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OUTPUT-NULLING INVARIANT AND CONTROLLABILITY SUBSPACES

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ABSTRACT

Invariant and controllability subspaces are studied in connection with the problem of zeroing the output of a finite-dimensional linear system with a transfer function matrix that is not necessarily zero for infinite frequencies. The concept of output-nulling invariant and controllability subspaces is introduced, and the existence of maximal subspaces with these properties is established. An application is given to characterizing the zeros of a transfer function matrix in terms of these maximal subspaces.

1. INTRODUCTION

The concepts of (A,B)-invariant subspaces controllability subspaces have proved useful in tackling a number of problems of linear system theory, see e.g. [1-4]. One of the crucial properties of these subspaces is that there is a maximal subspace of each type contained within a prescribed subspace of the whole state space. It turns out however that for linear systems with direct feedthrough defined by the quadruple $\{A,B,C,D\}$, i.e. (in standard notation)

$$\dot{x} = Ax + Bu \quad y = Cx + Du \quad D \neq 0 \tag{1}$$

this property is not the appropriate one for deducing helpful results. The purpose of this paper is to explain some other properties of (A,B)-invariant and controllability subspaces which allow derivation of results for (1); these properties replace the maximality property referred to above.

We tackle invariant subspaces first, motivating the problem, summarising known results, and then obtaining results pertinent to (1). The same procedure is then followed with controllability subspaces, and we conclude the new material of the paper by giving a characterization of the zeros of a system of the type (1) using certain invariant and controllability subspaces.

2. (A,B)-INVARIANT SUBSPACES - PROBLEM MOTIVATION

Consider the system

$$\dot{x} = Ax + Bu \tag{2}$$

and suppose that the initial state vector $x(0)$ resides in some subspace V of the state-space X. Suppose further that we wish for there to be a control $u(\cdot)$ defined for $t > 0$ which will ensure that $x(t) \in V$ for all $t \geq 0$. What are the conditions on V?

Clearly, we require $\dot{x} \in V$; alternatively, for all $x(0) \in V$, we require that there exist $v_1 \in V$ and $u \in U$, the space of controls, with

$$v_1 = Ax(0) + Bu$$

Put another way,

$$AV \subset V + B \tag{3}$$

where $B = \{Bu | u \in U\}$. Equation (3) is the defining equation for an (A,B)-invariant subspace.

Suppose now that associated with (2) we have the output mapping equation

$$y = Cx \tag{4}$$

It is then relevant to ask when we can also insist that $y(t) = 0$ for all t, i.e. that $V \subset N(C)$, the null-space of C. It is this question which gives rise to an examination of the (A,B)-invariant subspaces contained in another space.

If however, the output mapping equation is

$$y = Cx + Du \qquad D \neq 0 \tag{5}$$

then to insist that $y(t) \equiv 0$ is not the same as demanding that $V \subset N(C)$; now we want V to have the property that for arbitrary $v \in V$, not only are there $v_1 \in V$ and $u \in U$ with

$$v_1 = Av + Bu$$

but also that

$$Cv + Du = 0$$

For only if this condition holds can we ensure that the $x(\cdot)$ trajectory remains in V while yielding identically zero outputs.

3. (A,B)-INVARIANT SUBSPACES - REVIEW OF KNOWN RESULTS

Here, we quickly summarize results concerning invariant subspaces (as per, for example, [1]) which will either be used, or be paralleled, in the sequel.

1. The defining equation (3) holds if and only if for some F, V is $(A+BF)$-invariant, i.e.

$$(A+BF)V \subset V \tag{6}$$

2. Given a subspace W of X, there is a maximal (A,B)-invariant subspace, call it $\bar{V}$, such that $\bar{V} \subset W$, i.e. if V is an (A,B)-invariant subspace with $V \subset W$, then $V \subset \bar{V}$. [This can be easily proved using (3)]. (7)

3. $\bar{V}$ can be obtained by the following algorithm:*

$$\begin{aligned} V_0 = X \quad V_i &= W \cap A^{-1}(V_{i-1} + B) \\ &= V_{i-1} \cap A^{-1}(V_{i-1} + B) \end{aligned} \tag{8}$$

$\bar{V} = V_k$ where k is the first integer (bounded by dim X) for which $V_k = V_{k+1}$

To tie this back in some measure to the motivating material, take W in (7) to be $N(C)$. Let $\bar{V}$ be the maximal invariant subspace in W, and let $\bar{F}$ be such that $(A+B\bar{F})\bar{V} \subset \bar{V}$, with $\bar{F}$ existing by (6). Now use the feedback control $u = \bar{F}x$ in (2). There results $\dot{x} = (A+B\bar{F})x$, which shows that if $x(0) \in \bar{V}$, then $\dot{x}(0) \in \bar{V}$ and indeed $x(t) \in \bar{V}$ for all t. Further, since $\bar{V} \in N(C)$, $y(t) \equiv 0$ for all t.

4. OUTPUT-NULLING INVARIANT SUBSPACES

For the system with direct feedthrough defined in (1), suppose V is a space such that for arbitrary $v \in V$ there exists some $u \in U$ and $v_1 \in V$ with

$$\begin{aligned} v_1 &= Av + Bu \\ Cv + Du &= 0 \end{aligned} \tag{9}$$

We say that V is an <u>output-nulling invariant subspace</u>. The reason for the output-nulling terminology should be clear; V is also an invariant subspace by virtue of the first defining equation. A tidier description is immediately possible.

