SU-SEL-66-009

A Control Theory Viewpoint of Positive Reality

by

B. D. O. Anderson

February 1966

Technical Report No. 6558-4

Prepared under
National Science Foundation Grant GK 237

SYSTEMS THEORY LABORATORY

STANFORD ELECTRONICS LABORATORIES

STANFORD UNIVERSITY · STANFORD, CALIFORNIA

SEL-66-009

A CONTROL THEORY VIEWPOINT OF POSITIVE REALITY

by

B. D. O. Anderson

Technical Report No. 6558-4

Prepared under

National Science Foundation Grant GK 237

Systems Theory Laboratory
Stanford Electronics Laboratories
Stanford University
Stanford, California

February 1966

ABSTRACT

The paper is an attempt to link a well known concept of network theory, that of the positive real matrix, to the control systems concept of a minimal state space realization of a matrix of rational transfer functions. Necessary and sufficient conditions for a matrix to be positive real are given in terms of the matrices describing the state space realization.

CONTENTS

I. INTRODUCTION

The concept of a positive real function is now an old one of network theory, and more recently the concept has been usefully used in other system theoretic investigations, such as the development of the Popov criterion for the stability of a feedback system containing a nonlinearity [1]. In view of this and other connections between network and control theory, it seems possible that the concept of a positive real matrix could be employed fruitfully in control systems investigations. An n x n matrix A(s) is called positive real if [2]

a) A(s) is analytic in the right half plane

b) $A^*(s) = A(s^*)$ in the right half plane

c) $A'(s^*) + A(s)$ is nonnegative definite in the right half plane.

Here the superscript asterisk denotes complex conjugation; the prime denotes matrix transposition.

This paper is concerned with developing a criterion for a matrix of rational transfer functions to be positive real. The criterion is a systems theoretic one, in the sense that it is formulated in terms of the parameters of a control system realization of the matrix. Such a result has been developed elsewhere for the scalar case, and then applied to the development of the Popov criterion [1]. In a future paper we shall carry out an extension of the Popov criterion to cover a broad spectrum of control systems with multiple nonlinearities, and this will use the principal theorem of this paper.

A generalization of the criterion for a scalar to be a positive real function has been stated for a subclass of positive real matrices, namely, those which are zero at $s=\infty$ [3]. In this reference, no proof was given; an outline proof was suggested, but the filling in of the details presented more difficulties than suspected, and a correct proof has not been available hitherto.

After a review of some concepts associated with system realizations in section 2, we present in section 3 a full proof of the above mentioned result, for the case where the positive real matrix has no imaginary axis poles. Positive real matrices which have only imaginary axis poles are then considered, followed by a hitherto unstated result for general positive real matrices, (i.e. those which are not necessarily zero at $s=\infty$, and which have imaginary axis poles permitted). This result appears as Theorem 3, and includes all preceding results as special cases.

Of course Theorem 3 alone could be proved, but the motivation for it would then be lost. As it stands, it is a natural outgrowth of the earlier more motivated theorems.

Notation in the paper is straightforward; capital letters will be used for matrices, small letters for vectors. Other symbols will be explained as required.

2. PRELIMINARIES CONCERNING CONTROL SYSTEMS

In this section we review some of the concepts associated with linear time-invariant dynamical systems; these concepts will appear in the discussion of positive real matrices.

We suppose that $M(s)$ is an $m \times n$ matrix of rational transfer function, with $M(\infty)=0$. Then a triple $\{F,G,H\}$ is termed a realization of M [4], [5] if

$$M(s) = H'(sI-F)^{-1}G \tag{1}$$

This is because $M(s)$ is the transfer function relating an input vector u to an output vector y in the following state space representation of M:

$$\dot{x} = Fx + Gu \tag{2a}$$

$$y = H'x \tag{2b}$$

Here x is a p-vector, the state, u is an n-vector, the input, y is an m-vector, the output; F is $p \times p$, G is $p \times n$ and H is $p \times m$.

