], and then it was recognized that this algorithm with variations was also applicable to the time-varying synthesis problem [8], and with other variations to the covariance factorization problem [6]. An algorithm was in fact suggested in [6] for finding a P matrix satisfying (1.12) under additional differentiability and constancy of rank assumptions. In our work, we show that the Anderson-Moylan algorithm is precisely Kelley's transformation executed in a particular co-ordinate basis and in showing this, we derive the generalized Legendre-Clebsch conditions in a reasonably straightforward manner.

In connection with the optimal control problem associated with free $x(t_0)$, the Anderson-Moylan algorithm, considered in isolation from the Kelley transformation procedure, can be shown to yield the optimal performance index. Linking it with the Kelley transformation procedure yields the optimal controls as well.

Essential to the development that follows is the reduction of the given problem at least once to a standard form, via co-ordinate transformations of the input and state spaces. Details of these transformations can be found in [6] and [8]. The transformation to standard form in conjunction with the problem posed, be it the nonnegativity or finiteness problem, may involve a reduction in the dimension of the control space of the original problem i.e. the given problem may be replaced by an equivalent though simply related problem of the same form but of lower control space dimension. We will not discuss the details here.

In order to reduce to standard form, we must assume that $R(t)$ has constant rank, say p, on $[0, T]$; second, that the hypotheses of either the nonnegativity or finiteness problems hold, and, third, that another matrix constructed from the coefficient matrices also has constant rank on $[0, T]$. A sufficient differentiability requirement is that F, G, Q, H and R be continuously differentiable. These conditions are tantamount to demanding that the structure of the problem does not change on the interval of interest; for example, the order of the singular strip is constant throughout $[t_0, T]$.

The standard form that we assume is that

$$R = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} G_{11} & 0 \\ G_{21} & I_{m-p} \end{bmatrix} \tag{1.13}$$

where I_p is the $p \times p$ identity matrix ($0 \le p \le m$), and the partitions are consistent with the dimensions of R and G. We also assume that the state variable x is partitioned as $[x_1' \quad x_2']'$ where x_2 is $m-p$ dimensional and that the control u is partitioned as $[u_1' \quad u_2']'$ where u_2 is $m-p$ dimensional. We call u_1 and u_2 the nonsingular and singular controls respectively; this nomenclature arises because given the form of R in (1.13), the quadratic term $u_1'u_1$ occurs in (1.1) while the vector u_2 occurs at most linearly through the term $x'Hu$. Moreover, the singular controls all independently influence the state x (as is clear from (1.13)) and hence the cost functional (1.1).

We now outline the layout of the paper. Section II develops the general Kelley transformation for the finiteness (nonnegativity) problem in standard form, producing the generalized Legendre-Clebsch conditions and a set each of necessary and sufficient conditions for the existence of a solution to the finiteness (nonnegativity) problem. These results are then applied to the linear-quadratic optimal control problem. Minimizing controls and the corresponding minimum cost are also calculated. The results in Section III link an algorithm used in the dual problem of covariance factorization with the algorithm outlined in Section II. Section IV contains concluding remarks.

II Direct Approach

In this section we are primarily interested in the general linear-quadratic control problem which can be stated as:

> Find necessary and sufficient conditions for $V[x_0, u(\cdot)]$ to be bounded below independently of $u(\cdot) \in U$, subject to (1.2) with initial condition x_0. Moreover, when the lower bound exists, find a minimizing (optimal) control and the corresponding minimal (optimal) cost. (2.1)

Initially, however, we extend Kelley's transformation to the vector case, assuming that the problem has already been reduced to standard form. On application of this transformation to the finiteness (nonnegative) problem we obtain a set of necessary and sufficient conditions consisting of the existence of a solution of another finiteness (nonnegative) problem of lower state dimension, a set of end-point conditions, and the relevant generalized Legendre-Clebsch condition corresponding to equation (1.4). Finally, in this section, we give a solution to the singular linear-quadratic control problem.