<u>Lemma 1</u> V is an output-nulling invariant subspace if any only if

$$\begin{bmatrix} A \\ C \end{bmatrix} V \subset \begin{bmatrix} I \\ 0 \end{bmatrix} V + \begin{bmatrix} B \\ D \end{bmatrix} U \tag{10}$$

Proof: Equation (10) says precisely: given arbitrary $v \in V$, there exists $v_1 \in V$, $u \in U$ with

$$\begin{bmatrix} A \\ C \end{bmatrix} v = \begin{bmatrix} I \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} B \\ D \end{bmatrix} u$$

Modulo a sign change, this is the same as the defining equations (9).

<u>Remark</u> Observe that with $D = 0$, an output-nulling invariant subspace is still a well-defined object - any invariant subspace in $N(C)$ qualifying.

We can now establish parallels to each of (6), (7) and (8).

<u>Lemma 2</u> V is an output-nulling invariant subspace if and only if for some F,

$$\begin{aligned} (A+BF)V &\subset V \\ (C+DF)V &= 0 \end{aligned} \tag{11}$$

Proof: If (11) hold, then for any $v \in V$, we have $v_1 \in V$ with $(A+BF)v = v_1$, $(C+DF)v = 0$. Take $u \in U = -Fv$. Then $Av = v_1 + Bu$, $Cv = Du$, i.e. (10) holds.

Now assume that (10) holds. We shall construct an F such that (11) holds. Let V be a matrix whose columns are a basis for V. Then (10) guarantees that

$$\begin{bmatrix} A \\ C \end{bmatrix} V = \begin{bmatrix} V & B \\ 0 & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \tag{12}$$

has a solution $\begin{bmatrix} X \\ Y \end{bmatrix}$. Thus

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} V & B \\ 0 & D \end{bmatrix}^{\#} \begin{bmatrix} A \\ C \end{bmatrix} V = \begin{bmatrix} F_1 V \\ F_2 V \end{bmatrix}$$

for some F_1 and F_2. Using this in (12) yields

$$\begin{aligned} AV &= VF_1V + BF_2V \\ CV &= DF_2V \end{aligned}$$

Taking $F = -F_2$ yields

$$\begin{aligned} (A+BF)V &= VF_1V \\ (C+DF)V &= 0 \end{aligned}$$

which are equivalent to (11).

<u>Remark</u> The control $u(t) = Fx(t)$ for the system (1) has the property that if $x(0) \in V$, then $x(t) \in V$ and $y(t) = 0$ for all t, when V satisfies (11).

<u>Lemma 3</u> For a given $\{A,B,C,D\}$ quadruple, there is a maximal output-nulling invariant subspace $\bar{V}$.

Proof: Let $\{V_i\}$ be the collection of sets satisfying (10). Then $V = \cup V_i$ satisfies (10), and is clearly maximal.

<u>Remark</u> Though this proof is very straightforward, it is interesting to note that a proof based directly on (11) is not immediately obvious.

<u>Lemma 4</u> For a given $\{A,B,C,D\}$ quadruple, the maximal output-nulling invariant subspace $\bar{V}$ is obtainable by the following algorithm:

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* Let K be a matrix. Let M be a subspace with $M \subset KX$. Then $K^{-1}M = \{x \mid Kx \in M\}$.

$$V_0 = X \qquad V_i = V_{i-1} \cap \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} V_{i-1} + \begin{bmatrix} B \\ D \end{bmatrix} U \right\}$$

$\bar{V} = V_k$ where k is the least integer (bounded by dim X) for which $V_k = V_{k+1}$.

Proof: Suppose $W \supset \bar{V}$. Then

$$\begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} W + \begin{bmatrix} B \\ D \end{bmatrix} U \right\} \supset \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{V} + \begin{bmatrix} B \\ D \end{bmatrix} U \right\} \supset \bar{V}$$

the second inclusion following by (10). Hence

$$W \cap \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} W + \begin{bmatrix} B \\ D \end{bmatrix} U \right\} \supset \bar{V}$$

Applying this with $W = V_0 = X$ yields $V_1 \supset \bar{V}$, and repeating the argument shows that $V_i \supset \bar{V}$ for all i. On the other hand, if $V_k = V_{k+1}$, it is evident that

$$V_k = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} V_k + \begin{bmatrix} B \\ D \end{bmatrix} U \right\}$$

or

$$\begin{bmatrix} A \\ C \end{bmatrix} V_k = \begin{bmatrix} I \\ 0 \end{bmatrix} V_k + \begin{bmatrix} B \\ D \end{bmatrix} U$$

By (10), V_k is an output-nulling invariant subspace, and so $V_k \subset \bar{V}$.

Thus $V_k = \bar{V}$.

5. CONTROLLABILITY SUBSPACES - PROBLEM MOTIVATION

For the system (2), we have defined the concept of an (A,B)-invariant subspace V as one for which it is possible to find a control $u(\cdot)$ such that with $x(0) \in V$, $x(t) \in V$ for all $t \geq 0$. Suppose now that we further require that every $v \in V$ be reachable from the origin, with the associated trajectory lying in V. Such a V is termed a controllability subspace. We shall use R as the generic symbol for such a subspace.

Given the output mapping equation (4), with no direct feedthrough, we can study those controllability subspaces leading to $y(t) \equiv 0$, by looking at the controllability subspaces contained in $N(C)$. When direct feedthrough enters the picture, as in (5), the situation is more complicated.