For a given $M(s)$, there exist infinitely many sets $\{F,G,H\}$ constituting a realization [4]. Among the dimensions of all the possible F matrices, there is one which is minimal; any realization incorporating F of minimal dimension is termed a minimal realization. As is proved in for example [4], minimal realizations are uniquely determined to within an arbitrary basis for the state space. This means, see [4], that if $\{F_1, G_1, H_1\}$ and $\{F_2, G_2, H_2\}$ are two minimal realizations of $M(s)$, then there exists a nonsingular matrix T such that

$$F_2 = TF_1T^{-1} \tag{3a}$$

$$G_2 = TG_1 \tag{3b}$$

$$H_2 = (T')^{-1}H_1 \tag{3c}$$

The dimension of a minimal realization is termed the degree of $M(s)$, and is, naturally, a positive integer number uniquely determined by $M(s)$. Numerous definitions of degree have appeared over the last fifteen years [6],[7],[8], but recently these have been reconciled with one another [9]. For our purposes one of the more useful definitions is as follows. The degree of a pole $s=s_o$ of M, written $\delta(M;s_o)$, is the largest multiplicity it possesses as a pole of any minor of $M(s)$. The degree $\delta(M)$ of $M(s)$ equals the sum of the degrees of its distinct poles.

An immediate consequence of this definition is that if $M_1(s)$ and $M_2(s)$ have no poles in common,

$$\delta(M_1 + M_2) = \delta(M_1) + \delta(M_2) \tag{4}$$

A further property which we shall have occasion to use is that

$$\delta(M_1M_2) \le \delta(M_1) + \delta(M_2) \tag{5}$$

If M_1 is $n \times r$, M_2 is $r \times n$, $r \le n$, then equality holds in (5) if M_1 and M_2 have no common poles, and if the rank of M_1 at any pole of M_2 is r, and the rank of M_2 at any pole of M_1 is r

This can be seen by using the Smith-McMillan decomposition [6],[9] for M_1 and M_2. We have $M_1 = A_1\Gamma_1B_1$ and $M_2 = A_2\Gamma_2B_2$, where A_1, B_2 are $n \times n$ polynomial matrices with constant determinant; B_1, A_2 are $r \times r$ polynomial matrices with constant determinant; Γ_1 has its first r rows given by diag $[\epsilon_1/\psi_1, \ldots \epsilon_r/\psi_r]$ and its last $n-r$ rows zero; Γ_2 has its first r columns given by diag $[\lambda_1/\mu_1, \ldots \lambda_r/\mu_r]$ and its last $n-r$ columns zero. The ϵ_i etc. are polynomials. Let s_o be a pole of M_2 and suppose it is a υ_i-th order of zero of μ_i. Then it is known that $\delta(M_2;s_o) = \sum_1^r \upsilon_i$; it is also known that $\delta(M_1M_2;s_o) = \delta(\Gamma_1B_1A_2\Gamma_2;s_o)$, and an application of the first degree definition, (taking into account the properties of M_1 at poles of M_2) yields $\delta(\Gamma_1B_1A_2\Gamma_2;s_o) = \sum_1^r \upsilon_i$. Thus $\delta(M_2;s_o) = \delta(M_1M_2;s_o)$, from which the desired result follows readily.

3. PRINCIPAL RESULTS

We first recall a result familiar from network theory on the factorization of nonnegative definite parahermitian matrices, following which two lemmas are stated and proved. With this material in hand, we then pass on to the statement and proof of Theorem 1, concerning those Z(s) with no imaginary axis poles, and equalling zero at $s=\infty$. Theorem 2 follows, on lossless Z(s), or those Z(s) with imaginary axis poles only. Finally we present Theorem 3 dealing with general Z(s).

Given a positive real impedance matrix Z(s), we may form $Z(s) + Z'(-s)$. This matrix is termed parahermitian, for the reason that if it is transposed, and s is replaced by -s, it is left unaltered. On the $j\omega$-axis, the matrix is nonnegative definite hermitian. For such a matrix, it is known, see [10, Theorem 2], there exists a matrix W(s) such that

$$Z(s) + Z'(-s) = W'(-s)W(s) \tag{8}$$

where if $Z(s) + Z'(-s)$ has normal rank r, W is rxn, is analytic in the right half plane, together with its right inverse W^{-1}. This means that the rank of W in the right half plane is r. (For a full discussion see [10].)

Although there are many W satisfying Eq.(8), there is a unique W with the additional properties listed, (save for multiplication by an arbitrary constant orthogonal matrix). The choice of this W is critical for we can conclude that

$$\delta\left(W'(-s)\ W(s)\right) = \delta\left(W'(-s)\right) + \delta\left(W(s)\right) = 2\delta(W) \tag{9}$$

This result is a consequence of Eq.(5), and the fact that W has rank r in the right half plane, where all the poles of $W'(-s)$ are located, while $W'(-s)$ has rank r in the left half plane, where all the poles of $W(s)$ are located.

Until further notice we shall assume that $Z(\infty)=0$. From (8) it follows immediately that $W(\infty)=0$. We shall also assume initially that Z has poles only in the strict left half plane.

