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A CONTROL THEORIST LOOKS AT ABSTRACT NONSENSE

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The purpose of this part of the introduction is to give those at the category theory --rather than the control-- end of the spectrum some feel for the physical origin of some control problems, and some feel for some of the viewpoints taken by control theorists of these problems.

In the space available, far more must be omitted than can be included and the treatment must necessarily be very superficial. These facts notwithstanding, the material is of vital significance to any category theorist who wishes to make an honest, objective claim of applying category theory to control. In order that such category theorists might have their work accepted by the control fraternity, it is essential that the control fraternity be convinced that the category theory is not category theory for its own sake, nor another way of viewing known control results, nor a way of getting category theory generalizations of known control results with the generalizations possessing no control theoretic significance. There are already several distinct ways of viewing many control results, and control theorists will be understandably reluctant to add a further way if it provides merely a new view, rather than new control results.

Perhaps category theorists could even be warned that the module-theoretic approach to linear systems initiated nearly ten years ago by R. E. Kalman, though undoubtedly ingenious and aesthetically appealing to the more mathematically oriented of the control fraternity, has not yet received the close attention of a great many control theorists. This is certainly no reflection on the intrinsic merit of the work; it is the result of control theorists saying --not necessarily with irrefutable logic-- "this theory tells me all about realization and pole positioning, albeit in a nice way, but these are things I know about. There are many other control problems, solved and unsolved, which it does not tell me about. Why should I bother with it?". Perhaps the real point is that the control theorists simply should work harder. Nevertheless, with this experience, it is a little disappointing to find so much attention being paid in this conference to the realization problem, to the exclusion of other problems. A control theorist could validly ask "When is category theory going to start dealing with other control issues, such as feedback?".

PHYSICAL ORIGINS FOR LINEAR SYSTEMS

One readily understood physical situation which led electrical engineers to start the train of thought, the current development of which is explored in the present volume, is exemplified by a network of inductors, resistors and capacitors, with a terminal pair at which a voltage may be applied and a resultant current measured.

Here one has a situation which can be described in a fairly obvious manner by a differential equation of the type (see [1])

$$\sum_{i=0}^{n} a_i\, y^{(i)} \quad = \quad \sum_{j=0}^{m} b_j\, u^{(j)} \qquad (1)$$

(u = voltage or input or control, y = current or output, a_i, b_j scalar real constants). One can replace this equation set, using a standard device, by one of the form

$$\dot{x} = Fx + gu \qquad y = hx \qquad (2)$$

where $x = [y\ \dot{y}\ \cdots\ y^{(n-1)}]'$, F is an $n \times n$ real constant matrix and g is a real constant n-vector. In fact, an equivalent description is provided by

$$\dot{\bar{x}} = \bar{F}\bar{x} + \bar{g}u \qquad y = \bar{h}\bar{x} \qquad (3)$$

where $\bar{x} = Tx$ for some nonsingular T, $\bar{F} = TFT^{-1}$, $\bar{g} = Tg$, and $\bar{h} = [1\ 0\ \cdots\ 0]\ T^{-1}$. Parenthetically, we comment that this sort of change of variable (via change of coordinate basis) is widely used in linear systems theory. For an excellent introduction to this subject, consult [2] . For a deeper discussion of a subset of the topics covered in [2] , see [3] .

Electrical engineers would think of (1) as providing an input-output description of the network, and (2) or (3) as providing an internal description (because of the presence of the intervening variable x, linking u and y. It can be, as a result of special choice of T, that x has some physical significance as far as the internal behaviour of the network is concerned; for example, the entries of x may correspond to the capacitor voltages and inductor currents within the network (see [1] and also [4].)

