Proceedings of the First International Symposium: Category Theory Applied to Computation and Control, published by the Mathematics Department and the Department of Computer and Information Science, University of Massachusetts at Amherst, 1974.

REALIZATION OF MULTILINEAR AND[1] MULTIDECOMPOSABLE MACHINES

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INTRODUCTION

In [1], there appears a category theoretic view of linear systems over a vector space which at the same time extends to a number of other classes of systems, for example, group machines. Our object here is to present some results on multilinear systems in the same vein as [1]. For most of the paper, we shall pose the results in terms of vector spaces. However, at the end of the paper, we indicate generalizations applicable to a category more general than Vect.

[1] The research reported in this paper was supported in part by Grant #GJ35759 of the National Science Foundation and the Australian Research Grants Committee.

An outline of our results is as follows: we define a K-line multilinear machine in terms of linear, bilinear,...(K-1)-line multilinear machines via an internal description, and we show that the associated input-output map is multilinear. Then we tackle the converse problem: passage from a prescribed multilinear map to an internal description. We discuss questions of the reachability, observability, and minimality of realizations.

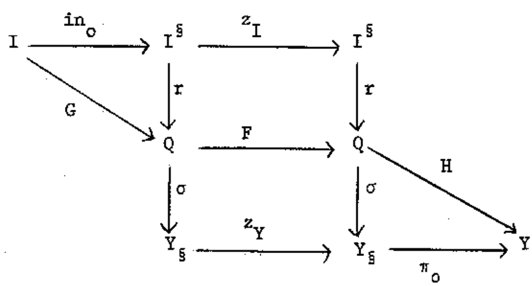
DEFINITION OF A MULTILINEAR SYSTEM

First, we define a linear system as a sextuple (Q,I,Y,F,G,H) where Q, I,Y are the state, input and output (vector) spaces, and $F:Q \to Q$, $G:I \to Q$, $H:Q \to Y$ are linear transformations; the system operates according to

$$q(k+1) = Fq(k) + Gu(k)$$
$$y(k) = Hq(k) \qquad (1)$$

where u,q,y are in I,Q,Y respectively.

In diagram form we have



(2)

Here, $I^§$ denotes the countably infinite copower of I, which one can identify with left infinite sequences of finite support $(\ldots,0,\ldots,0,i_k,i_{k-1},\ldots,i_o)$ with the injection $in_j: I \to I^§: i_j \mapsto (\ldots,0,\ldots,0,i_j,0,\ldots,0)$ into the (j+1) -th slot from the right; one can think of i_ℓ as $u(-\ell)$. Also, $Y_§$ denotes the countably infinite power of Y, which one can identify with right infinite sequences $(y_o,y_1,y_2\ldots)$ thinking of y_ℓ as $y(\ell+1)$. One has

$\pi_j: Y_\S \to Y: (y_o, y_1, y_2 \ldots) \to y_j$. The morphisms z_I and z_Y are shift operators, defined respectively by $z_i in_j = in_{j+1}$ and $\pi_{j+1} = \pi_j z_Y$; z_I has the effect of left shift with addition of zero in the zero position and z_Y of left shift, discarding the element in the zero position prior to the shift. The maps r and σ are the reachability map (mapping past input sequences into the present state) and the observability map (mapping the present state into a future output sequence, assuming zero future inputs). For a full discussion, see [1].

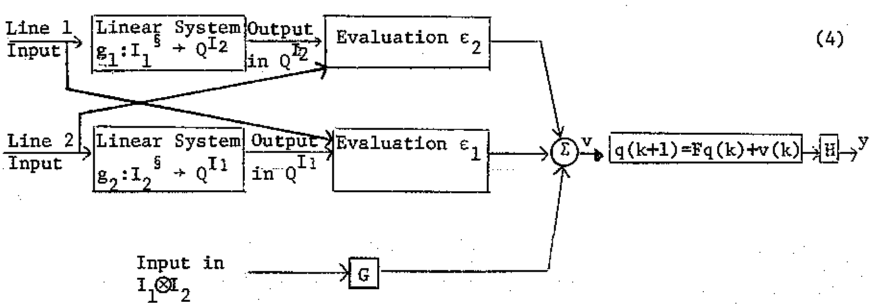
A multilinear system with K input lines numbered 1, 2,...,K is defined by an equation of the form

$$q(k+1) = Fq(k) + \sum \varepsilon_{i_{j+1} \ldots i_K} [y_{i_1 \ldots i_j}(k) \otimes u_{i_{j+1}}(k) \otimes \ldots \otimes u_{i_K}(k)] + G(u_1(k) \otimes \ldots \otimes u_K(k))$$