Now suppose that a minimal realization $\{F,G,H\}$ is known for Z, so that

$$Z(s) = H'(sI-F)^{-1}G \tag{10}$$

and let $\{A,K,L\}$ be a minimal realization for W, so that

$$W(s) = L'(sI-A)^{-1}K \tag{11}$$

The first lemma is concerned with the identification of A and F.

> Lemma 1. The matrices A and F are similiar and thus by a basis change we may identify them.

Proof: To establish the result, we shall compare two minimal realizations for $Z(s) + Z'(-s) = W'(-s)\ W(s)$. Because $\{F,G,H\}$ is a realization for $Z(s)$, direct calculation yields as a realization for $Z'(-s) + Z(s)$

$$\{F_1,G_1,H_1\} = \left\{\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ -H \end{bmatrix}, \begin{bmatrix} H \\ G \end{bmatrix}\right\} \tag{12}$$

Because $Z(s)$ has strict left half plane poles, $Z(s)$ and $Z'(-s)$ have no poles in common. Then from(4) it follows that $Z(s) + Z'(-s)$ has twice the degree of Z (as clearly $Z'(-s)$ has the same degree as Z), and thus $\{F_1,G_1,H_1\}$ is a minimal realization.

Turning now to W'(-s) W(s), we observe by direct calculation that

$$W'(-s)\ W(s) = K'(-sI - A')^{-1} L\ \ L'(sI - A)^{-1} K \tag{13a}$$

$$= H_2'(sI - F_2)^{-1} G_2 \tag{13b}$$

where

$$\{F_2, G_2, H_2\} = \left\{ \begin{bmatrix} A & 0 \\ LL' & -A' \end{bmatrix},\ \begin{bmatrix} K \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ -K \end{bmatrix} \right\} \tag{14}$$

By Eq.(9) and the minimality of $\{A, K, L\}$ as a realization of W it follows that $\{F_2, G_2, H_2\}$ is a minimal realization of W'(-s)W(s).

Now define P to be the (unique symmetric) see [11, p.92] solution of

$$P\ A + A'P = -L\ L' \tag{15}$$

(One way of determining P is to use the easily verifiable formula $P = \int_0^\infty \exp\ (A'\tau)\ L\ L'\ \exp(A\tau)d\tau$.) Now the matrices $\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$ are inverses, and another realization of W'(-s) W(s) alternative to $\{F_2, G_2, H_2\}$ is, see Eq.(3),

$$\{F_3, G_3, H_3\} = \left\{ \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} A & 0 \\ LL' & -A' \end{bmatrix} \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix},\ \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ -K \end{bmatrix} \right\} = \left\{ \begin{bmatrix} A & 0 \\ 0 & -A' \end{bmatrix},\ \begin{bmatrix} K \\ P\ K \end{bmatrix},\ \begin{bmatrix} PK \\ -K \end{bmatrix} \right\} \tag{16}$$

The realizations $\{F_1, G_1, H_1\}$ and $\{F_3, G_3, H_3\}$ are both minimal realizations of the same transfer function, Z(s) + Z'(-s) = W'(-s) W(s). Consequently they differ only with respect to the state space basis, and from (3a), it follows that F_1 and F_3 are similar matrices. But since

F and A have eigenvalues with negative real part (corresponding to the strict left half plane poles of Z and W) we can claim, observing the forms of F_1 and F_3 in Eqs. (12) and (16), that F and A are similar. This proves the lemma.

We turn now to the simple lemma 2, which establishes an algebraic property used in the proof of Theorem 1.

Lemma 2. If F is a p x p matrix with all negative real part eigenvalues, the only matrices which commute with

$$F_1 = \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \tag{12'}$$

are matrices

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \tag{17}$$

where T_1, T_4' commute with F.

Proof: Suppose that

$$\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

or

$$\begin{bmatrix} T_1F & -T_2F' \\ T_3F & -T_4F' \end{bmatrix} = \begin{bmatrix} F\,T_1 & F\,T_2 \\ -F'T_3 & -F'T_4 \end{bmatrix}$$

whence

$$F\,T_1 = T_1F$$

$$F'T_4 = T_4F'$$

$$F\,T_2 + T_2F' = 0$$

$$F\,T_3 + T_3F' = 0$$

The last two equations imply by the eigenvalue condition on F [11, p.90] that $T_2 = T_3 = 0$.

