A Control Theory Viewpoint of Positive Reality

by

B. D. O. Anderson

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SYSTEMS THEORY LABORATORY
STANFORD ELECTRONICS LABORATORIES
STANFORD UNIVERSITY • STANFORD, CALIFORNIA
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ABSTRACT

The paper is an attempt to link a well known concept of network theory, that of the positive real matrix, to the control systems concept of a minimal state space realization of a matrix of rational transfer functions. Necessary and sufficient conditions for a matrix to be positive real are given in terms of the matrices describing the state space realization.
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I. INTRODUCTION

The concept of a positive real function is now an old one of network theory, and more recently the concept has been usefully used in other system theoretic investigations, such as the development of the Popov criterion for the stability of a feedback system containing a nonlinearity [1]. In view of this and other connections between network and control theory, it seems possible that the concept of a positive real matrix could be employed fruitfully in control systems investigations. An n x n matrix $A(s)$ is called positive real if [2]

a) $A(s)$ is analytic in the right half plane
b) $A^*(s) = A(s^*)$ in the right half plane
c) $A'(s^*) + A(s)$ is nonnegative definite in the right half plane.

Here the superscript asterisk denotes complex conjugation; the prime denotes matrix transposition.

This paper is concerned with developing a criterion for a matrix of rational transfer functions to be positive real. The criterion is a systems theoretic one, in the sense that it is formulated in terms of the parameters of a control system realization of the matrix. Such a result has been developed elsewhere for the scalar case, and then applied to the development of the Popov criterion [1]. In a future paper we shall carry out an extension of the Popov criterion to cover a broad spectrum of control systems with multiple nonlinearities, and this will use the principal theorem of this paper.

A generalization of the criterion for a scalar to be a positive real function has been stated for a subclass of positive real matrices, namely, those which are zero at $s=0$ [3]. In this reference, no proof was given; an outline proof was suggested, but the filling in of the details presented more difficulties than suspected, and a correct proof has not been available hitherto.

After a review of some concepts associated with system realizations in section 2, we present in section 3 a full proof of the above mentioned result, for the case where the positive real matrix has no imaginary axis poles. Positive real matrices which have only imaginary axis poles are
then considered, followed by a hitherto unstated result for general positive real matrices, (i.e. those which are not necessarily zero at $s=\infty$, and which have imaginary axis poles permitted). This result appears as Theorem 3, and includes all preceding results as special cases.

Of course Theorem 3 alone could be proved, but the motivation for it would then be lost. As it stands, it is a natural outgrowth of the earlier more motivated theorems.

Notation in the paper is straightforward; capital letters will be used for matrices, small letters for vectors. Other symbols will be explained as required.

2. PRELIMINARIES CONCERNING CONTROL SYSTEMS

In this section we review some of the concepts associated with linear time-invariant dynamical systems; these concepts will appear in the discussion of positive real matrices.

We suppose that $M(s)$ is an $m \times n$ matrix of rational transfer function, with $M(\infty) = 0$. Then a triple $\{F, G, H\}$ is termed a realization of $M$ [4], [5] if

$$M(s) = H'(sI-F)^{-1}G$$

(1)

This is because $M(s)$ is the transfer function relating an input vector $u$ to an output vector $y$ in the following state space representation of $M$:

$$\dot{x} = Fx + Gu$$

(2a)

$$y = H'x$$

(2b)

Here $x$ is a $p$-vector, the state, $u$ is an $n$-vector, the input, $y$ is an $m$-vector, the output; $F$ is $p \times p$, $G$ is $p \times n$ and $H$ is $p \times m$.