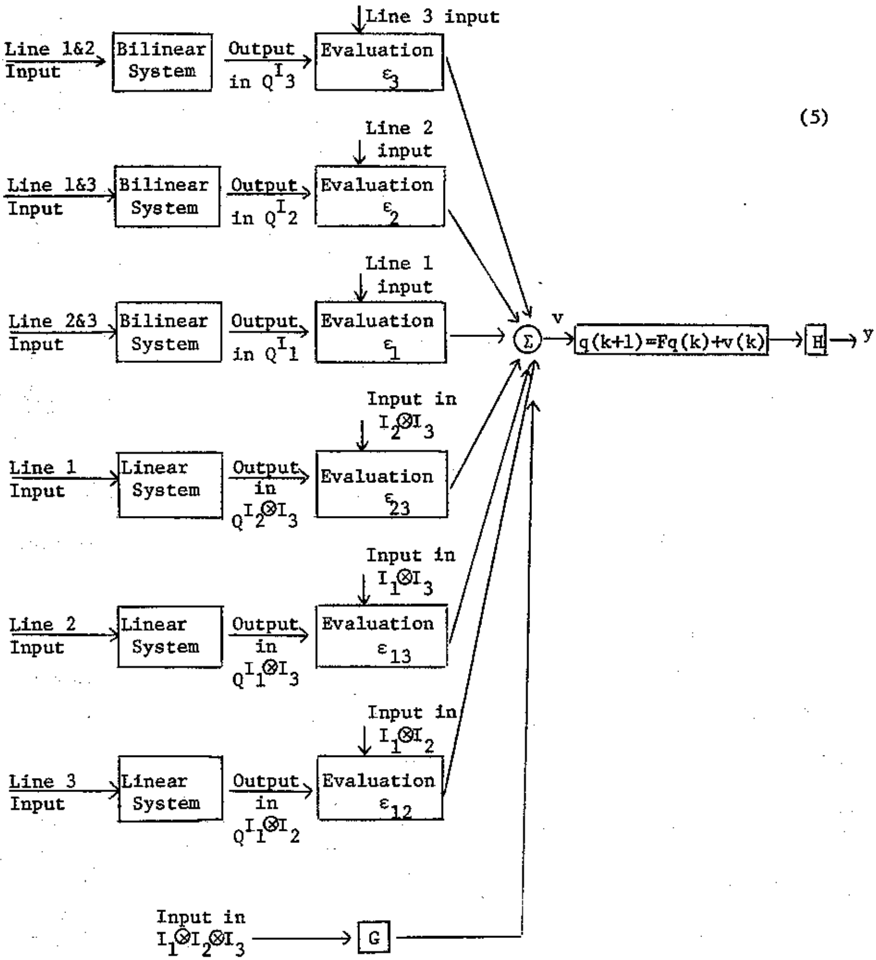
$$y(k) = Hq(k) \qquad (3)$$

where $I_1, \ldots, I_K$ are K input spaces, Q is the state space, Y is the output space; $u_j \varepsilon I_j$, $q \varepsilon Q$, $y \varepsilon Y$; $y_{i_1 \ldots i_j}(k)$ is the output of a known j-line multilinear system with input spaces $I_{i_1}, \ldots, I_{i_j}$, the summation is over every partition of $(1,2,\ldots,K)$ into disjoint nonempty sets $(i_1,\ldots,i_j)$, $(i_{j+1},\ldots,i_K)$; $y_{i_1 \ldots i_j}(k) \varepsilon \operatorname{Hom}(I_{i_{j+1}} \otimes \ldots \otimes I_{i_K}, Q)$ and $\varepsilon_{i_{j+1} \ldots i_K}$ is the canonical evaluation map; and F,G,H are linear maps with obvious domain and codomain.

Figure 1 illustrates the idea for K=2 while Figure 2 illustrates K=3



(4)



(5)

Figure 2

Each bilinear system has its own breakdown into linear systems.