The logical entity to study now becomes these controllability subspaces with the property that trajectories lying in the subspace [possibly resulting from a nonzero $u(\cdot)$] cause $y(t) \equiv 0$. We shall term such controllability subspaces output-nulling, giving a more precise definition.

6. CONTROLLABILITY SUBSPACES - REVIEW OF KNOWN RESULTS

We define a controllability subspace R as an (A,B)-invariant subspace for which every state in R is reachable from the origin with the associated trajectory lying in R. Then immediately

1. Every controllability subspace R is an (A,B)-invariant subspace.

Other important results are:

2. Let R be an (A,B)-invariant subspace for which $(A+BF)R \subset R$. Then R is a controllability subspace if and only if

$$R = \{A+BF | B \cap R\} \tag{14}$$

where $\{A|B\}$ denotes $B + AB + \ldots$

3. Given a subspace W of X, there is a maximal controllability subspace $\bar{R}$ with $\bar{R} \subset W$. (15)

4. The subspace $\bar{R}$ just defined is given by

$$\bar{R} = \{A+BF | B \cap \bar{V}\} \tag{16}$$

where $\bar{V}$ is the maximal (A,B)-invariant subspace in W and $(A+BF)\bar{V} \subset \bar{V}$.

5. $\bar{R}$ is also defined by

$$R_0 = 0 \qquad R_i = AR_{i-1} + B \cap \bar{V} \tag{17}$$

$\bar{R} = R_k$ where k is the first integer (bounded by dim X) for which $R_k = R_{k+1}$.

An interpretation of equation (14) which we shall draw on later is easily given. Consider equation (2) with $u = Fx + \bar{u}$, i.e.

$$\dot{x} = (A+BF)x + B\bar{u} \tag{18}$$

Now $B \cap R$ is the set of values of $B\bar{u}$, $\bar{u} \in U$, which are contained in R. (This is the same as the values of BGv for some fixed G and variable v). Then (14) says that R (the left side of (14)) is identical with the set of states reachable from the origin of

$$\dot{x} = (A+BF)x + BGv \tag{19}$$

Clearly this makes R a controllability subspace via the original definition; it is even easier to prove the converse, i.e. to deduce (14) given that R is a controllability subspace.

7. OUTPUT-NULLING CONTROLLABILITY SUBSPACES

Let R be a controllability subspace. Then there are (at least in general) a plurality of pairs of matrices (F,G) with the property that controls of the form $u = Fx + Gv$ with v arbitrary allow movement from one point of R to any other without leaving R. We say that a controllability subspace R is output-nulling if the set of $x \in R$ and set of controls defined by at least one but not necessarily all of the (F,G) pairs causes the output $y = Cx + Du$ to be identically zero. Equivalently, it is always possible to move the state between any two points in R maintaining zero output.

Remark In case $D = 0$, this definition reduces to one requiring R to lie in a certain subspace of X, viz. $N(C)$.

Remark It may be the case that although there exists a control taking the zero state to an arbitrary state of R, with the trajectory entirely in R and the output identically zero, there may be other controls still achieving transfer to an arbitrary state, and still with the trajectory entirely in R, but this time with the output not necessarily zero. Suppose for example that $C + DF = 0$, $B = [b \;\; b]$ for some column vector b with $[A+BF, b]$ a completely controllable pair, and $D = [1 \;\; 0]$. Taking $G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ leads to

$$\dot{x} = (A+BF)x + bv \qquad y = (C+DF)x + v = v$$

The controllability subspace R becomes identical with the entire state-space X, but the output-nulling property is not observed with this G. On the other hand, with $G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we obtain

$$\dot{x} = (A+BF)x + bv \qquad y = (C+DF)x = 0$$

and now the output-nulling property is evident.

We have the following characterization of output-nulling controllability subspaces.

Lemma 5 R is an output-nulling controllability subspace if and only if for some F we have

$$R = \{A+BF|\hat{B} \cap R\} \tag{20}$$

$$(C+DF)R = 0 \tag{21}$$

where $\hat{B}$ is the range of the map B restricted to $N(D)$.

Proof: Suppose R is output-nulling, with controls of the form $u = Fx + Gw$ causing $y = Cx + Du = 0$ for arbitrary $x \in R$. Now $0 = y = (C+DF)x + DGw$ for arbitrary $x \in R$ and w implies $(C+DF)R = 0$ and $DG = 0$.

Now the controllability property of R implies that the set of states of the system $\dot{x} = (A+BF)x + BGv$ reachable from the origin is precisely R, i.e. $R = \{A+BF|\text{range}(BG)\}$. Now $\text{range}(BG) \subset R$ and because $DG = 0$, $\text{range}(BG) \subseteq \hat{B}$. So $R \subset \{A+BF|\hat{B} \cap R\}$. Also, $\{A+BF|\hat{B} \cap R\} \subset \{A+BF|R\} = R$. So $R = \{A+BF|\hat{B} \cap R\}$. Thus (20) and (21) hold.