For a given $M(s)$, there exist infinitely many sets $\{F, G, H\}$ constituting a realization [4]. Among the dimensions of all the possible $F$ matrices, there is one which is minimal; any realization incorporating $F$ of minimal dimension is termed a minimal realization. As is proved in
for example [4], minimal realizations are uniquely determined to within an arbitrary basis for the state space. This means, see [4], that if \( \{F_1,G_1,H_1\} \) and \( \{F_2,G_2,H_2\} \) are two minimal realizations of \( M(s) \), then there exists a nonsingular matrix \( T \) such that

\[
\begin{align*}
F_2 &= TF_1T^{-1} \\
G_2 &= TG_1 \\
H_2 &= (T')^{-1}H_1
\end{align*}
\]

The dimension of a minimal realization is termed the degree of \( M(s) \), and is, naturally, a positive integer number uniquely determined by \( M(s) \). Numerous definitions of degree have appeared over the last fifteen years [6],[7],[8], but recently these have been reconciled with one another [9]. For our purposes one of the more useful definitions is as follows. The degree of a pole \( s=s_0 \) of \( M \), written \( \delta(M;s_0) \), is the largest multiplicity it possesses as a pole of any minor of \( M(s) \). The degree \( \delta(M) \) of \( M(s) \) equals the sum of the degrees of its distinct poles.

An immediate consequence of this definition is that if \( M_1(s) \) and \( M_2(s) \) have no poles in common,

\[
\delta(M_1 + M_2) = \delta(M_1) + \delta(M_2)
\]

A further property which we shall have occasion to use is that

\[
\delta(M_1M_2) \leq \delta(M_1) + \delta(M_2)
\]

If \( M_1 \) is \( n \times r \), \( M_2 \) is \( r \times n \), then equality holds in (5) if \( M_1 \) and \( M_2 \) have no common poles, and if the rank of \( M_1 \) at any pole of \( M_2 \) is \( r \), and the rank of \( M_2 \) at any pole of \( M_1 \) is \( r \).

This can be seen by using the Smith-McMillan decomposition [6],[9] for \( M_1 \) and \( M_2 \). We have \( M_1 = A_1 \Gamma_1 B_1 \) and \( M_2 = A_2 \Gamma_2 B_2 \), where \( A_1, B_2 \) are \( n \times n \) polynomial matrices with constant determinant; \( B_1, A_2 \) are \( r \times r \) polynomial matrices with constant determinant; \( \Gamma_1 \) has its first \( r \) rows given by \( \text{diag} [\epsilon_1/\mu_1, \ldots, \epsilon_r/\mu_r] \) and its last \( n-r \) rows zero; \( \Gamma_2 \) has its first \( r \) columns given by \( \text{diag} [\lambda_1/\mu_1, \ldots, \lambda_r/\mu_r] \) and its last \( n-r \) columns.
zero. The $c_i$ etc. are polynomials. Let $s_0$ be a pole of $\mathbf{M}_2$ and suppose it is a $u_i$-th order of zero of $\mathbf{u}_1$. Then it is known that

$$\delta(\mathbf{M}_2; s_0) = \sum_{i=1}^{u_1} u_i;$$

it is also known that $\delta(\mathbf{M}_1 \mathbf{M}_2; s_0) = \delta(\mathbf{1}_B \mathbf{A}_2 \mathbf{T}_2; s_0)$, and an application of the first degree definition, (taking into account the properties of $\mathbf{M}_1$ at poles of $\mathbf{M}_2$) yields $\delta(\mathbf{1}_B \mathbf{A}_2 \mathbf{T}_2; s_0) = \sum_{i=1}^{u_1} u_i$. Thus $\delta(\mathbf{M}_2; s_0) = \delta(\mathbf{M}_1 \mathbf{M}_2; s_0)$, from which the desired result follows readily.

3. PRINCIPAL RESULTS

We first recall a result familiar from network theory on the factorization of nonnegative definite parahermitian matrices, following which two lemmas are stated and proved. With this material in hand, we then pass on to the statement and proof of Theorem 1, concerning those $\mathbf{Z}(s)$ with no imaginary axis poles, and equalling zero at $s = 0$. Theorem 2 follows, on lossless $\mathbf{Z}(s)$, or those $\mathbf{Z}(s)$ with imaginary axis poles only. Finally we present Theorem 3 dealing with general $\mathbf{Z}(s)$.