It is clearly of interest to be able to compute the output at a given time resulting from the input applied up to that time, assuming the network is initially unexcited. In fact, one has

$$y(0) = \int_{-\infty}^{0} w(\tau)u(\tau)d\tau \qquad (4)$$

(provided that the integral exists), where $w(\tau)$ is given by $\bar{h}\exp(-\bar{F}\tau)\bar{g}$. Thus there is defined a map $u(\tau),\ \tau \in (-\infty,0] \longmapsto y(0)$. Such a map is also an example of an input-output description ([1,2]), and the function $w(\cdot)$ is almost the <u>impulse response</u> of linear systems theory; here, (4) is almost the <u>convolution</u> formula ([1,2]).

Generally, from a system user's point of view, the input-output description is the most relevant. From a system designer's point of view, an internal description may be relevant, since generally it is needed in specifying how a system may physically be constructed.

Passing from an input-output description to an internal description is the act of <u>solving the realization problem</u>. Questions such as the following occur:

1) What $w(\tau)$ can be expressed in the form $w(\tau) = h\exp(-F\tau)g$?

2) How may one find any triple F, g, h realizing $w(\tau)$?

3) Are there especially interesting F, g, h realizing $w(\tau)$?

In realization theory, for those $w(\tau)$ expressible as $h\exp(-F\tau)g$, "especially interesting" has come to mean "with F matrix of minimum dimension", and the associated triples (F,g,h) are termed <u>minimal</u> or <u>canonical</u> (see [2,3]).

Minimal realizations are precisely those which are simultaneously completely reachable and completely observable. A realization is completely controllable if, with x initially zero, the map $u(\tau), \tau \in (-\infty, 0] \longmapsto x(0)$ defined by (3) -- well defined if, for example, u has compact support, and x(-M) is zero where supp $u \in [-M, 0]$ -- is onto; in other words, there exists a control taking the zero state to any nonzero state. A realization is completely observable if, with $u(\tau) = 0$ for $\tau \geq 0$, the map $x(0) \longmapsto y(\tau), \quad \tau \in [0, \infty)$ is injective; in other words, no two distinct states can yield the same output or, in theory (and actually in practice), the state x(0) is computable from the output $y(\tau), \tau \in [0, \infty)$.

The controllability property has many equivalent statements. For example ([2,3])

a) rank $[g \; Fg \; \cdots \; F^{n-1}g] = n$ (which is a helpful statement for checking controllability)

b) $w'e^{Ft}g = 0$ for all t implies $w = 0$

c) the eigenvalues of $F + gk'$ can be assigned arbitrarily via choice of k, provided that complex eigenvalues occur in complex conjugate pairs (this result is of major significance in the design of <u>feedback</u> <u>control</u> <u>laws</u> which are mentioned further below).

Likewise, the observability property has many equivalent statements. To begin with, it is dual to the controllability property. To fully expound the duality would not be worthwhile here; we merely comment that $[F, h]$ is a completely observable pair if and only if its adjoint pair $[F', h']$ is completely controllable. A second important property of observable systems is that one can construct a dynamic observer, the output of which is a vector $\hat{x}$, the input of which is both $u(\cdot)$ and $y(\cdot)$, such that $\lim_{t\to\infty} \hat{x}(t) - x(t) = 0$, with the convergence being exponential, and otherwise arbitrarily fast. The observer is actually itself a linear finite-dimensional system. (For some discussion of the principal types of observer, developed by Luenberger and Kalman-Bucy, see [5]).

Of course, not all the controllability equivalences carry over to more general category-theoretic situations; thus one might expect to carry over the epi property of controllability, but not the eigenvalue positioning property.

Let us pause to note a key notion of control—that of feedback. Consider the equation (3) in which we have said u is an input or control. Suppose u is derived as the sum of an externally applied control, v say, and a linear function $k'\bar{x}$ (k a real vector) of the state $\bar{x}$. Then we can say that there is feedback around the system defined by (3) which can be written as

$$\dot{\bar{x}} = \bar{F}\bar{x} + \bar{g}(v + k'\bar{x}) = (\bar{F} + \bar{g}k')\bar{x} + \bar{g}v$$

Feedback can be used to modify the style of behaviour of a system and it need not be linear as above. The principal design task in many control system problems is that of selecting a feedback law to cause the system with feedback —the closed loop system— to have a certain type of behaviour; so feedback is a very central concept —if not the central concept— in control.