Assume that we are given (1.1) and (1.2) with R and G given by (1.13), and that we are interested in the finiteness problem. Construct the Mayer form of the problem by introducing the scalar variable w_0 defined by

$$\dot{w}_0 = x'Qx + 2x'Hu + u'Ru, \quad w_0(t_0) = 0. \quad (2.2)$$

Equations (1.2) and (2.2) now define a set of $(n+1)$ differential equations in the variables w_0 and x. Recalling the standard form of R and G, and the resultant partitioning of u and x, it is clear that (2.2) involves u_2 linearly but not quadratically. Moreover, from the partitioned form of (1.2), with obvious definitions of F_{11}, etc.,

$$\dot{x}_1 = F_{11}x_1 + F_{12}x_2 + G_{11}u_1 \quad (2.3)$$

$$\dot{x}_2 = F_{21}x_1 + F_{22}x_2 + G_{21}u_1 + u_2 \quad (2.4)$$

we see that (2.3) is influenced by u_2 only indirectly via (2.4). Clearly, if (2.2) did not contain a u_2 term at all, the original problem could intuitively be replaced by one with state x_1 (of lower dimension than x) and controls x_2 and u_1, since u_2 is essentially x_2 differentiated. With this in mind, we define on $[t_0, T]$ the nonsingular transformation

$$z_0 = w_0 - 2x_1'H_{12}x_2 - x_2'H_{22}^{S}x_2 \quad (2.5)$$

$$z_1 = x_1 \quad (2.6)$$

$$z_2 = x_2, \quad (2.7)$$

H_{22}^{S} being the symmetric part of H_{22}. This transformation can be derived by essentially following the development of the scalar case in [13] and [20].

It is now convenient to introduce the notation

$$\hat{Q} = Q_{11} - H_{12}F_{21} - F_{21}'H_{12}'$$

$$\hat{H}_1 = Q_{12} - F_{11}'H_{12} - H_{12}F_{22} - \dot{H}_{12} - F_{21}'H_{22}^{S}$$

$$\hat{H}_2 = H_{11} - H_{12}G_{21} \quad (2.8)$$

$$\hat{R}_1 = Q_{22} - F_{12}'H_{12} - H_{12}'F_{12} - F_{22}'H_{22}^{S} - H_{22}^{S}F_{22} - \dot{H}_{22}^{S}$$

$$\hat{R}_2 = H_{21} - H_{12}'G_{11} - H_{22}^{S}G_{21};$$

H_{22}^{A} is the anti-symmetric part of H_{22}

and then define

$$\hat{x} = x_1, \quad \hat{u} = [\hat{u}_1' \quad \hat{u}_2']' = [x_2' \quad u_1']'$$

$$\hat{F} = F_{11}, \quad \hat{G} = [F_{12} \quad G_{11}], \quad \hat{H} = [\hat{H}_1 \quad \hat{H}_2] \quad (2.9)$$

$$\hat{R} = \begin{bmatrix} \hat{R}_1 & \hat{R}_2 \\ \hat{R}_2' & I \end{bmatrix}.$$

Using (2.5)-(2.9), in conjunction with (1.1), it proves possible to write

$$V[x_0, u(\cdot)] = [\hat{x}'S_{11}\hat{x} + 2\hat{x}'(S_{12} + H_{12})\hat{u}_1 + \hat{u}_1'(S_{22} + H_{22})\hat{u}_1]_{t=T} + \int_{t_0}^{T} \{\hat{x}'\hat{Q}\hat{x} + 2\hat{x}'\hat{H}\hat{u} + \hat{u}'\hat{R}\hat{u}\}\,dt + \int_{t_0}^{T} \hat{u}_1'H_{22}^{A}\,u_2\,dt$$

$$-[2x_1'H_{12}x_2 + x_2'H_{22}^S x_2]_{t=t_0}. \qquad (2.10)$$

Also, (1.2) becomes

$$\dot{\hat{x}} = \hat{F}\hat{x} + \hat{G}\hat{u}, \quad \hat{x}(t_0) = \hat{x}_0. \qquad (2.11)$$

Given the above, we can state a set of necessary and sufficient conditions for $V^*[x_0]$ to be finite for all x_0, given that the problem is in standard form. We do not prove the result here though some remarks on the proof follow the theorem statement.