Given a positive real impedance matrix $\mathbf{Z}(s)$, we may form $\mathbf{Z}(s) + \mathbf{Z}(-s)$. This matrix is termed parahermitian, for the reason that if it is transposed, and $s$ is replaced by $-s$, it is left unaltered. On the $j\omega$-axis, the matrix is nonnegative definite hermitian. For such a matrix, it is known, see [10, Theorem 2], there exists a matrix $\mathbf{W}(s)$ such that

$$\mathbf{Z}(s) + \mathbf{Z}'(-s) = \mathbf{W}'(-s)\mathbf{W}(s)$$

where if $\mathbf{Z}(s) + \mathbf{Z}'(-s)$ has normal rank $r$, $\mathbf{W}$ is $r \times n$, is analytic in the right half plane, together with its right inverse $\mathbf{W}^{-1}$. This means that the rank of $\mathbf{W}$ in the right half plane is $r$. (For a full discussion see [10].)

Although there are many $\mathbf{W}$ satisfying Eq. (8), there is a unique $\mathbf{W}$ with the additional properties listed, (save for multiplication by an arbitrary constant orthogonal matrix). The choice of this $\mathbf{W}$ is critical for we can conclude that
This result is a consequence of Eq.(5), and the fact that \( W \) has rank \( r \) in the right half plane, where all the poles of \( W'(-s) \) are located, while \( W'(-s) \) has rank \( r \) in the left half plane, where all the poles of \( W(s) \) are located.

Until further notice we shall assume that \( Z(\omega) = 0 \). From (8) it follows immediately that \( W(\omega) = 0 \). We shall also assume initially that \( Z \) has poles only in the strict left half plane.

Now suppose that a minimal realization \( \{F, G, H\} \) is known for \( Z \), so that

\[
Z(s) = H'(sI-F)^{-1}G
\]  
\[ (10) \]

and let \( \{A, K, L\} \) be a minimal realization for \( W \), so that

\[
W(s) = L'(sI-A)^{-1}K
\]  
\[ (11) \]

The first lemma is concerned with the identification of \( A \) and \( F \).

Lemma 1. The matrices \( A \) and \( F \) are similar and thus by a basis change we may identify them.

Proof: To establish the result, we shall compare two minimal realizations for \( Z(s) + Z'(-s) = W'(-s)W(s) \). Because \( \{F, G, H\} \) is a realization for \( Z(s) \), direct calculation yields as a realization for \( Z'(-s) + Z(s) \)

\[
\begin{align*}
\{F_1, G_1, H_1\} &= \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \\
&= \begin{bmatrix} G & H \\ -H & G \end{bmatrix}
\end{align*}
\]  
\[ (12) \]

Because \( Z(s) \) has strict left half plane poles, \( Z(s) \) and \( Z'(-s) \) have no poles in common. Then from (4) it follows that \( Z(s) + Z'(-s) \) has twice the degree of \( Z \) (as clearly \( Z'(-s) \) has the same degree as \( Z \)), and thus \( \{F_1, G_1, H_1\} \) is a minimal realization.
Turning now to $W'(s) W(s)$, we observe by direct calculation that

$$W'(s) W(s) = K'(-sI - A')^{-1} L' (sI - A)^{-1} K \tag{13a}$$

$$= H'_2(sI - F'_2)^{-1} G_2 \tag{13b}$$

where

$$\{F'_2, G'_2, H'_2\} = \left\{ \begin{bmatrix} A & 0 \\ LL' & -A' \end{bmatrix}, \begin{bmatrix} K \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -K \end{bmatrix} \right\} \tag{14}$$

By Eq. (9) and the minimality of $[A,K,L]$ as a realization of $W$ it follows that $\{F'_2, G'_2, H'_2\}$ is a minimal realization of $W'(-s)W(s)$.

Now define $P$ to be the (unique symmetric) see [11, p.92] solution of

$$PA + A'P = -L L' \tag{15}$$

(One way of determining $P$ is to use the easily verifiable formula

$$P = \int_0^\infty \exp(A'\tau) L L' \exp(At) d\tau.$$  

Now the matrices $\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$ are inverses, and another realization of $W'(-s)W(s)$ alternative to $\{F'_2, G'_2, H'_2\}$ is, see Eq. (3),

$$\{F'_3, G'_3, H'_3\} = \left\{ \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} A & 0 \\ LL' & -A' \end{bmatrix}, \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} 0 \\ O \end{bmatrix}, \begin{bmatrix} 0 \\ -K \end{bmatrix} \right\} \tag{16}$$

The realizations $\{F'_1, G'_1, H'_1\}$ and $\{F'_3, G'_3, H'_3\}$ are both minimal realizations of the same transfer function, $Z(s) + Z'(-s) = W'(-s) W(s)$. Consequently they differ only with respect to the state space basis, and from (3a), it follows that $F'_1$ and $F'_3$ are similar matrices. But since
F and A have eigenvalues with negative real part (corresponding to the strict left half plane poles of Z and W) we can claim, observing the forms of F₁ and F₃ in Eqs. (12) and (16), that F and A are similar. This proves the lemma.