VARIATIONS OF THE BASIC SYSTEM EQUATIONS

Let us note other physical situations which can be modeled by variations on (1) - (4).

Discrete-time systems (see e.g. [3]). As discussed in part 3 of this introduction, it is possible to derive discrete-time equations of the form

$$x_{k+1} = Ax_k + bu_k$$
$$y_k = cu_k \tag{5}$$

from, say, (3). (Category theorists should be warned that there are prejudices within the control theory community as to whether one should use F, g and h or A, b and c in linear system equations. In the author's view, the arguments favoring either one over the other have little or no objective validity.) If one attempts to solve (3) on a digital computer, some time sampling is obviously necessary (strictly, level quantization is also needed, but this is overlooked). Sampling (3) leads to a set-up describeable by (5); u_k, y_k and x_k are sample values of u, y, x roughly, or averages over a sampling interval of the continuous time quantities. A, b, c can be found from F, g, h.

Non linear systems (see e.g. [6,7]). Many physical systems are nonlinear. The result is that one may have to work with equations such as

$$\dot{x} = f(x) + g(x) \cdot u \tag{6}$$

or, more generally,

$$\dot{x} = f(x,u) \tag{7}$$

and

$$y = h(x) \tag{8}$$

<u>Linear</u> <u>time-varying</u> <u>systems</u>. There are at least two quite distinct ways in which one might end up studying a time-varying system, even a discrete time-varying system. Some systems important in electrical engineering applications are intentionally made time-varying, for example, parametric amplifiers, where one has periodically varying F, g and/or h replacing the constant matrices of earlier (see [8]). Sampling such systems yields a time-varying discrete-time system.

A second source for such systems arises when one studies a perturbation of a nonlinear system about a nominal control and trajectory. Let a control $\bar{u}(\cdot)$ (the nominal control) give rise to a trajectory $\bar{x}(\cdot)$ via $\dot{\bar{x}} = f(\bar{x},\bar{u})$. Suppose an actual control $u(t) = \bar{u}(t) + \delta u(t)$ is applied, where δu is small. If the resultant trajectory is $x = \bar{x} + \delta x$ with small δx, then δx and δu may be approximately related by a <u>time-varying</u> differential equation, $(\dot{\delta x}) = F(t)(\delta x) + g(t)(\delta u)$. The entries of $F(\cdot)$ and $g(\cdot)$ are obtained by evaluating entries of partial derivatives of f, which vary along the trajectory. This sort of device is important in optimal control, see e.g. [3,9].

<u>Infinite</u> <u>dimensional</u> <u>systems</u>. Systems such as telephone channels, waveguides ([10]) and reactors are examples of infinite dimensional systems of great interest to engineers. Many such systems are linear and have external descriptions via a $w(\tau)$ as in (4), but do not have a finite-dimensional description such as (3). Since the engineering design problem for such systems clearly exists (given an input-output specification, build a system), and since system descriptions of a level adequate to build a system are generally of the internal description type (whatever that may be in this case), we see that there is some sort of realization problem

here too.

Vector generalizations (see [1,2,3,4,5,6]). It is always possible to conceive of a number of input channels and a number of output channels. The generalization is not always trivial, even for linear, finite-dimensional systems.

The major tool through the 1960's and early 1970's in linear system theory and, to an extent, control theory generally, has been linear algebra. Perhaps because engineers on the whole are more comfortable in less abstract situations, or perhaps simply because engineering problems demand solutions specified by numbers, matrices rather than linear transformations figure largely. References containing many of the standard devices (or, as category theorists might prefer to say, tricks) are [11, 12] . Recently, [13] has appeared; it would provide a nonspecialist with the matrix theory scenario for linear systems.

Let us now discuss some of these ideas in a little more detail.