With these lemmas in hand, we are now in a position to state

Theorem 1. Let Z(s) be an n x n matrix of rational transfer functions with $Z(\infty)=0$. Let {F,G,H} be a minimal realization for Z. Let all poles of Z lie in the strict left half plane. Then necessary and sufficient conditions for Z(s) to be positive real are: there exists a symmetric, positive definite matrix P and a matrix L such that

$$P F + F'P = -L L' \tag{18}$$

and

$$P G = H \tag{19}$$

Proof of Sufficiency. Of the three conditions listed in section 1 which Z must satisfy, the only one which needs to be verified is the third, viz., $Z'(s^*) + Z(s)$ is nonnegative definite in the right half plane. We have

$$\begin{aligned} Z'(s^*) + Z(s) &= G'(s^*I - F')^{-1}H + H(sI - F)^{-1}G \\ &= G'\{(s^*I - F')^{-1}P + P(sI - F)^{-1}\}G \\ &= G'(s^*I - F')^{-1}\{P(sI - F) \\ &\quad + (s^*I - F')P\}(sI - F)^{-1}G \\ &= G'(s^*I - F')^{-1}P^{\frac{1}{2}}(s+s^*)P^{\frac{1}{2}}(sI - F)^{-1}G \\ &\quad + G'(s^*I - F')^{-1}L L'(sI - F)^{-1}G \end{aligned}$$

Since this is of the form $A'(s^*)\ (s+s^*)\ A(s) + B'(s^*)\ B(s)$ it is clearly nonnegative definite in the right half plane.

Proof of Necessity. We represent $Z(s) + Z'(-s)$ as $W'(-s)\ W(s)$, where W as the properties mentioned before and has a minimal realization {F, K, L}, where lemma 1 justifies the use of F. Then, as established in lemma 1, the following are both minimal realizations for $Z(s) + Z'(-s)$:

$$\{F_1, G_1, H_1\} = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ -H \end{bmatrix}, \begin{bmatrix} H \\ G \end{bmatrix} \right\} \qquad (12)$$

and

$$\{F_3, G_3, H_3\} = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} K \\ PK \end{bmatrix}, \begin{bmatrix} PK \\ -K \end{bmatrix} \right\} \qquad (16')$$

where

$$P\,F + F'P = -L\,L' \qquad (15')$$

Since these are minimal realizations of the same transfer function matrix, they differ only with respect to the state space basis, but because the "F" matrices are the same, we observe using Eqs.(3) that

$$\begin{bmatrix} G \\ -H \end{bmatrix} = T \begin{bmatrix} K \\ PK \end{bmatrix} \qquad \begin{bmatrix} PK \\ -K \end{bmatrix} = T' \begin{bmatrix} H \\ G \end{bmatrix} \qquad (20)$$

where T is a matrix commuting with $\begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}$.

Using lemma 2, we conclude that

$$\begin{aligned} G &= T_1 K \qquad & PK &= T_1' H \\ -H &= T_4 PK \qquad & -K &= T_4' G \end{aligned} \qquad (21)$$

where T_1, T_4 commute with F and F'. But now since $\{F,K,L\}$ is a realization for W, so is $\{F, T_1 K, (T_1')^{-1} L\}$ where T_1 is any matrix commuting with F. Thus by Eq.(21) we can assume that the realization for W is of the form $\{F,G,L\}$, identifying the matrices G and K. Then the first part of (21) becomes

$$G = T_1 G \qquad PG = T_1' H \qquad (22)$$

Now since T_1 commutes with F,

$$[G,FG,\ldots] = [T_1G,FT_1G,\ldots] = [T_1G,T_1FG,\ldots] = T_1[G,FG,\ldots]$$

The matrix $[G,FG,\ldots]$ has rank p where p is the dimension of F, since $\{F,G,H\}$ is a minimal realization of $Z(s)$ and all minimal realizations are completely controllable. Hence $T_1 = I$, and thus $PG = H$.

It merely remains to show that P is positive definite; but this follows from the result $P = \int_0^\infty \exp(F'\tau)\, L\, L'\, \exp(F\tau)d\tau$. Clearly P is symmetric. Moreover $Px=0$ implies $L'\exp(F\tau)x=0$, which is contradicted by the minimality (and hence complete observability) of $\{F,G,L\}$ as a realization of W.

Q. E. D.

In preparation for dealing with matrices which may possess imaginary axis poles, we consider now purely lossless positive real matrices (i.e. those whose only poles are on the imaginary axis). In the sequel we shall later write a general positive real $Z(s)$ as the sum $Z_1(s) + Z_2(s)$ of a lossless positive real $Z_1(s)$ and a matrix $Z_2(s)$ with strictly left half plane poles. The matrix $Z_2(s)$ is also positive real.