We turn now to the simple lemma 2, which establishes an algebraic property used in the proof of Theorem 1.

Lemma 2. If F is a p x p matrix with all negative real part eigenvalues, the only matrices which commute with

\[ F_1 = \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \]  \hspace{1cm} (12')

are matrices

\[ T = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \]  \hspace{1cm} (17)

where T₁, T₄ commute with F.

Proof: Suppose that

\[
\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}
\]

or

\[
\begin{bmatrix} T_1F & -T_2F' \\ T_3F & -T_4F' \end{bmatrix} = \begin{bmatrix} FT_1 & FT_2 \\ -F'T_3 & -F'T_4 \end{bmatrix}
\]

whence

\[
FT_1 = T_1F
\]

\[
F'T_4 = T_4F'
\]

\[
FT_2 + T_2F' = 0
\]

\[
FT_3 + T_3F' = 0
\]
The last two equations imply by the eigenvalue condition on $F$ \cite[p.90]{11} that $T_2 = T_3 = 0$.

With these lemmas in hand, we are now in a position to state

**Theorem 1.** Let $Z(s)$ be an $n \times n$ matrix of rational transfer functions with $Z(\infty) = 0$. Let $\{F, G, H\}$ be a minimal realization for $Z$. Let all poles of $Z$ lie in the strict left half plane. Then necessary and sufficient conditions for $Z(s)$ to be positive real are: there exists a symmetric, positive definite matrix $P$ and a matrix $L$ such that

$$PF + F'P = -LL' \quad (18)$$

and

$$PG = H \quad (19)$$

**Proof of Sufficiency.** Of the three conditions listed in section 1 which $Z$ must satisfy, the only one which needs to be verified is the third, viz., $Z'(s^*) + Z(s)$ is nonnegative definite in the right half plane. We have

$$Z'(s^*) + Z(s) = G'(sI - F')^{-1}H + H(sI - F)^{-1}G$$

$$= G'[(sI - F')^{-1}P + P(sI - F)^{-1}]G$$

$$= G'(sI - F')^{-1}P(sI - F)^{-1}G$$

$$+ (sI - F')P(sI - F)^{-1}G$$

$$= G'(sI - F')^{-1}P^2(sI + s*)P^2(sI - F)^{-1}G$$

$$+ G'(sI - F')^{-1}L L'(sI - F)^{-1}G$$

Since this is of the form $A'(s^*) (s + s*) A(s) + B'(s^*) B(s)$ it is clearly nonnegative definite in the right half plane.

**Proof of Necessity.** We represent $Z(s) + Z'(-s)$ as $W'(-s) W(s)$, where $W$ as the properties mentioned before and has a minimal realization $\{F, K, L\}$, where lemma 1 justifies the use of $F$. Then, as established in lemma 1, the following are both minimal realizations for $Z(s) + Z'(-s)$:
where

\[ P F + F'P = -L L' \]  \hspace{1cm} (15')

Since these are minimal realizations of the same transfer function matrix, they differ only with respect to the state space basis, but because the "F" matrices are the same, we observe using Eqs. (3) that

\[
\begin{bmatrix}
G \\
-H
\end{bmatrix}
= T
\begin{bmatrix}
K \\
PK
\end{bmatrix}
\]
\[
\begin{bmatrix}
PK \\
-K
\end{bmatrix}
= T'
\begin{bmatrix}
H \\
G
\end{bmatrix}
\]  \hspace{1cm} (20)

where \( T \) is a matrix commuting with \( \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \).