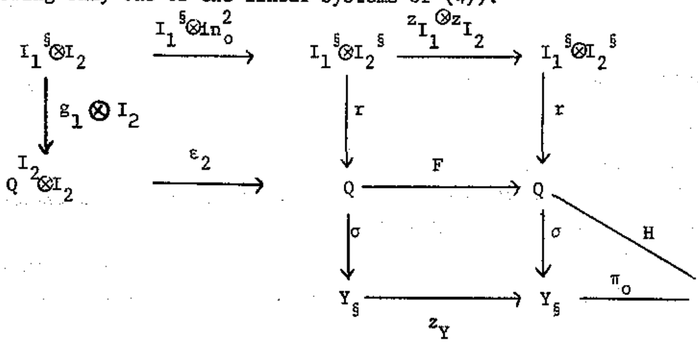
OBTAINING INPUT-OUTPUT DESCRIPTION FROM INTERNAL DESCRIPTION

Theorem 1 A K-line multilinear system defines a linear map $f: I_1^{\S} \otimes I_2^{\S} \ldots \otimes I_K^{\S} \to Y$ which can be extended to a map $f^{\blacktriangle}: I_1^{\S} \otimes \ldots \otimes I_K^{\S} \to Y_{\S}$

such that $f^{\blacktriangle}$ possesses the dynamorphic property $f^{\blacktriangle}(z_{I_1}\otimes\ldots\otimes z_{I_K}) = z_Y f^{\blacktriangle}$, and $\pi_o f^{\blacktriangle} = f$.

Remark One can also work with a multilinear map $\tilde{f} : I_1{}^{\S}\otimes\ldots\otimes I_K{}^{\S} \to Y$ which is associated with f in a canonical way.

In the bilinear case, one defines $f^{\blacktriangle}$ as $\sigma\cdot r$ from the following diagram (showing only one of the linear systems of (4)).

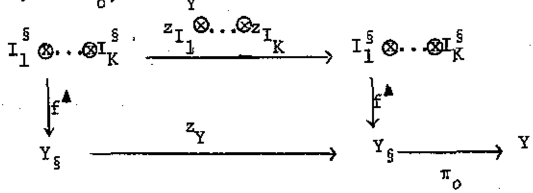


(6)

The map σ is constructed as for linear systems in [1]. Notice that given the definition (3) is completely equivalent. A set-up like that of (4) has been analyzed in [2-4], with the structure obtained via an analysis of the Nerode equivalence relation for a bilinear map $\tilde{f}: I_1{}^{\S}\times I_2{}^{\S} \to Y$. Let us now see how this structure will arise in an apparently quite different way, by generalizing some ideas of [1].

OBTAINING INTERNAL DESCRIPTION FROM INPUT-OUTPUT DESCRIPTION

Suppose $f : I_1^{\S}\otimes\ldots\otimes I_K^{\S} \to Y$; with no attention being paid to reachability or observability. An internal description is obtained as follows: take $Q = Y_{\S}$, $r = f^{\blacktriangle}$, $H = \pi_o$, $F = z_Y$



(7)

By using the appropriate zero injection, $I_{i_1}^{\S} \otimes \ldots \otimes I_{i_j}^{\S} \otimes I_{i_{j+1}} \otimes \ldots \otimes I_{i_K} \to I_1^{\S} \otimes \ldots \otimes I_K^{\S}$, obtain a map $g^{\bullet}_{i_1 \ldots i_j} : I_{i_1}^{\S} \otimes \ldots \otimes I_{i_j}^{\S} \otimes I_{i_{j+1}} \otimes \ldots \otimes I_{i_K} \to Y_{\S}$. Then $g_{i_1 \ldots i_j} : I_1^{\S} \otimes \ldots \otimes I_{i_j}^{\S} \to \mathrm{Hom}(I_{i_{j+1}} \otimes \ldots \otimes I_{i_K}, Y_{\S})$ such that

$\varepsilon_{i_{j+1} \ldots i_K}(g_{i_1 \ldots i_j} \otimes I_{i_{j+1}} \otimes \ldots \otimes I_{i_K}) = g^{\bullet}_{i_1 \ldots i_j}$ is uniquely definable.