Theorem 2. Let a positive real $Z(s)$ have pure imaginary poles, with $Z(\infty)=0$. Let

$$Z(s) = H'(sI - F)^{-1}G \tag{10}$$

where $\{F,G,H\}$ is a minimal realization for Z.
Then there exists a symmetric positive definite P such that

$$PF + F'P = 0 \tag{23}$$

$$PG = H \tag{24}$$

Proof: First note that if P satisfies the above equations, then $P^* = (T')^{-1}P\,T^{-1}$ satisfies the corresponding equations for the minimal realization $\{TFT^{-1}, TG, (T^{-1})'H\}$. Consequently if we exhibit a symmetric positive definite P for any one realization, it follows that a symmetric positive definte P exists for all minimal realizations. Our procedure will in fact be to choose a realization $\{F, G, H\}$ for which P has a particularly obvious form.

The form of Z(s) has been established, e.g. [2] as

$$Z(s) = \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} \tag{25}$$

where the ω_i are all different. By realizing separately each term $(A_i s + B_i)(s^2 + \omega_i^2)^{-1}$ with minimal $\{F_i, G_i, H_i\}$ and selecting a P_i such that (23) and (24) are satisfied, one can obtain a minimal $\{F, G, H\}$ and a P satisfying (23) and (24) by $F = \dot{+}_i F_i$ (where $\dot{+}$ denotes direct sum), $G' = [G_1', G_2', \dots]$, $H' = [H_1', H_2', \dots]$ and $P = \dot{+}_i P_i$. Consequently we shall consider

$$Z(s) = \frac{As + B}{s^2 + \omega_o^2} \tag{26}$$

In [2, chapter 6] it is pointed out that if 2k is the degree of Z in (26), there exist k complex vectors x_i such that

$$x_i^{*\prime} x_i = 1, \quad x_i' x_i = \mu_i, \quad 0 < \mu_i \leq 1 \quad (\mu_i \text{ real}) \tag{27}$$

and

$$Z(s) = \sum_{i=1}^{k} \left[\frac{x_i x_i^{*\prime}}{p - j\omega_o} + \frac{x_i^* x_i'}{p + j\omega_o} \right] \tag{28}$$

Direct sum techniques allow us to restrict consideration to

$$Z(s) = \frac{x\, x^{*\prime}}{p - j\omega_o} + \frac{x^* x'}{p + j\omega_o} \tag{29}$$

It is easy then to verify that if

$$y_1 = \frac{x + x^*}{\sqrt{2}}, \qquad y_2 = j\,\frac{x - x^*}{\sqrt{2}} \tag{30}$$

we have

$$Z(s) = [y_1, y_2] \frac{1}{s^2 + \omega_o^2} \begin{bmatrix} s & \omega_o \\ -\omega_o & s \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \tag{31}$$

and then

$$\{F,G,H,P\} = \left\{ \begin{bmatrix} 0 & -\omega_o \\ \omega_o & 0 \end{bmatrix}, \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \tag{32}$$

Q.E.D.

All preliminaries are now in hand to give the final theorem, which applies to general positive real matrices. Since any positive real matrix can be written as the sum of Ds and $Z(s)$ where D is non-negative definite and $Z(s)$ is also positive real, but with $Z(\infty)$ finite, we shall lose no generality in restricting the theorem statement to such $Z(s)$.

Theorem 3. Let $Z(s)$ be an $n \times n$ matrix of rational transfer functions with $Z(\infty)$ finite. Let $\{F,G,H\}$ be a minimal realization for $Z(s) - Z(\infty)$. Let Z have poles which are either in the left half plane, or lie on the $j\omega$-axis and are simple. Then necessary and sufficient conditions for $Z(s)$ to be positive real are: there exists a symmetric, positive definite P, and matrices W_o and L such that

$$PF + F'P = -L L' \tag{33}$$

$$PG = H - L W_o \tag{34}$$

$$W_o' W_o = Z(\infty) + Z'(\infty) \tag{35}$$

Proof of sufficiency. It only remains to verify the positive real behavior of $Z'(s^*) + Z(s)$ in the right half plane. We have

$$\begin{aligned} Z'(s^*) + Z(s) &= Z'(\infty) + Z(\infty) + G'(s^*I-F')^{-1}H \\ &\quad + H'(sI-F)^{-1}G \\ &= W_o'W_o + G'[(s^*I-F')^{-1}P + P(sI-F)^{-1}]G \\ &\quad + G'(s^*I-F')^{-1}L\,W_o + W_o'\,L'(sI-F)^{-1}G \end{aligned}$$