LINEAR DISCRETE-TIME SYSTEMS

One can start by postulating that a system is of the form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \qquad (9)$$

where A, B, C are real constant matrices of dimension n x n, n x p, m x n respectively. Imagine p input channels; at time k, they inject u_k into the system. The state of the system at time k is x_k; the output is y_k.

One can conclude that

$$y_1 = \sum_{i=0}^{M} CA^iBu_{-i} \qquad (10)$$

where one assumes that the input sequence is of support bounded on the left, and the state is zero prior to nonzero input appearing. See [3].

Now one dresses this up a bit and observes that (10) is but a map from $R^m[z]$, the set of m-vectors whose entries are polynomials in z with real coefficients into the reals, if we recognize that the sequence $(0 \cdots 0\ u_{-M}\ u_{-(M-1)} \cdots u_0)$ is isomorphic to the polynomial

$$\sum_{i=0}^{M} z^i\, u_{-i}$$

Call this map $f : R^m[z] \longrightarrow R$.

Just as $R^m[z]$ is introduced to help talk about input sequences, we can introduce $R^p[[z^{-1}]]$, formal power series in z^{-1} with p-vector coefficients, to talk about output sequences.

Suppose an input sequence $(\cdots 0\ u_{-m}\ u_{-m+1} \cdots u_0\ 00 \cdots 0 \cdots)$ is applied. Look at the output string $(y_1,y_2,y_3 \cdots)$ (disregard all outputs earlier than time instant 1). The output strings are clearly isomorphic to $R^p[[z^{-1}]]$: $(y_1,y_2,y_3 \cdots) \longmapsto \sum y_i z^{-i}$.

Now it is clear that $f : (0 \cdots 0\ u_{-m} \cdots u_0) \longmapsto y_1$ can be extended to $f^* : (0 \cdots 0\ u_{-m} \cdots u_0) \longmapsto (y_1,y_2,y_3 \cdots)$ by direct calculation using (9). But there are two other distinct ways of looking at this extension.

First, as first pointed out by Kalman (see e.g. [3]), f^* <u>is an</u>

$R[z]$ -<u>module</u> <u>homomorphism</u>. (That $R^m[z]$ is an $R[z]$ -module with appropriate module action is easily seen; that $R^p[[z^{-1}]]$ is also an $R[z]$-module is a little harder to see: the action involves multiplication of each component by a polynomial in z and deletion of nonnegative powers of z).

Second, f^* is a morphism $(R^m)^{\xi} \longrightarrow (R^p)_{\xi}$ where $(R^m)^{\xi}$ is the copower (weak direct sum) of an infinite sequence of copies of R^m and $(R^p)_{\xi}$ is the product of an infinite sequences of copies of R^p, with a commutative diagram of the following sort holding.

$$\begin{tikzcd}
(R^m)^{\xi} \arrow[r, "z_u"] \arrow[d, "f^*"'] & (R^m)^{\xi} \arrow[d, "f^*"] \\
(R^p)_{\xi} \arrow[r, "z_y"'] & (R^p)_{\xi}
\end{tikzcd}$$

where z_u and z_y are morphisms defineable just using coproduct and product properties. See [14].

Realization is the problem of passing from f to an (A,B,C) triple as in (9).

From the $R[z]$ -module viewpoint, one proceeds as follows. From f, set up f^*. Define the state-space X in which x resides by $X = R^m[z]/\ker f^*$. Then technical, but conceptually straightforward calculations allow the construction of an A,B,C triple, first as linear transformations, and then, via coordinate basis specialization, as matrices. By using standard module-decomposition theorems, one can get various particular structures for A, B and C. With this definition of X, realizations are guaranteed universal.

As an interesting sidelight to these calculations, one can observe that X, stripped of its module structure, is identical with the set of Nerode equivalence classes which may be associated with f. This is a pleasing result, but it would probably have been equally pleasing if instead, for example, the Nerode equivalence classes generated X.