Theorem 2.1 Assume continuity of F, G, Q and R, continuous differentiability of H and that the finiteness problem is in standard form. Further, with quantities x, $\hat{u}$, $\hat{F}$, $\hat{G}$, $\hat{H}$, $\hat{R}$, $\hat{S}$ as defined above, set

$$\hat{V}[\hat{x}_0, \hat{u}(\cdot)] = \hat{x}'(T)\hat{S}\hat{x}(T) + \int_{t_0}^{T} \{\hat{x}'\hat{Q}\hat{x} + 2\hat{x}'\hat{H}\hat{u} + \hat{u}'\hat{R}\hat{u}\}dt \qquad (2.12)$$

with $\hat{S} = S_{11} - [(S_{12} + H_{12})'(S_{22} + H_{22})^{\#}(S_{12} + H_{12})]_{t=T}$.

Then $V^*[x_0]$ is finite for each x_0 subject to (1.2) if and only if

(a) $\hat{V}^*[\hat{x}_0]$ is finite for each $\hat{x}_0$, subject to (2.11) (with u_2 calculable from (2.4))

(b) $H_{22}(t)$ is symmetric for each $t \in [t_0, T]$

(c) $S_{22} + H_{22}(T) \geq 0$

(d) $N(S_{22} + H_{22}(T)) \subseteq N(S_{12} + H_{12}(T))$.

Remarks:

(i) The notations $\#$ and N denote pseudo-inverse and null space respectively.

(ii) As stated, this theorem refers to the finiteness problem. Obviously, similar statements hold for the nonnegativity problem.

(iii) Recall that to put the given problem into standard form, it is necessary to make some assumptions on the ranks and differentiability of matrices constructible from the coefficient matrices F, G, etc.

(iv) This theorem is independent of the controllability assumption (1.10). However, (1.10) guarantees the retention of the controllability property in passing to the standard form and thence to the "hat" problem defined in Theorem 2.1 .

(v) Condition (b) follows from a second order analysis of the right side of (2.10) and is just the generalized Legendre-Clebsch condition corresponding to (1.4). Conditions (c) and (d) are a result of minimizing the first term on the right side of (2.10).

(vi) A necessary condition for $\hat{V}^*[\hat{x}_0]$ to be finite for all $\hat{x}_0$ is $\hat{R}(t) \geq 0$ on $[t_0, T]$. This is the generalized Legendre-Clebsch condition corresponding to (1.5).

Theorem 2.1 says that our original singular problem in standard form is equivalent to an identical though possibly nonsingular problem of lower state dimension (condition (a) of Theorem 2.1) plus side conditions ((b), (c) and (d) of Theorem 2.1). If after reducing to standard form $\hat{R}$ is again singular on the interval $[t_0, T]$ and the various differentiability and rank assumptions hold, the process can be repeated to produce yet a lower dimensional problem and further side conditions. Now, since the state dimension is lowered at each application of Theorem 2.1 and the control dimension is lowered at each reduction to standard form, the process must end when either the state dimension shrinks to zero, or the problem becomes nonsingular, or G and H become zero in standard form. However, should the controllability assumption (1.10) be in force, this third possibility cannot occur [see Remark (iv) above].

In case the state dimension shrinks to zero, necessary and sufficient conditions are trivial. In case a nonsingular problem is obtained, necessary and sufficient conditions are given by the classical Jacobi conjugate point condition in the form of a Riccati equation having no escape times on the interval $(t_0, T]$. Finally for G and H zero in standard form a necessary and sufficient sufficient condition is the nonnegativity of $R(t)$ on $[t_0, T]$.