Using lemma 2, we conclude that

\[
G = T_F K  \hspace{1cm} PK = T' H
\]
\[
-H = T_F PK  \hspace{1cm} -K = T'_G
\]  \hspace{1cm} (21)

where \( T_F, T'_G \) commute with \( F \) and \( F' \). But now since \( \{F, K, L\} \) is a realization for \( W \), so is \( \{F, T_F K, (T'_F)^{-1} L\} \) where \( T_F \) is any matrix commuting with \( F \). Thus by Eq. (21) we can assume that the realization for \( W \) is of the form \( \{F, G, L\} \), identifying the matrices \( G \) and \( K \). Then the first part of (21) becomes

\[
G = T_F G  \hspace{1cm} PG = T'_G
\]  \hspace{1cm} (22)

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Now since $T_L$ commutes with $F$,

$$[G, FG, \ldots] = [T_L G, FT_L G, \ldots]$$
$$= [T_L G, T_L FG, \ldots]$$
$$= T_L [G, FG, \ldots]$$

The matrix $[G, FG, \ldots]$ has rank $p$ where $p$ is the dimension of $F$, since $\{F, G, H\}$ is a minimal realization of $Z(s)$ and all minimal realizations are completely controllable. Hence $T_L = I$, and thus $PG = H$.

It merely remains to show that $P$ is positive definite; but this follows from the result $P = \int_0^\infty \exp(F't) L L' \exp(Ft) dt$. Clearly $P$ is symmetric. Moreover $P x = 0$ implies $L' \exp(Ft) x = 0$, which is contradicted by the minimality (and hence complete observability) of $\{F, G, L\}$ as a realization of $W$. Q. E. D.

In preparation for dealing with matrices which may possess imaginary axis poles, we consider now purely lossless positive real matrices (i.e. those whose only poles are on the imaginary axis). In the sequel we shall later write a general positive real $Z(s)$ as the sum $Z_1(s) + Z_2(s)$ of a lossless positive real $Z_1(s)$ and a matrix $Z_2(s)$ with strictly left half plane poles. The matrix $Z_2(s)$ is also positive real.

**Theorem 2.** Let a positive real $Z(s)$ have pure imaginary poles, with $Z(\infty) = 0$. Let

$$Z(s) = H'(sI - F)^{-1} G$$

where $\{F, G, H\}$ is a minimal realization for $Z$. Then there exists a symmetric positive definite $P$ such that

$$PF + F'P = 0$$
$$PG = H$$

**Proof:** First note that if $P$ satisfies the above equations, then $P* = (T')^{-1} P T^{-1}$ satisfies the corresponding equations for the minimal realization $\{TFT^{-1}, TG, (T^{-1})' H\}$. Consequently if we exhibit a symmetric positive definite $P$ for any one realization, it follows that a symmetric
positive definite $P$ exists for all minimal realizations. Our procedure will in fact be to choose a realization $\{F,G,H\}$ for which $P$ has a particularly obvious form.

The form of $Z(s)$ has been established, e.g. [2] as

$$Z(s) = \sum \frac{A_is + B_i}{s^2 + \omega_i^2}$$  \hspace{1cm} (25)

where the $\omega_i$ are all different. By realizing separately each term $(A_is + B_i)(s^2 + \omega_i^2)^{-1}$ with minimal $[F_i,G_i,H_i]$ and selecting a $P_i$ such that (23) and (24) are satisfied, one can obtain a minimal $[F,G,H]$ and a $P$ satisfying (23) and (24) by $F = \hat{F}_i$ (where $\hat{\cdot}$ denotes direct sum), $G' = [G_1', G_2', \ldots]$, $H' = [H_1', H_2', \ldots]$ and $P = \hat{P}_i$. Consequently we shall consider

$$Z(s) = \frac{As + B}{s^2 + \omega_0^2}$$  \hspace{1cm} (26)

In [2, chapter 6] it is pointed out that if $2k$ is the degree of $Z$ in (26), there exist $k$ complex vectors $x_i$ such that

$$x_i^*x_i = 1, \quad x_i^*x_1 = \mu_i, \quad 0 < \mu_i < 1 \quad (\mu_i \text{ real})$$  \hspace{1cm} (27)

and

$$Z(s) = \sum_{i=1}^{k} \left[ \frac{x_i^*x_i}{p - j\omega_0} + \frac{x_i^*x_i}{p + j\omega_0} \right]$$  \hspace{1cm} (28)