To proceed using the categorical approach, one needs an $\mathcal{E}-\mathcal{M}$ factorization system (see the first part of this introduction). One sets $X = \mathrm{Im}(f^*)$. Then $A : X \longrightarrow X$ follows by a standard property of $\mathcal{E}-\mathcal{M}$ factorizations, $B = r\, in_0$ and $C = \pi_1 \sigma$ where $\sigma r = f^*$ is the $\mathcal{E}-\mathcal{M}$ factorization of f^*, in_0 is the coproduct injection corresponding to $u_0 \longmapsto (\cdots 0 \cdots 0 u_0)$ and π_1 is the product projection corresponding to $(y_1, y_2, y_3 \cdots) \longmapsto y_1$.

The maps f and f^* can be thought of as being defined by the collection of matrices M_i such that

$$y_1 = \sum_{i \geq 0} M_i u_{-i} \qquad (11)$$

The realization problem (compare (10)) is then one of finding A,B,C such that $CA^iB = M_i$ for all nonnegative integers i. Interpretation of the procedure for realizing f^* can be given in terms of the M_i. See expecially [3].

EXTENSIONS USING MODULE THEORY IDEAS

There is no single decisively preeminent extension of the module theoretic approach to linear system realization. Rather, there are a number of examples; perhaps their multitude indicates the richness of the theory as a realization theory. It would seem however that the application of the ideas to control problems other than realization has to this point been minimal.

First, one can make the observation that the construction of the state-space X as $R^m[z]/\ker f^*$ does not use finite-dimensionality of the underlying linear system. Second, one can observe that the fact that R is the real field is inessential; realization over other fields is relevant to coding theory, e.g. [15]. Third, one can even replace R by a ring (an important result here is that if the ring is a Noetherian integral domain with identity, finite dimensional realizations exists if and only if they exists over the associated quotient ring, [16]). Such extensions, were they without applications, would be unappetizing to control theorists. Applications exist however; for example, Brockett and Willems ([17]) used the ring of circulant matrices to get efficient algorithms for solving problems arising from a class of linear partial differential equations. Johnson ([18]) shows that discretization of the one-dimensional heat equation yields a linear difference equation over $R^*[y]$, the quotient ring of $R[x,y]$ modulo the principal ideal generated by $xy - 1$.

EXTENSIONS USING CATEGORY THEORY IDEAS

In these Proceedings, the paper by Wyman ([19]) allows the ring k to be arbitrary, and replaces $k[z]$ by an arbitrary k-algebra of operators. The construction of f^* proceeds by an adjointness argument. Full development of the examples is unfortunately inhibited by length constraints. However, one class of examples is provided by those partial difference equations stemming from two-dimensional partial differential equations. Such equations arise in studying the processing of seismic signals —see recent issues of journals such as the IEEE Transactions on Geoscience Electronics and IEEE Transactions on Audio and Electroacoustics. However, one must still seriously ask whether, from the point of view of applications, the problem justifies the use of tools which most electrical engineers would regard as exotic.

Elsewhere, the paper of Arbib and Manes ([14]) shows that in formal terms, the realization problem can be viewed using coproducts, products and $\mathcal{E}-\mathcal{M}$ factorizations. It then follows that one can recover, for example, some of the group machine ideas of Brockett and Willsky ([20]). Finite-dimensionality is to be discussed in [21] . Further, the possibility of replacing the real field with an arbitrary ring becomes clear.

CONTINUOUS-TIME LINEAR SYSTEMS

Before discussing some of the approaches to the realization problem for continuous-time linear systems ([1,2,4,5,6]) let us write several facts distinguishing them from discrete-time linear systems.

1. $y = \dot{u}$ is the equation of a capacitor ([1]); if $u(\cdot)$ has a step discontinuity, but remains bounded, y becomes infinite.

2. One may seek to rule out infinite inputs and outputs; however, engineers find it useful to work with an input-output description of a linear system, viz. its impulse response (very closely related to $w(\tau)$ in (4), see [1,2]). The impulse response is useful because, from it, one can in theory obtain the output due

any input. However, as the name suggests, it is the response which would result from a Schwartz impulse function ([22]), which is not finite.