We now switch our attention back to the control problem. To calculate the minimizing control and the corresponding minimal cost, we work backwards from either the nonsingular, or zero state dimension problem, or zero input dimension problem minimizing at each successive state. For the purpose of illustration, support first that after one transformation the problem is nonsingular, i.e. $\hat{R}(t) > 0$ on $[t_0, T]$. Then the necessary and sufficient condition for $\hat{V}^*[\hat{x}_0]$ finite for all $\hat{x}_0$ is that the Riccati equation

$$-\dot{\hat{P}} = \hat{P}\hat{F} + \hat{F}'\hat{P} + \hat{Q} - (\hat{P}\hat{G}+\hat{H})'\hat{R}^{-1}(\hat{P}\hat{G}+\hat{H}), \quad \hat{P}(T) = \hat{S} \qquad (2.13)$$

where $\hat{P}$ is a symmetric square matrix of appropriate dimension, has no escape times on the interval $[t_0, T]$. From standard linear regulator theory we know that the optimal control for the cost term $\hat{V}[\hat{x}_0, \hat{u}(\cdot)]$ subject to (2.11) is

$$\hat{u}^*(t) = \hat{L}(t)\hat{x}(t) \quad \text{for} \quad t \in [t_0, T] \qquad (2.14)$$

where $\hat{L} = -\hat{R}^{-1}(\hat{G}'\hat{P} + \hat{H}')$ and the corresponding minimum cost is

$$\hat{V}^*[\hat{x}_0] = \hat{x}_0'\hat{P}(t_0)\hat{x}_0. \qquad (2.15)$$

However, we also need to separately minimize the end point term occurring in $V[x_0, u(\cdot)]$ in (2.10); this gives as the optimal value for the control $\hat{u}_1$ at $t = T$,

$$\hat{u}^*(T) = \hat{R}\hat{x}(T) \tag{2.16}$$

where $\hat{R} = -[(S_{22} + H_{22})^{\#}(S_{12} + H_{12})']_{t=T}$ and the corresponding minimal cost for the end-point term of $\hat{x}'(T)\hat{S}\hat{x}(T)$. The optimal value for $\hat{u}_2(T)$ is seen to be indeterminate, the most convenient value being that defined by (2.14). Now considering the optimal control at t_0 we see that $\hat{u}_1^*(t_0)$ is specified as $x_2(t_0)$. Again, we also have $\hat{u}_2^*(t_0)$ arbitrary; the most convenient value being that defined by (2.14).

We now combine the optimal cost and control from the separate optimization problems to obtain the optimal cost and control, in terms of the hat quantities, for the problem (2.1). From (2.15) and (2.10), we can write the optimal value for $V[x_0, u(\cdot)]$ as

$$[x_1'(t_0)\ \ x_2'(t_0)] \begin{bmatrix} \hat{P}(t_0) & -H_{12}(t_0) \\ -H_{12}'(t_0) & -H_{22}(t_0) \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} \tag{2.17}$$

while the optimal control constructed from (2.14) and (2.16) is

$$\begin{aligned} \hat{u}^*(t) &= \hat{L}(t)\hat{x}(t) \quad \text{for } t \in (t_0, T) \\ \hat{u}_1^*(t_0) &= x_2(t_0) \\ \hat{u}_1^*(T) &= \hat{R}\hat{x}(T) \\ \hat{u}_2^*(t_0) &\text{ and } \hat{u}_2^*(T) \text{ determined as discussed above.} \end{aligned} \tag{2.18}$$

The computation of the optimal control for the problem (2.1) is then completed by using (2.4) to determine u_2^* from x_1^*, x_2^* and u_1^* (or in hat notation, $\hat{x}_1^*$, $\hat{u}_1^*$ and $\hat{u}_2^*$); part of the control u_1^* is already determined by u_2^*. The possible occurrence of delta functions in the optimal control u^* at both the initial and final points of the interval $[t_0, T]$ is now apparent since from (2.18) there is the possibility of jumps in the optimal control $\hat{u}^*$ at the end points of $[t_0, T]$. To prevent the possibility of delta functions occurring within the interval $[t_0, T]$, we demand that $\hat{L}(t)$, which is constructed from $\hat{P}$, $\hat{R}$ and $\hat{G}$, be continuously differentiable throughout the interval. Finally, the arbitrary nature of $\hat{u}_2^*(t_0)$ and $\hat{u}_2^*(T)$ introduces non-uniqueness into the choice of optimal control, in the form of non-uniqueness in the delta functions at t_0 and T.