Conversely, we start with (20) and (21). Since $\hat{B} \cap R \subset B \cap R \subset R$, we have

$$R = \{A+BF|\hat{B} \cap R\} \subset \{A+BF|B \cap R\} \subset \{A+BF|R\} = R$$

So $R = \{A+BF|B \cap R\}$ making R a controllability subspace. Now let $\hat{B}$ be a matrix whose columns span $\hat{B}$; then $\hat{B} = BG_1$ for some G_1 with the columns of G_1 spanning $N(D)$, this following by the definition of $\hat{B}$. Further, there exists a matrix G_2 such that the columns of BG_1G_2 span $\hat{B} \cap R$. Equation (20) then implies that the set of states reachable from the origin of

$$\dot{x} = (A+BF)x + BG_1G_2v$$

is precisely R. These states are reachable using the control $u = Fx + Gu$ (where $G = G_1G_2$) on $\dot{x} = Ax + Bu$. The output becomes $y = Cx + Du = (C+DF)x + DG_1G_2v$, which is zero on account of (21) and the fact that columns of G_1 span $N(D)$. Therefore R is output-nulling.

Remark If D is nonsingular, or simply has rank equal to the number of its columns, there are no output-nulling controllability subspaces, since $\hat{B}$ is empty.

Remark The reason for the replacement of $B \cap R$ by $\hat{B} \cap R$ in (20) is roughly the following. The output-nulling condition imposes a constraint on the instantaneous value of the control, as well as the state, viz. the control must be of the form $Fx + Gu$ where F does not concern us here, while $DG = 0$. Thus the closed loop system is of the form $\dot{x} = (A+BF)x + BGv$ and the set $\{BGv\}$ is evidently contained in $\hat{B}$. [When $D = 0$, i.e. there is no direct feedthrough, $\hat{B}$ becomes identical with B, and we are in the standard situation]. Evidently, the state vectors reachable from the origin of $\dot{x} = (A+BF)x + BGv$, i.e. R, lie in $\{A+BF|\hat{B}\}$, which is (in general) a smaller space than $\{A+BF|B\}$.

As one might expect, output-nulling controllability subspaces are also output-nulling (A,B)-invariant subspaces:

Lemma 6 Let R be an output-nulling controllability subspace. Then it is an output-nulling invariant subspace.

Proof: For some F, $R = \{A+BF|\hat{B} \cap R\}$ ensures that R is (A+BF)-invariant:

$$(A+BF)R \subset R$$

In association with (21), this proves the result.

The crucial result on output-nulling controllability subspaces associated with a prescribed quadruple is that there is a maximal one.

Theorem 1: Consider the system (1). Then there is an associated maximal output-nulling controllability subspace defined by

$$\bar{R} = \{A+BF|\hat{B} \cap \bar{V}\} \tag{22}$$

where $\bar{V}$ is the associated maximal output-nulling (A,B)-invariant subspace, $(A+BF)\bar{V} \subset \bar{V}$, $(C+DF)\bar{V} = 0$ and $\hat{B}$ is as defined in the preceding lemma.

Proof: We show first that $\bar{R}$ is an output-nulling controllability subspace. Equation (22) implies that $\hat{B} \cap \bar{V} \subset \bar{R}$, so that $\hat{B} \cap \bar{V} \subset \hat{B} \cap \bar{R}$. Also,

$$\bar{R} = \{A+BF|\hat{B} \cap \bar{V}\} \subset \{A+BF|\bar{V}\} = V$$

so that $\hat{B} \cap \bar{R} \subset \hat{B} \cap \bar{V}$. Hence $\hat{B} \cap \bar{R} = \hat{B} \cap \bar{V}$ and

$$\bar{R} = \{A+BF|\hat{B} \cap \bar{R}\} \tag{20}$$

Since $\bar{R} \subset \bar{V}$ and $(C+DF)\bar{V}$, we have also

$$(C+DF)\bar{R} = 0 \tag{21}$$

The preceding lemma then establishes that $\bar{R}$ is an output-nulling controllability subspace. [Notice that from the point of view of computing $\bar{R}$, (22) is the equation to use, not (20)]

To prove the maximality property, we make the preliminary observation [following from (22)] that if

$$R_i = \sum_{j=0}^{i-1} \{(A+BF)^j|\hat{B} \cap \bar{V}\}$$

or

$$R_0 = 0 \quad R_i = (A+BF)R_{i-1} + \hat{B} \cap \bar{V} \tag{23}$$

then $R_i \subset \bar{R}$ and $\bar{R} = R_k$ where k is the first integer (bounded by dim X) for which $R_k = R_{k+1}$.

Let us now suppose $\tilde{R}$ is an arbitrary output-nulling controllability subspace, so that $(C+D\tilde{F})\tilde{R} = 0$ and $\tilde{R} = \{A+B\tilde{F}|\hat{B} \cap \tilde{R}\}$ for some $\tilde{F}$. Define the sequence $\tilde{R}_i$ by $\tilde{R}_0 = 0$ and

$$\tilde{R}_i = (A+B\tilde{F})\tilde{R}_{i-1} + \hat{B} \cap \tilde{R} \tag{24}$$

Then $\tilde{R} = \tilde{R}_j$ where j is the first integer for which $\tilde{R}_j = \tilde{R}_{j+1}$. We shall show that for each i, $\tilde{R}_i \subset R_i$. From this, the desired maximality property follows as $\tilde{R} \subset \bar{R}$, these subspaces being identical with the limit of the $\tilde{R}_i$ and R_i sequence.