$$= W_o'W_o + G'(S^*I-F')^{-1}[P(s+s^*) - PF - F'P](sI-F)^{-1}G$$
$$+ G'(s^*I-F')^{-1}L\, W_o + W_o'\, L'(sI-F)^{-1}G$$
$$= W_o'W_o + G'(s^*I-F')^{-1}LW_o + W_o'\, L'(sI-F)^{-1}G$$
$$+ G'(s^*I-F')^{-1}L\, L'(sI-F)^{-1}G$$
$$+ G'(s^*I-F')^{-1}P^{1/2}(s+s^*)P^{1/2}(sI-F)^{-1}G$$
$$= [W_o' + G'(s^*I-F')^{-1}L][W_o + L'(sI-F)^{-1}G]$$
$$+ G'(s^*I-F')^{-1}P^{1/2}(s+s^*)P^{1/2}(sI-f)^{-1}G \qquad (36)$$

which is plainly nonnegative definite in the right half plane.

Proof of necessity. Initially, suppose Z(s) has strict left half plane poles. We shall consider the general case later on, with the aid of Theorem 2. We may write (see Eq.(8) and associated remarks)

$$Z(s) + Z'(-s) = W'(-s)W(s) \qquad (8)$$

where

$$W(s) = \frac{\Phi(s)}{N(s)} = \Gamma + \frac{\Phi_1(s)}{N(s)}$$

and W has all the properties stated earlier. Here N(s) is the least common denominator of all elements of W, when all possible cancellations have been made; Φ and Φ_1 and matrices whose elements are polynomial in s, while the maximum power of s in any polynomial of Φ_1 is less than the maximum power of s in N(s). The matrix Γ is simply $W(\infty)$, and we define this matrix to be W_o.

It immediately follows from (8) that

$$W_o'W_o = Z(\infty) + Z'(\infty) \qquad (35)$$

Now suppose $\{A,K,L\}$ is a minimal realization for $W(s) - W(\infty)$. Then two minimal realizations for $Z(s) + Z'(-s) - W_o'W_o$ and $W'(-s)W(s) - W_o'W_o$ are readily found to be

$$\{F_1, G_1, H_1\} = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} G \\ -H \end{bmatrix}, \begin{bmatrix} H \\ G \end{bmatrix} \right\} \qquad (38)$$

and

$$\{F_2, G_2, H_2\} = \left\{ \begin{bmatrix} A & 0 \\ L\,L' & -A' \end{bmatrix}, \begin{bmatrix} K \\ L\,W_o \end{bmatrix} \begin{bmatrix} L\,W_o \\ -K \end{bmatrix} \right\} \qquad (39)$$

the minimality of the second following from the identity of its dimension with the first. As for lemma 1 it is possible to argue that A can be taken as F, with another equivalent realization being

$$\{F_3, G_3, H_3\} = \left\{ \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix}, \begin{bmatrix} K \\ PK + L\,W_o \end{bmatrix}, \begin{bmatrix} PK + L\,W_o \\ -K \end{bmatrix} \right\}$$

where P is the unique positive definite solution of

$$PF + F'P = -L\,L' \qquad (33)$$

Following Theorem 1, we conclude the existence of matrices T_1, T_4' commuting with F such that

$$G = T_1 K \qquad PK + L\,W_o = T_1' H \qquad (40)$$

$$-H = T_4 PK + T_4 L\,W_o \qquad -K = T_4' G$$

The first of these four equations allows us to identify K with G in the minimal realization of $W(s) - W(\infty)$, by making a suitable basis change. Applying complete controllability of the $\{F,G,H\}$ realization, we obtain $T_1 = I$, and

$$PG + L\,W_o = H \tag{34}$$

Now let us relax the restriction on the poles of $Z(s)$. We write $Z(s)$ as the sum of $Z_1(s)$ with purely imaginary axis poles and $Z_2(s)$ with strict left half plane poles, where both Z_1 and Z_2 are positive real. For Z_1, we select $\{F_1,G_1,H_1\}$ and P_1, using Theorem 2, such that

$$P_1\,F_1 + F_1'\,P_1 = 0 \tag{23'}$$

$$P_1\,G_1 = H_1 \tag{24'}$$

and for Z_2 we select $\{F_2,G_2,H_2\}$ and P_2, using the material just proved, such that

$$P_2\,F_2 + F_2'\,P_2 = -L_2\,L_2' \tag{33'}$$

$$P_2\,G_2 = H_2 - L_2 W_o \tag{34'}$$

$$W_o' W_o = Z_2(\infty) + Z_2'(\infty) \tag{35'}$$

Then it is easily verified that Eqs.(33), (34), (35) are satisfied by taking

$$\begin{aligned} P &= P_1 \dotplus P_2 \\ F &= F_1 \dotplus F_2 \\ G' &= [G_1',\ G_2'] \\ H' &= [H_1',\ H_2'] \\ L' &= [0\ ,\ L_2'] \end{aligned} \tag{41}$$