Suppose, second, that the transformed problem has zero state dimension. Then $\hat{V}[\hat{x}_0, \hat{u}(\cdot)]$ is just $\int_{t_0}^{T} \hat{u}'\hat{R}\hat{u}\, dt$; a necessary and sufficient condition for this to be bounded below being $\hat{R} \geq 0$ on $[t_0, T]$. For $\hat{R} > 0$, the optimal (and clearly unique) control is $\hat{u}^*(t) \equiv 0$. However, for $\hat{R}$ of rank s along $[t_0, T]$, a transformation to standard form makes it clear that only the first s components of $\hat{u}$ are required to be set to zero, the remaining components of $\hat{u}$ being arbitrary. Calculation of the optimal control and cost can now be carried out along the lines of the procedure discussed for the case of the transformed problem being nonsingular.

Finally, for G and H being zero in standard form, the minimum value is easily calculated. The corresponding optimal control is then calculated as in the previous paragraph, depending on the rank of $R(t)$.

Any of the three cases discussed above could arise as the first step in the backward procedure required to calculate the optimal control and cost for a problem where more than one transformation is needed to obtain a nonsingular, zero state dimension, or zero input dimension problem. A straightforward extension of the above discussion applies to the calculation of the optimal control and cost for a singular problem from the optimal quantities for a singular problem of lower (but nonzero) state dimension. The major point of interest here is the possibility of the occurrence of derivatives of delta functions at the end-points.

III Indirect Approach

In the solution of the linear-quadratic control problem presented in Section II, the reduction in state dimension and the calculation of the optimal control appear in a reasonably straightforward manner, with the computation of the optimal cost completing the solution of the problem. Here we present an alternative derivation of the results of Section II employing the Anderson-Moylan algorithm in conjunction with a variation of Theorem 1.1. By this method manipulations are made on the matrix measure involving P in (1.12) in order to compute a solution P which we can show to define the performance index; the calculation of the state transformation and optimal control are not part of the main algorithm.

We now connect the finiteness problem with the type of necessary and sufficient conditions stated in Theorem 1.1.

<u>Theorem 3.1</u> Assume continuity of Q, H, R, F and G on $[t_0, T]$. Then $V[x_0, u(\cdot)]$ subject to (1.2), is bounded below for each x_0 independently of $u(\cdot) \in U$ if and only if there exists an $n \times n$ matrix $P(t)$, symmetric, and of bounded variation on each interval $[t_1, t_2] \subset [t_0, T]$ such that (1.11) and (1.12) hold.

<u>Remarks</u>:

(i) This theorem can be proved by the same regularization procedures as can be used to prove Theorem 1.1 (see the discussion following the statement of that theorem) or by adapting and extending slightly the results in [18].

(ii) The gap between the necessary conditions and the sufficient conditions of Theorem 1.1 has been removed in Theorem 3.1 by assuming the existence of the infimum of $V[x_0, u(\cdot)]$ for all x_0 rather than just $x_0 = 0$ as in Theorem 1.1. Moreover, no controllability assumption is needed here.

(iii)The linkage between Theorems 1.1 and 3.1 is really very close. For example, it follows easily that if $V[x_0, u(\cdot)] \geq 0$ for $x_0 = 0$, then $V^*[x_1, t_1] > -\infty$ for all $t_1 \in (t_0, T]$. Converseley, an "extendability" result (see [6] for a restricted version) also holds.

In general, there can be many matrices $P(\cdot)$ satisfying (1.12) with $[t_1, t_2] \subset [t_0, T]$. One particular one however is readily picked out.

Lemma 3.2 Suppose Q, H, R, F and G are continuous on $[t_0, T]$ and (with notation as defined in the proof of Theorem 3.1) $V^*[x_0]$ exists for all x_0. Then $V^*[x_1, t_1] = x_1'P^*(t_1)x_1$ for some symmetric $P^*(\cdot)$ of bounded variation on $[t_0, T]$, satisfying (1.11) and (1.12) for arbitrary $[t_1, t_2] \subset [t_0, T]$, and such that any other $P(t)$ meeting these conditions is such that $P(t) \leq P^*(t)$.