Suppose that $\tilde{R}_i \subset R_i$ is established for $i = 0,1,\ldots,p$. We shall prove that $\tilde{R}_{p+1} \subset R_{p+1}$. To do this, we shall use the fact that for all k

$$B(F-\tilde{F})\tilde{R}_k \subset \hat{B} \cap \bar{V} \tag{25}$$

Equation (25) will be established below. Assuming its validity for the moment, we have

$$\begin{aligned} R_{p+1} &= (A+BF)R_p + \hat{B} \cap \bar{V} && \text{by (23)} \\ &\supset (A+BF)\tilde{R}_p + \hat{B} \cap \bar{V} && \text{by inductive hypothesis} \\ &= (A+B\tilde{F})\tilde{R}_p + B(F-\tilde{F})\tilde{R}_p + \hat{B} \cap \bar{V} \\ &= (A+B\tilde{F})\tilde{R}_p + \hat{B} \cap \bar{V}, && \text{by (25)} \\ &= \tilde{R}_{p+1} && \text{by (24)} \end{aligned}$$

This completes the induction, and it remains to establish (25).

Since $\tilde{R}$ is an output-nulling controllability subspace, it is an output-nulling invariant subspace, and is accordingly contained in the maximal such subspace $\bar{V}$. Now $(C+DF)\bar{V} = 0$, so $(C+DF)\tilde{R} = 0$. Also, $(C+D\tilde{F})\tilde{R} = 0$ and so $D(F-\tilde{F})\tilde{R} = 0$. Further, $\tilde{R}_k \subset \tilde{R}$, and so $D(F-\tilde{F})\tilde{R}_k = 0$. Hence

$$B(F-\tilde{F})\tilde{R}_k \subset \hat{B} \tag{26}$$

Next, $(A+B\tilde{F})\tilde{R}_k \subset \tilde{R} \subsetneq \bar{V}$, and $(A+BF)\tilde{R}_k \subset (A+BF)\bar{V} \subset \bar{V}$. Thus $B(F-\tilde{F})\tilde{R}_k \subset \bar{V}$. Together with (26), this yields (25).

<u>Remark</u> The result of the above theorem parallels (15) through (17), with the constructive procedure in the proof using (23) closest to that used in (17). Actually, it is possible to show that the subspaces R_i defined in (17) are identical with

$$\sum_{j=0}^{i-1} \{(A+BF)^j \mid B \cap \bar{V}\}$$

for any F such that $(A+BF)\bar{V} \subset \bar{V}$. In particular then, the R_i of (17) are also defined by

$$R_0 = 0 \quad R_i = (A+BF)R_{i-1} + B \cap \bar{V}.$$

This might lead one to suspect that for the output-nulling problem, one could define the R_i of (23) by

$$R_0 = 0 \quad R_i = AR_{i-1} + \hat{B} \cap \bar{V} \tag{27}$$

(This formula might be more attractive as a tool for computing $\bar{R}$ than (23), since no F is required). Equation (27) is however simply not correct. What one can say though is that if F_1 and F_2 have the property that $(A+BF_\alpha)\bar{V} \subset \bar{V}$ and $(C+DF_\alpha)\bar{V} = 0$ for $\alpha = 1,2$, then the same R_i are defined by both F_1 and F_2 in (23). This follows by an argument similar to that used in proving the maximality of $\bar{R}$.

8. DUAL RESULTS

A dual concept to that of the largest (A,B)-invariant subspace contained in a given subspace of the state space is developed in [3], and is applied to illustrating a further property of the maximal controllability subspace. Here, we develop various dual ideas in the context of the system (1) with direct feedthrough. The basic approach is to take one of the concepts already established and apply it to the dual of (1), viz.

$$\dot{x}_d = A'x_d + C'u_d \quad y_d = B'x_d + D'u_d \tag{28}$$

and then to reinterpret in terms of (1). The material of the section culminates in a further characterization of the subspace $\bar{R}$ of the previous section.

Below, we shall use the notation $S^\perp$ to denote the subspace X/S of X, when S is itself a subspace of X. Thus members of $S^\perp$ are orthogonal to members of S. C' will denote Range (C'). We make the observation that

$$(MS)^\perp = (M')^{-1}S^\perp \tag{29}$$

for any nonsquare M, and

$$(S_1 + S_2)^{\perp} = S_1^{\perp} \cap S_2^{\perp} \qquad (30)$$

for any two subspaces S_1, S_2 of X. These facts may be easily checked.

Dual of invariant subspace

Dual of (3). The invariant subspaces for (28) are defined by $A'S \subset S + C'$, whence

$$(A'S)^{\perp} \supset (S + C')^{\perp}$$

or

$$A^{-1}S^{\perp} \supset S^{\perp} \cap (C')^{\perp}$$

or

$$T \supset A[T \cap N(C)] \qquad (31)$$

Here, $T = X/S$. The interpretation is: under zero input conditions, any state $x(t) \in T$ such that $Cx = 0$ yields $\dot{x} \in T$. This does not seem a particularly fruitful idea.

Dual of (6). The invariant subspaces S for (28) can also be defined by $(A' + C'K)S \subset S$ for some K. This yields $(A + K'C)^{-1}T \supset T$ or

$$(A + K'C)T \subset T \qquad (32)$$

Equation (32) shows that with output-to-state feedback, one can arrange that under conditions of zero input, trajectories of (1) starting in T remain in T. Equation (31) can be used to check the existence of feedback laws satisfying (32), and as the basis for computing them.

Dual of maximal invariant subspace

Dual of (7). There exists a least subspace, call it $\bar{T}$, containing a given subspace W of X. From (31), this is seen to be $\cap T_i$, where each T_i satisfies (31) and $T_i \supset W$, and the intersection is over all such T_i.