Moreover with $\{F_1, G_1, H_1\}$ and $\{F_2, G_2, H_2\}$ minimal realizations for $Z_1(s)$ and $Z_2(s)$, $\{F, G, H\}$ is a minimal realization for $Z(s)$; this is because the degree of Z is the sum of the degrees of Z_1 and Z_2, the latter having no common poles, while the dimension of F is the sum of the dimensions of F_1 and F_2. One should at this stage verify that Eqs. (33), (34), (35) are valid under a state space coordinate transformation, as they have merely been established for a particular class of F (i.e. those of the form $F_1 \dotplus F_2$). This is easy to do however, along the lines given in Theorem 2 for a more particular case.

Q. E. D.

4. CONCLUSIONS

The significance of the theorems in their own right is self-evident. They provide a conceptual link between basic concepts of control theory and network theory. Their proofs have a number of interesting features, such as the necessity to use the particular $W(s)$ in the factorization of $Z(s) + Z'(-s)$, the heavy reliance on the concept of degree in the network and control theory senses, the canonical $\{F, G, H\}$ representation (believed new) of a lossless Z. Hopefully the results themselves as well as their proofs will help forge another link in the growing chain [3], [12] between control and network theory.

The results do have an immediate application to the study of control systems containing a number of memoryless nonlinearities. This material is discussed in [13].

ACKNOWLEDGMENT

This work was carried out under National Science Foundation Grant GK 237. The author would also like to acknowledge the support of the Services Canteens Trust Fund, an agency of the Australian Government, and the United States Educational Foundation in Australia for a Fulbright Grant.

REFERENCES

1. R. E. Kalman, "Lyapunov Functions for the Problem of Lur'e in Automatic Control," Proceedings of the National Academy of Sciences, vol.49, No.2, February 1963, pp.201-205.

2. R. W. Newcomb, Linear Multiport Synthesis, McGraw Hill, New York 1966.

3. R. E. Kalman, "On a New Characterization of Linear Passive Systems," Proceedings of the First Allerton Conference on Circuit and System Theory, University of Illinois, November 1963, pp. 456-470.

4. R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," Journel of the Society for Industrial and Applied Mathematics Control, Series A, vol 1, No. 2, 1963, pp.152-192.

5. B.D.O. Anderson and R.W. Newcomb, "A Canonical Simulation of a Transfer Function Matrix," IEEE Transactions on Automatic Control, April 1966, to appear.

6. B. McMillan, "Introduction to Formal Realizability Theory," Bell System Technical Journal, vol. 31, No. 2, March 1952, pp. 217-279 and vol. 31, No. 3, May 1952, pp. 541-600.

7. R. J. Duffin and D. Hazony, "The Degree of a Rational Matrix Function," Journal of the Society for Industrial and Applied Mathematics, vol. 11, No. 3, September 1963, pp. 645-658.

8. D. Hazony, Elements of Network Synthesis, Reinhold, New York, 1963

9. R. E. Kalman, " Irreducible Realizations and the Degree of a Matrix of Rational Functions," Journel of the Society for Industrial and Applied Mathematics, vol. 13, No. 2, June 1965, pp.520-544.

10. D. C. Youla, "On the Factorization of Rational Matrices," IRE Trans. on Information Theory, vol. IT7, No. 3, July 1960, pp.172-189.

11. C. C. Macduffee, The Theory of Matrices, Springer, Berlin, 1933.

12. D. C. Youla and P. Tissi, "N-Port Synthesis via Reactance Extraction Part I," Electrophysics Memo PIBMRI-1309-66, January 1966, Polytechnic Institute of Brooklyn, Brooklyn, New York.

13. B. D. O. Anderson, "Stability of Control Systems Containing Multiple Nonlinearities," Report No. SEL-66-012 (TR No. 6560-2), Stanford Electronics Laboratories, Stanford, California.