As in the previous sections, assume that G and R are given in standard form and that the corresponding partitioning of the various matrices and vectors hold. Substituting into (1.12), defining $w' = [x_1' \ \ x_2' \ \ u_1']$ and with the obvious definition of dY, we obtain

$$\int_{t_1}^{t_2} w'dYw + 2\int_{t_1}^{t_2} x_1'(P_{12} + H_{12})u_2dt + 2\int_{t_1}^{t_2} x_2'(P_{22} + H_{22})u_2dt \geq 0. \quad (3.1)$$

We can conclude from (3.1) that

$$P_{12}(t) + H_{12}(t) = 0 \quad \text{on} \quad (t_0, T) \qquad (3.2)$$

$$P_{22}(t) + H_{22}(t) = 0 \quad \text{on} \quad (t_0, T). \qquad (3.3)$$

We have now identified the blocks P_{12} and P_{22} of P uniquely for any P satisfying (1.12). Moreover, P is symmetric on (t_0, T) implying by (3.3) the symmetry of H_{22} as a necessary condition.

We can also show that the equalities (3.2) and (3.3) extend to the point t_0 in case $P(t) = P^*(t)$, where $P^*(\cdot)$ is as defined above. The Riemann-Stieltjes integral inequality shows that all jumps in any $P(\cdot)$ satisfying the inequality must be nonnegative, i.e. $P(t_-) \leq P(t) \leq P(t_+)$. It is then clear that at $t = t_0$ taking $P_{12}(t_0) = -H_{12}(t_0)$, $P_{22}(t_0) = -H_{22}(t_0)$ is consistent with the Riemann-Stieltjes integral inequality, and by the maximal property of $P^*(\cdot)$, and in particular $P^*(t_0)$, we must then have $P^*_{12}(t_0) = -H_{12}(t_0)$, $P^*_{22}(t_0) = -H_{22}(t_0)$. A study of the right hand end-point allows us to conclude that $\mathcal{N}[S_{22} + H_{22}(T)] \subseteq \mathcal{N}[S_{12} + H_{12}(T)]$ and $S_{22} + H_{22}(T) \geq 0$.

Further, returning to (3.1) and the definition of dY we can partition dY in an obvious manner with

$$dY_{11} = dP_{11} + (Q_{11} + P_{11}F_{11} + F_{11}'P_{11} + P_{12}F_{21} + F_{21}'P_{12})dt$$

$$dY_{12} = dP_{12} + (Q_{12} + P_{11}F_{12} + P_{12}F_{22} + F_{11}'P_{12} + F_{21}'P_{22})dt$$

$$dY_{22} = dP_{22} + (Q_{22} + P_{22}F_{22} + F_{22}'P_{22} + P_{12}'F_{12} + F_{12}'P_{12})dt \qquad (3.4)$$

$$dY_{13} = (P_{11}G_{11} + P_{12}G_{21} + H_{11})dt$$

$$dY_{23} = (P_{12}'G_{11} + P_{22}G_{21} + H_{21})dt .$$

Now, assuming the differentiability of H and defining $\hat{P} = P_{11}$, we combine the definitions (2.9) with the above to obtain

$$dY = \begin{bmatrix} d\hat{P} + (\hat{P}\hat{F} + \hat{F}'\hat{P} + \hat{Q})dt & (\hat{P}\hat{G} + \hat{H})dt \\ (\hat{P}\hat{G} + \hat{H})'dt & \hat{R}\,dt \end{bmatrix}. \qquad (3.5)$$

Observing that the Riemann-Stieltjes integral of (3.5) has the same form as the original integral (1.12), we attempt to find the relevant minimization problem (of the same form as (3.1)) corresponding to (3.5). However, given our development of Section II it is clear that with the transformation (2.5) - (2.7), the definitions of $\hat{x}$ and $\hat{u}$ as in (2.9) and $\hat{S}$ as defined in Section II, the minimization problem is just that described in the first part of Theorem 2.1. Thus