Dual of (8). $\bar{T}$ can be obtained via

$$T_0 = 0 \qquad T_i = W + A[T_{i-1} \cap N(C)]$$
$$= T_{i-1} + A[T_{i-1} \cap N(C)] \qquad (33)$$

$\bar{T} = T_k$ where k is the first integer for which $T_k = T_{k+1}$. The proof is easy.

Dual of output-nulling invariant subspace

Dual of (10). Output-nulling invariant subspaces S for (28) have the property that

$$\begin{bmatrix} A' \\ B' \end{bmatrix} S \subset \begin{bmatrix} I \\ 0 \end{bmatrix} S + \begin{bmatrix} C' \\ D' \end{bmatrix} U \qquad (34)$$

which implies that

$$\left(\begin{bmatrix} A' \\ B' \end{bmatrix} S\right)^{\perp} \supset \left(\begin{bmatrix} I \\ 0 \end{bmatrix} S + \begin{bmatrix} C' \\ D' \end{bmatrix} U\right)^{\perp}$$

or, with $T = S^{\perp}$,

$$[A \quad B]^{-1}T \supset [I \quad 0]^{-1}T \cap N[C \quad D]$$

This yields

$$T \supset [A \quad B]\left\{\begin{bmatrix} T \\ U \end{bmatrix} \cap N[C \quad D]\right\}$$

i.e.

$$T \supset A[T \cap N(C)] + \hat{B} \qquad (35)$$

This is the defining equation for the dual of an output-nulling invariant subspace. In case D = 0, (35) reduces to (33) (with W replaced by B), which is as one would expect.

One interpretation of (35) is: suppose $x \in T$, $Cx = 0$. Then use of any u such that $Du = 0$ causes $\dot{x} = Ax + Bu$ to be in T.

Dual of (11). Starting with the fact that for some K, one has

$$(A' + C'K)S \subset S \qquad (B' + D'K)S = 0$$

one obtains

$$(A + K'C)T \subset T \qquad T \supset \text{Range } (B + K'D) \quad (36)$$

This has the following interpretation: suppose output-to-state feedback is applied to (1), yielding

$$\dot{x} = Ax + Bu + K'y + (A + K'C)x + (B + K'D)u \qquad (37)$$

Then any trajectory commencing in T remains in T, irrespective of u.

Evidently (35) implies the existence of K satisfying (36) and conversely.

Remark Subspaces T satisfying (36) also satisfy the corresponding equations for (1) with arbitrary state feedback, i.e. (36) hold if and only if for all F of suitable dimension

$$[(A + BF) + K (C + DF)]T \subset T$$
$$T \supset \text{Range } (B + K D) \qquad (38)$$

Equation (38) is an immediate consequence of (36). Alternatively, it follows from the observation that trajectories of (37) starting in T remain in T irrespective of u, and in particular must remain in T when u is the sum of a feedback term Fx and some arbitrary quantity v. Equation (38) implies the equivalence of (35) with

$$T \supset (A + BF)[T \cap N(C + DF)] + \hat{B} \qquad (39)$$

Dual of maximal output nulling invariant subspace

Dual of Lemma 3 Evidently (35) shows that there is a smallest space $\bar{T}$ that is the dual of the largest output-nulling invariant subspace of (28). Further, $\bar{T}$ is the intersection of all T satisfying (35).

Dual of Lemma 4 $\bar{T}$ is obtainable via

$$T_0 = 0 \quad T_i = T_{i-1} + A[T_{i-1} \cap N(C)] + \hat{B} \qquad (40)$$

with $\bar{T} = T_k$ where k is the first integer for which $T_k = T_{k+1}$.

Proof: Suppose $W \subset \bar{T}$. Then $A[W \cap N(C)] + \hat{B} \subset A[\bar{T} \cap N(C)] + \hat{B} \subset \bar{T}$, so that

$$W + A[W \cap N(C)] + \hat{B} \subset \bar{T}$$

Apply this with $W = 0$ to conclude $T_1 \subset \bar{T}$, with $W = T_1$ to conclude $T_2 \subset \bar{T}$ etc. If $T_k = T_{k+1}$,

$$T_k + A[T_k \cap N(C)] + \hat{B} \subset T_{k+1} = T_k,$$

so that

$$A[T_k \cap N(C)] + \hat{B} \subset T_k$$

This shows that $T_k \supset \bar{T}$ by the minimality property of $\bar{T}$. Hence $T_k = \bar{T}$.

In view of the remark leading to (38) and (39), we see that (40) can be replaced, for any F of appropriate dimension, by

$$T_0 = 0 \quad T_i = T_{i-1} + (A+BF)[T_{i-1} \cap N(C+DF)] + \hat{B} \qquad (41)$$

with $\bar{T} = T_k$ where k is the first integer for which $T_k = T_{k+1}$.

Further characterization of maximal output-nulling controllability subspace

Now we can state and prove an important result which the presentation of the duality results has been leading up to. The result constitutes an extension of Lemma 1.1 of [3].