3. $y = u$ is the equation of a resistor ([1]); y is only nonzero if u is. So one cannot look at the output arising only after the cessation of input, for it is identically zero. In other words, whereas in discrete time systems, one could study the relation between input sequences on $(-\infty,0]$ and outputs on $[1,\infty)$, one is really driven to study at the least continuous-time inputs on $(-\infty,0]$ and outputs on $[0,\infty)$. (Justification is also provided by, e.g., $y = \dot{u}$).

4. In a physical situation impulse functions or their derivatives cannot really exist. Models of the form $y = \dot{u}$ cease to be physically valid when either u or y becomes too large ([1]).

5. Let $u(t)$, $t \in (-\infty,\infty)$ be the value of some system variable at time t. A physical measuring instrument can never measure $u(t)$ precisely. Instead, it will measure something like

$$\int_{t-\epsilon}^{t+\epsilon} \varphi(\lambda)\, u(\lambda)\, d\lambda$$

for some function $\varphi(\cdot)$ and some number ϵ, both determined by the instrument.

The Schwartz distribution theory provides the answer to many of these problems ([22]). One can restrict attention for example to those situations in which all inputs and outputs are in $\mathcal{D}_+$, the set of functions with support bounded on the left and which are infinitely differentiable, see e.g. [23] . Or one can take the view that the inputs and possibly the outputs need not even be bounded, although they must be distributions (implying a measureability property in the sense of point 5 above).

The first attempt at a realization theory paralleling the module theoretic approach to discrete time linear systems appears to be that of Kalman and Hautus ([24]). They start by postulating that the system defines a linear map $f : \mathcal{E}'^{\,m}_{(-\infty,0]} \longrightarrow R$ (here $\mathcal{E}'$ is the space of distributions with compact support). Then f is extended to an $\mathcal{E}'_{(-\infty,0]}$ homomorphism $f^* : \mathcal{E}'^{\,m}_{(-\infty,0]} \longrightarrow \mathcal{E}^{p}_{[0,\infty)}$ (here, $\mathcal{E}$ denotes C^∞ functions). Then one looks at the Nerode equivalence classes, observing them to be isomorphic to $\mathcal{E}'^{\,m}_{(-\infty,0]}/\ker f^*$, which is an $\mathcal{E}'_{(-\infty,0]}$ homomorphic image. Finally, one can examine the consequences of finite-dimensionality.

There are difficulties with this approach. First, even such a system as $\dot{y} = u$ is not included. Second, though outputs are C^∞ and the states are equivalence classes of outputs, the states have to be viewed as being distributions.

More recently, Kamen ([25]) considered maps

$f^* : \mathcal{E}'^{\,m}_{(-\infty,0]} \longrightarrow \mathcal{D}'^{\,p}_{(0,\infty)}$ (Here $\mathcal{D}'$ is the space of distributions). Kamen abandons the attempt to start with a map f with codomain R in view of the difficulties that arise. However, his definition of f^* also fails to capture $y = u$ or $y = \dot{u}$, both of interest to engineers, because the output interval is $(0,\infty)$ rather than $[0,\infty)$. One finds that f^* is an $\mathcal{E}'_{(-\infty,0]}$-module homomorphism, and as a result, can obtain structure theorems for some finite-dimensional problems. These appear to be promising in connection with the study of lumped-distributed systems ([26]), such as those obtained by interconnection of a finite number of resistors, inductors, capacitors and transmission lines. In these Proceedings, Kamen ([27]) takes some of these ideas further. The paper largely speaks for itself, but there is one point we wish to make here. To get around the difficulty of overlapping input and output intervals in the definition of f, the technique adopted demands that concatenation of two inputs only be permitted with the inputs are separated by an interval of length a (a is arbitrary, > 0) during which the input is zero. This is slightly unfortunate, but perhaps less serious a restriction than those of earlier papers.

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