The mapping G is defined as composition of $f^{\blacktriangle}$ with the injection $in_o^1 \otimes \ldots \otimes in_o^K : I_1 \otimes \ldots \otimes I_K \to I_1^{\S} \otimes \ldots \otimes I_K^{\S}$; these definitions yield the right commutative diagrom from which (by induction on K) the internal description can be recovered.

REACHABILITY AND OBSERVABILITY

Definitions of reachability and observability are obtainable via induction on the number of lines of a multilinear system.

Define a reachable multilinear system as one where r is epi and in case $K \geq 1$, the linear, bilinear,...(K-1)-line multilinear systems within it are also reachable. (Notice that if the second condition fails, the first need not; it is only the input-output properties of the linear,...(K-1)-line systems which can affect r, see (4) or (6)).

An observable system is one where σ is mono, and the linear, bilinear... (K-1)-line multilinear systems within it are also observable.

MINIMAL REALIZATION

A minimal realization of a map $f : I_1^{\S} \otimes \ldots \otimes I_K^{\S} \to Y$ is one which is reachable and observable.

Such a realization can be obtained as follows. Factor $f^{\blacktriangle}$ as σ r where r is epi, σ is mono; take $Q = r(I_1^{\S} \otimes \ldots \otimes I_K^{\S})$ (Effectively, $Q = \mathrm{Im}(f^{\blacktriangle})$. Using the appropriate zero injection, define

$g^{\bullet}_{i_1 \ldots i_j} : I_{i_1}^{\S} \otimes \ldots \otimes I_{i_j}^{\S} \otimes I_{i_{j+1}} \otimes \ldots \otimes I_{i_K} \to Q$ and then $g_{i_1 \ldots i_j} : I_{i_1}^{\S} \otimes \ldots \otimes I_{i_j}^{\S} \to \mathrm{Hom}(I_{i_{j+1}} \otimes \ldots \otimes I_{i_K}, Q)$. Induction allows construction of a minimal realization for each such $g_{i_1 \ldots i_j}$.

Define the state-space set $\mathcal{Q}$ of a K-line multilinear system by: $Q \in \mathcal{Q}$, and elements of the state-space set of all linear, bilinear,...(K-1)-linear multilinear system within the K-line multilinear system are in $\mathcal{Q}$. Then we have the expected result:

Theorem The elements, appropriately ordered, of the state-space sets of two minimal realizations of the same $f = I_1^{\S} \otimes \ldots \otimes I_K^{\S} \to Y$ are isomorphic.

CATEGORIES OTHER THAN VECT

Let K be a symmetric closed strict monoidal category with countable co-products and products. One can consider diagrams such as (5) and observe they define, via a generalized sort of recursion, a morphism $f = I_1^{\S} \otimes \ldots \otimes I_K^{\S} \to Y$, and one can start with such a morphism and show that there exist diagrams of the form (5) which define realizations of f. (One uses the adjointness property of the tensor product to pass between $\dot{g}_{i_1 \ldots i_j}$ and $g_{i_1 \ldots i_j}$ in the construction, which otherwise is the same mutatis mutandis, as for Vect). If $\mathcal{E}$-$\mathcal{M}$ factorizations exist in K, one can define reachable and observable realizations of a prescribed f. By analogy with [1], we might call these systems multidecomposable.

REFERENCES

1. M.A. Arbib and E.G. Manes, "Foundations of System Theory:Decomposable Systems", Automatica, May 1974.

2. M.A. Arbib, "A Characterization of Multilinear Systems", IEEE Transaction on Automatic Control, Vol. AC-14, No. 6, Dec. 1969, pp. 699-702.

3. G. Marchesini and G. Picci, "Some Results on the Abstract Realization Theory of Multilinear Systems", in Theory and Applications of Variable Structure Systems, ed. R.R. Mohler and A. Ruberti, Academic Press, New York, 1972, pp. 109-135.

4. E. Fornassini and G. Marchesini, "On the Internal Structure of Bilinear Input-Output Maps", Proceedings of NATO Conference on Control and System Theory, Imperial College, London, 1973.

5. R.E. Kalman, P.L. Falb and M.A. Arbib, Topics in Mathematical System Theory, McGraw Hill Book Co., New York, 1969.