Theorem 2 For the system (1), let $\bar{V}$ and $\bar{R}$ be the maximal output-nulling invariant and controllability subspaces, and let $\bar{T}$ be the minimal space satisfying (35), so that $\bar{T}^{\perp}$ is the maximal output-nulling invariant subspace associated with (28). Then

$$\bar{R} = \bar{T} \cap \bar{V} \qquad (42)$$

Proof: Let F be such that $(A + BF)\bar{V} \subset \bar{V}$ and $(C + DF)\bar{V} = 0$. In the course of the proof, we shall use the following fact:

$$\bar{V} = N(C + DF) \cap (A + BF)^{-1}\bar{V} \qquad (43)$$

This comes about in the following way. First, any $x \in \bar{V}$ is clearly in $N(C + DF)$ and $(A + BF)^{-1}\bar{V}$ by virtue of the output-nulling invariant subspace property. Second, if there exists $x \in N(C + DF) \cap (A + BF)^{-1}\bar{V}$ for which $x \notin \bar{V}$, define $\hat{V} = \bar{V} + x$. Then one easily has $(C + DF)\hat{V} = 0$ and $(A + BF)\hat{V} \subset \hat{V}$ contradicting the maximality property of $\bar{V}$. So (43) holds.

Now recall that in the course of proving theorem 1, we showed that $\bar{R}$ is the limit of the sequence

$$R_0 = 0 \quad R_i = (A + BF)R_{i-1} + \hat{B} \cap \bar{V} \qquad (23)$$

To prove (42), it is evidently sufficient to show that $R_i = T_i \cap \bar{V}$ with T_i given by (41). Evidently this holds for $i = 1$. Assume it holds for $i = 2, 3, \dots j$. Then

$$R_{j+1} = R_j + (A + BF)R_j + \hat{B} \cap \bar{V} \quad \text{[Using (23) and } R_j \subset R_{j+1}]$$

$$= T_j \cap \bar{V} + (A + BF)(T_j \cap \bar{V}) + \hat{B} \cap \bar{V} \quad \text{[by induction]}$$

$$= T_j \cap \bar{V} + (A + BF)(T_j \cap N(C + DF) \cap (A + BF)^{-1}\bar{V}) + \hat{B} \cap \bar{V} \quad \text{[by (43)]}$$

$$= \{T_j + (A + BF)[T_j \cap N(C + DF)] + \hat{B}\} \cap \bar{V}$$

$$= T_{j+1} \cap \bar{V}$$

The theorem then follows by induction.

One can go on from this point to study duals of controllability subspaces, output-nulling controllability subspaces, and maximal subspaces of these classes, but the ideas do not seem particularly fruitful and are not particularly difficult to work out.

9. AN APPLICATION TO THE CHARACTERIZATION OF THE ZEROS OF A SYSTEM

Roughly speaking, notions of invariant and controllability subspaces are as useful for systems with direct as with no direct feedthrough. Let us illustrate an application to proving a result which, at the present time, is little known - the result for the no feedthrough case having been proved only recently and at some length.

To describe the result, some preliminaries are necessary. We work with the system (1); let $\bar{V}$ and $\bar{R}$ be the maximal output-nulling invariant and controllability subspaces, and think of X as $\bar{R} \oplus \bar{V}/\bar{R} \oplus X/\bar{V}$. Let the co-ordinate basis be so chosen that (to within isomorphism of the spaces)

$$\bar{R} = \left\{\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}\right\} \quad \bar{V}/\bar{R} = \left\{\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}\right\} \quad X/\bar{V} = \left\{\begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}\right\}$$

where x_1, x_2 and x_3 have the appropriate dimensions. Let F be such that $(A+BF)\bar{V} \subset \bar{V}$ and $(C+DF)\bar{V} = 0$. Since also $(A+BF)\bar{R} \subset \bar{R}$, we have

$$A+BF = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \quad (44)$$

and since $(C+DF)\bar{V} = 0$, we must have

$$(C+DF) = [0 \quad 0 \quad C_3]$$

Now let G_1 be a matrix with columns spanning the null-space of D and let H_1 be a matrix such that $[G_1 \vdots H_1]$ is nonsingular. Let $[G_2 \vdots H_2]$ be a nonsingular matrix such that columns of BG_1G_2 span $\hat{B} \cap \bar{V}$ and G_2 has a maximal number of columns. Then

$$BG_1[G_2 \vdots H_2] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ 0 & B_{32} \end{bmatrix}$$

with B_{32} possessing rank equal to the number of its columns. The bottom left zero comes because BG_1G_2 has columns in $\bar{V}$. If the rank condition on B_{32} failed, then we should have $B_{32}\alpha = 0$ for some $\alpha = 0$ and so $B_1G_1H_2\alpha$ would be contained in $\hat{B} \cap \bar{V}$. Then $H_2\alpha$ would have to be of the form $G_2\beta$ in view of the maximality property of G_2, and then $[G_2 \; H_2]$ would have $[\beta' \quad \alpha']'$ as a nonzero nullvector, which is a contradiction.

Next, observe that $\hat{B} \cap \bar{V} \subset \bar{R}$, so that $B_{21} = 0$, while (A_{11}, B_{11}) must be a completely controllable pair. All this means that

$$\begin{bmatrix} -sI+(A+BF) & B[G_1 \vdots H_1] \begin{bmatrix} [G_2 \vdots H_2] & 0 \\ 0 & I \end{bmatrix} \\ C+DF & D[G_1 \vdots H_1] \begin{bmatrix} [G_2 \vdots H_2] & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} \quad (45)$$

$$= \begin{bmatrix} -sI+A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ 0 & -sI+A_{22} & A_{23} & 0 & B_{22} & B_{23} \\ 0 & 0 & -sI+A_{33} & 0 & B_{32} & B_{33} \\ 0 & 0 & C_3 & 0 & 0 & D_3 \end{bmatrix}$$

with (A_{11}, B_{11}) completely controllable, B_{32} of rank equal to the number of its columns, and D_3 with the *same* property [since G_1 spans $N(D)$]. The main result is the following

Theorem 3 The non-unity invariant factors of $sI-A_{22}$ [defined in (44)] are the same as the non-unity numerator polynomials of the Smith-McMillan canonical form of the transfer function matrix $D + C(sI-A)^{-1}B$.