Lemma 3.3 With the same assumptions as in Theorem 2.1, there exists an $n \times n$ matrix $P(t)$, symmetric and of bounded variation on $[t_0, T]$ such that with arbitrary $[t_1, t_2] \subset [t_0, T]$, (1.11) and (1.12) hold if and only if there exists a matrix $\hat{P}(t)$ of appropriate dimension, symmetric and of bounded variation on $[t_0, T]$ such that

(a) $\hat{P}(T) = \hat{S}$ (3.6)

(b) (3.7)

$$\int_{t_1}^{t_2} [\hat{x}'\hat{u}'] \begin{bmatrix} d\hat{P} + (\hat{P}\hat{F} + \hat{F}'\hat{P} + \hat{Q})dt & (\hat{P}\hat{G} + \hat{H})dt \\ (\hat{P}\hat{G} + \hat{H})dt & \hat{R}\,dt \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{u} \end{bmatrix} \geq 0$$

for all $\hat{u}(\cdot)$ subject to (2.11) with arbitrary $x(t_1)$

(c) $H_{22}(t)$ is symmetric on $[t_0, T]$

(d) $S_{22} + H_{22}(T) \geq 0$

(e) $\mathcal{N}(S_{22} + H_{22}(T)) \subseteq \mathcal{N}(S_{12} + H_{12}(T))$.

Again, as in the previous section , the application of Lemma 3.3 may need to be made a number of times terminating with either a zero dimensional $\hat{P}$ in which case P would be completely and uniquely identified by a series of equalities such as (3.2) and (3.3), or a problem with $\hat{P}$ of positive dimension with $\hat{R}$ nonsingular, or in transforming from nonstandard to standard form a problem with G and H zero may arise. For the second case we can show that (3.6) and (3.7) have a solution $\hat{P}$ if and only if the Riccati equation (2.13) has no escape times on $[t_0, T]$, and, moreover, the unique solution $\hat{P}$ of the Riccati equation is one of many possible solutions of (3.6) and (3.7). Further, $\hat{P}$ is the maximal solution of (3.6) and (3.7) in the sense that for any solution $\hat{P}_1$ of (3.6) and (3.7) we have $\hat{P}_1 \leq \hat{P}$. Finally, $\hat{P}$ as calculated from the Riccati equation is connected to the optimal cost via the standard quadratic form. Tracing back to the original control problem, the solution P of (1.12) so generated defines the optimal cost for each x_0 for problem (3.1).

For the third case when G and H are zero, we have noted earlier what $\hat{P}$ is. Again, one can trace back to a solution P of the original control problem.

IV Conclusions

We begin by reviewing briefly the contributions of the paper. First, we have given an algorithmic procedure for computing a matrix the existence of which is guaranteed by the nonnegativity of a certain functional. Indirectly, this gives a procedure for checking the nonnegativity of the functional. Second, we have shown how this algorithm can also be used in computing the optimal performance index and optimal control (the latter possibly not being unique) for linear-quadratic singular optimal control problems. Several key properties of the algorithm are: its capacity to handle vector control problems; its linkage with, on the one hand, other and possibly less complete approaches to the singular control problem, and on the other hand, with the singular time-varying covariance factorization problem; its disadvantage viz., a requirement that the ranks of certain matrices remain constant over the interval of interest, and that certain matrices enjoy differentiability properties.

Extension to the case of fixed (or partly fixed) end-point problems should hot be particularly difficult. What is interesting to explore here is the connection between fixed and free end-point problems, using ideas analogous to those arising in the extendability result alluded to in the remarks following Theorem 3.1.

There is another possible approach to the optimal control problem which we have not mentioned to this point. By a standard completion of the square device, one can characterise the optimal control, if it exists, in open-loop form as the solution of a linear Fredholm integral equation which is only of the second kind in case the optimal control problem is nonsingular. A solution procedure for the dual singular problem (arising in détection theory) is studied in [23] and could presumably be modified to deal with the control problem.

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