DISTRIBUTION LIST FOR
TECHNICAL REPORT 6558-4
by Brian Anderson

No. of Copies	
	U. of Aberdeen Dept. of Natural Philosophy Marischal College Aberdeen, Scotland
1	Attn: Mr. R. V. Jones
	U. of Arizona EE Dept. Tucson, Ariz.
1	Attn: R. L. Walker
1	Attn: D. J. Hamilton
	U. of British Columbia Vancouver 8, Canada
1	Attn: Dr. A. C. Soudack
	Calif. Institute of Tech. Pasadena, Calif.
1	Attn: Prof. R. W. Gould
1	Attn: D. Braverman (EE Dept)
	Calif. Inst. of Technology 4800 Oak Grove Dr. Pasadena 3, Calif.
1	Attn: Library Jet Propulsion Lab
	U. of Calif. Berkeley 4, Calif.
1	Attn: Prof. R. M. Saunders (EE Dept)
1	Attn: Dr. R. K. Wakerling Rad. Lab. Info. Div. Bldg. 30, Rm. 101
	U. of Calif. Los Angeles 24, Calif.
1	Attn: C. T. Leondes, Prof. of Engrg, Engrg Dept.
1	Attn: R. S. Elliot, Electromagnetics Div., College of Engrg
	U. of Calif., San Diego School of Science and Engrg La Jolla, Calif.
1	Attn: Physics Dept.

No. of Copies	
	Carnegie Institute of Tech. Schenley Park Pittsburg 13, Pa.
1	Attn: Dr. E.M. Williams EE Dept
	Case Inst. of Technology Engrg Design Center Cleveland 6, Ohio
1	Attn: Dr. J. B. Neswick, Dir.
	Columbia Rad. Lab. Columbia University 538 W. 120th St.
1	New York, N.Y. 10027
	Columbia U. Dept. of EE New York, N.Y. 10027
1	Attn: Prof. Ralph J. Schwarz
	Cornell U Cognitive Systems Res. Program Ithaca, N.Y.
1	Attn: F. Rosenblatt Hollister Hall
	Thayer School of Engrg Dartmouth College Hanover, N.H.
1	Attn: John W. Strohbehn
	Drexel Inst. of Technology Philadelphia 4, Pa.
1	Attn: F.B. Haynes, EE Dept
	U. of Florida Engrg. Bldg., Rm. 336 Gainesville, Fla.
1	Attn: M.J. Wiggins, EE Dept
	Georgia Inst. of Technology Atlanta 13, Ga.
1	Attn: Mrs. J. H. Crosland, Lib
1	Attn: F. Dixon, Engr. Exp. Station

No. of Copies	
	Northwestern U. 2422 Oakton St. Evanston, Ill.
1	Attn: W.S. Toth Aerial Measurements Lab
	U. of Notre Dame South Bend, Ind.
1	Attn: E. Henry, EE Dept.
	U. of Notre Dame Dept. of EE Notre Dame, Ind. 46556
1	Attn: W. B. Berry
	Ohio State U. 2024 Niel Ave. Columbus 10, Ohio
1	Attn: Prof. E.M. Boone, EE Dept
	Oregon State U. Corvallis, Ore.
1	Attn: H.J. Oorthuys, EE Dept
	Polytechnic Institute 333 Jay St. Brooklyn, N.Y.
1	Attn: L. Shaw, EE Dept
	Polytechnic Inst. of Brooklyn Graduate Center, Rt 110 Farmingdale, N.Y.
1	Attn: Librarian
	Purdue U. Lafayette, Ind.
1	Attn: Library, EE Dept
	Rensselaer Polytechnic Inst. School of Engrg Troy, N.Y.
1	Attn: Lib., Serial Dept. K. E. Mortenson
	U. of Saskatchewan College of Engrg Saskatoon, Canada
1	Attn: Prof. R.E. Ludwig
	Syracuse U. Syracuse 10, N.Y.
1	Attn: EE Dept

No. of Copies	
	U. of Texas Serials Acquisitions Austin, Texas
1	Attn: John Womack, Ser. Lib.
	Uppsala U. Inst. of Physics Uppsala, Sweden
1	Attn: Dr. P.A. Tove
	U. of Toledo Dept. of EE Toledo 6, Ohio
1	Attn: James B. Farison
	U. of Utah Salt Lake City, Utah
1	Attn: R.W. Grow, EE Dept
	U. of Virginia Charlottesville, Va.
1	Attn: J.C. Wyllie Alderman Lib.
	U. of Washington Seattle 5, Wash.
1	Attn: A.E. Harrison, EE Dept
	Worchester Polytechnic Inst. Worchester, Mass.
1	Attn: Dr. H.H. Newell
	Yale U. New Haven, Conn.
1	Attn: Sloane Physics Lab
1	Attn: EE Dept.
1	Attn: Dunham Lab Engrg. Lib.