Remark For a discussion of the Smith-McMillan canonical form, see [5]. This reference and [6] also make clear the connection between the numerator polynomials of the form and the zeros of a transfer function matrix. Reference [7] contains the first known proof of the theorem in case $D = 0$, while reference [8] has a shorter proof. Reference [9], with $D = 0$, shows that the characteristic polynomial of A_{22} divides the characteristic polynomial of all (possibly one-sided) inverses of (1) when these inverses are described in a certain state-space format.

Proof of Theorem: The non-unity Smith-McMillan form numerator polynomials are, see [5, Chapter 3, Theorem 4.1] the non-unity invariant factors of

$$M = \begin{bmatrix} -sI+A & B \\ C & D \end{bmatrix}.$$

The invariant factors of M are unaltered when M is post-multiplied by a nonsingular matrix, [10]. Post-multiplication of M by

$$\begin{bmatrix} I & 0 \\ F & K \end{bmatrix}$$

where $K = [G_1 \quad H_1] \begin{bmatrix} [G_2 \vdots H_2] & 0 \\ 0 & I \end{bmatrix}$

yields the matrix on the left side of (45). Rearrangement of columns leaves unaltered the invariant factors. *Thus the non-unity Smith-McMillan* numerator polynomials are the non-unity invariant factors of

$$\tilde{M} = \begin{bmatrix} -sI+A_{11} & B_{11} & A_{12} & A_{13} & B_{12} & B_{13} \\ 0 & 0 & -sI+A & A_{23} & B_{22} & B_{23} \\ 0 & 0 & 0 & -sI+A_{33} & B_{32} & B_{33} \\ 0 & 0 & 0 & C_3 & 0 & D_3 \end{bmatrix} \quad (46)$$

Now the greatest common divisor of all minors of $[-sI+A_{11} \vdots B_{11}]$ is 1, since $[A_{11}, B_{11}]$ is completely controllable, see [5, Chapter 2, Theorem 6.2]. We can also show that

$$\begin{bmatrix} -sI+A_{33} & B_{32} & B_{33} \\ C_3 & 0 & D_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

implies v_1, v_2 and $v_3 = 0$. For suppose to the contrary; then $v_1 \neq 0$, for if $v_1 = 0$, the rank *constraints* on B_{32} and D_3 will force $v_2 = v_3 = 0$. Analogously to arguments in [8], one can then show that the space spanned by $\bar{V}$ and $[0 \quad 0 \quad v_1']'$ is an output-nulling invariant subspace, that this contradicts the maximality of $\bar{V}$, and so $v_1 = 0$ after all. *Using the characterization of invariant fact*ors as ratios of greatest common divisors of minor of certain orders then shows as in [8], that the non-unity invariant factors of $\tilde{M}$ are those of $sI-A_{22}$.

10. CONCLUSIONS

Although we have shown that the concept of output-nulling invariant and controllability subspaces constitutes an extension to linear systems with direct feedthrough of known results for systems without feedthrough, we have yet to check that the myriad of known results for the no feedthrough case extend. We have given one example; others are very easy to establish - for example results on invertibility, and disturbance localization [1]. Decoupling results might be harder to obtain, though we would still expect them.

REFERENCES

[1] A.S. Morse and W.M. Wonham, "Status of Noninteracting Control", *IEEE Trans. Auto. Control*, Vol. AC-16, No. 6, December, 1971, pp 568-581.

[2] W.M. Wonham and A.S. Morse, "Decoupling and Pole Assignment in Linear Multivariable Systems: A Geometric Approach", *SIAM J. Control*, Vol. 8, No. 1 February 1970, pp. 1.18.

[3] A.S. Morse, "Structural Invariants of Linear Multivariable Systems", *SIAM J. Control*, Vol. 11, No. 3, August 1973, pp. 446-465.

[4] A.S. Morse, "Structure and Design of Linear Model Following Systems", *IEEE Trans. Auto. Control*, Vol. AC-18, No.4, August 1973, pp. 346-354.

[5] H.H. Rosenbrock, State-Space and Multivariable Theory, Thomas Nelson and Sons Ltd., London 1970.

[6] W.A. Wolovich, "On the Numerators and Zeros of Rational Transfer Matrices", *IEEE Trans. Auto. Control*, Vol. AC-18, No.5, October 1973, pp. 542-544.

[7] B.C. Moore and L.M. Silverman, "A Time Domain Characterization of the Invariant Factors of a System Transfer Function", *Proceedings of 1974 Joint Automatic Control Conference*.

[8] B.D.O. Anderson, "A Note on Transmission Zeros of a Transfer Function Matrix", submitted *IEEE Trans. Auto. Control*.

[9] G. Bengtsson, "A Theory of C ontrol of Linear Multivariable Systems", Report 7341, Division of Automatic Control, *Lund Institute of Technology*, November, 1973.

[10] S. Perlis, Theory of Matrices, Addison-Wesley, Reading, 1958.

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