Direct sum techniques allow us to restrict consideration to

$$Z(s) = \frac{x x^*}{p - j\omega_0} + \frac{x x^*}{p + j\omega_0}$$  \hspace{1cm} (29)

It is easy then to verify that if

$$y_1 = \frac{x + x^*}{\sqrt{2}}, \quad y_2 = j \frac{x - x^*}{\sqrt{2}}$$  \hspace{1cm} (30)
we have
\[
Z(s) = \begin{bmatrix} y_1, y_2 \end{bmatrix} \begin{bmatrix} \frac{1}{s^2 + \omega_0^2} & s \omega_0 \\ -s\omega_0 & s \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}
\] (31)

and then
\[
\{F, G, H, P\} = \left\{ \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}, \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\] (32)

Q.E.D.

All preliminaries are now in hand to give the final theorem, which applies to general positive real matrices. Since any positive real matrix can be written as the sum of $D$s and $Z(s)$ where $D$ is non-negative definite and $Z(s)$ is also positive real, but with $Z(\infty)$ finite, we shall lose no generality in restricting the theorem statement to such $Z(s)$.

**Theorem 3.** Let $Z(s)$ be an $n \times n$ matrix of rational transfer functions with $Z(\infty)$ finite. Let $\{F, G, H, P\}$ be a minimal realization for $Z(s) - Z(\infty)$. Let $Z$ have poles which are either in the left half plane, or lie on the $j\omega$-axis and are simple. Then necessary and sufficient conditions for $Z(s)$ to be positive real are: there exists a symmetric, positive definite $P$, and matrices $W_0$ and $L$ such that

\[
PF + F'P = -L L'
\] (33)

\[
PG = H - L W_0
\] (34)

\[
W_0'W_0 = Z(\omega) + Z'(\omega)
\] (35)

**Proof of sufficiency.** It only remains to verify the positive real behavior of $Z'(s*) + Z(s)$ in the right half plane. We have

\[
Z'(s*) + Z(s) = Z'(\infty) + Z(\infty) + G'(s*I-F')^{-1}H \\
+ H'(sI-F)^{-1}G
\]

\[
= W_0'W_0 + G'[(s*I-F')^{-1}P + P(sI-F)^{-1}]G \\
+ G'(s*I-F')^{-1}L W_0 + W_0' L'(sI-F)^{-1}G
\]
which is plainly nonnegative definite in the right half plane.

Proof of necessity. Initially, suppose \( Z(s) \) has strict left half plane poles. We shall consider the general case later on, with the aid of Theorem 2. We may write (see Eq. (8) and associated remarks)

\[
Z(s) + Z'(-s) = W'(-s)W(s)
\]

where

\[
W(s) = \frac{\Phi(s)}{N(s)} = \Gamma + \frac{\Phi_\perp(s)}{N(s)}
\]

and \( W \) has all the properties stated earlier. Here \( N(s) \) is the least common denominator of all elements of \( W \), when all possible cancellations have been made; \( \Phi \) and \( \Phi_\perp \) are matrices whose elements are polynomial in \( s \), while the maximum power of \( s \) in any polynomial of \( \Phi_\perp \) is less than the maximum power of \( s \) in \( N(s) \). The matrix \( \Gamma \) is simply \( W(\infty) \), and we define this matrix to be \( W_0 \).

It immediately follows from (8) that

\[
W_0'W_0 = Z(\infty) + Z'(\infty)
\]
Now suppose \( \{A, K, L\} \) is a minimal realization for \( W(s) - W(m) \). Then two minimal realizations for \( Z(s) + Z'(-s) - W'W_0 \) and \( W'(-s)W(s) - W'W_0 \) are readily found to be

\[
\begin{align*}
\{F_1G_1, H_1\} &= \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ -H \end{bmatrix}, \begin{bmatrix} H \end{bmatrix} \right\} \\
\{F_2G_2, H_2\} &= \left\{ \begin{bmatrix} A & 0 \\ L & -A' \end{bmatrix}, \begin{bmatrix} K \\ L \end{bmatrix}, \begin{bmatrix} L \end{bmatrix}, \begin{bmatrix} W_0 \\ -K \end{bmatrix} \right\}
\end{align*}
\]  

(38)

and

the minimality of the second following from the identity of its dimension with the first. As for lemma 1 it is possible to argue that \( A \) can be taken as \( F \), with another equivalent realization being

\[
\begin{align*}
\{F_3G_3, H_3\} &= \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} K & \end{bmatrix}, \begin{bmatrix} PK + L W_0 \end{bmatrix} \right\} \\
\{F_4G_4, H_4\} &= \left\{ \begin{bmatrix} K & \end{bmatrix}, \begin{bmatrix} PK + L W_0 \end{bmatrix}, \begin{bmatrix} -K \end{bmatrix} \right\}
\end{align*}
\]

(39)

where \( P \) is the unique positive definite solution of

\[
PF + F'P = -L L'
\]

(33)

Following Theorem 1, we conclude the existence of matrices \( T_1, T_4 \) commuting with \( F \) such that

\[
\begin{align*}
G &= T_1K \\
PK + L W_0 &= T_1' H \\
-H &= T_4 PK + T_4 L W_0 \\
-K &= T_4' G
\end{align*}
\]

(40)
The first of these four equations allows us to identify \( K \) with \( G \) in the minimal realization of \( W(s) - W(\infty) \), by making a suitable basis change. Applying complete controllability of the \([F,G,H]\) realization, we obtain \( T_1 = I \), and

\[
P G + L W_o = H \tag{34}
\]

Now let us relax the restriction on the poles of \( Z(s) \). We write \( Z(s) \) as the sum of \( Z_1(s) \) with purely imaginary axis poles and \( Z_2(s) \) with strict left half plane poles, where both \( Z_1 \) and \( Z_2 \) are positive real. For \( Z_1 \), we select \([F_1,G_1,H_1]\) and \( P_1 \), using Theorem 2, such that

\[
P_1 F_1 + F_1 P_1 = 0 \tag{23'}
\]

\[
P_1 G_1 = H_1 \tag{24'}
\]

and for \( Z_2 \) we select \([F_2,G_2,H_2]\) and \( P_2 \), using the material just proved, such that

\[
P_2 F_2 + F_2 P_2 = -L_2 L_2' \tag{33'}
\]

\[
P_2 G_2 = H_2 - L_2 W_o \tag{34'}
\]

\[
W_o W_o = Z_2(\infty) + Z_2'(\infty) \tag{35'}
\]

Then it is easily verified that Eqs. (33), (34), (35) are satisfied by taking

\[
P = P_1 + P_2
\]

\[
F = F_1 + F_2
\]

\[
G' = [G_1', G_2']
\]

\[
H' = [H_1', H_2']
\]

\[
L' = [0, L_2'] \tag{41}
\]
Moreover with \( \{F_1, G_1, H_1\} \) and \( \{F_2, G_2, H_2\} \) minimal realizations for \( Z_1(s) \) and \( Z_2(s) \), \( \{F, G, H\} \) is a minimal realization for \( Z(s) \); this is because the degree of \( Z \) is the sum of the degrees of \( Z_1 \) and \( Z_2 \), the latter having no common poles, while the dimension of \( F \) is the sum of the dimensions of \( F_1 \) and \( F_2 \). One should at this stage verify that Eqs. (33), (34), (35) are valid under a state space coordinate transformation, as they have merely been established for a particular class of \( F \) (i.e. those of the form \( F_1 + F_2 \)). This is easy to do however, along the lines given in Theorem 2 for a more particular case.

Q. E. D.

4. CONCLUSIONS

The significance of the theorems in their own right is self-evident. They provide a conceptual link between basic concepts of control theory and network theory. Their proofs have a number of interesting features, such as the necessity to use the particular \( W(s) \) in the factorization of \( Z(s) + Z'(-s) \), the heavy reliance on the concept of degree in the network and control theory senses, the canonical \( \{F, G, H\} \) representation (believed new) of a lossless \( Z \). Hopefully the results themselves as well as their proofs will help forge another link in the growing chain [3], [12] between control and network theory.

The results do have an immediate application to the study of control systems containing a number of memoryless nonlinearities. This material is discussed in [13].

ACKNOWLEDGMENT

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