# SMOOTHING OF NOISY
# RANDOM-TELEGRAPH-TYPE SIGNALS

David J. Clements and Brian D.O. Anderson
Department of Electrical Engineering
University of Newcastle, New South Wales
Australia

Abstract

For a class of Markov processes in additive gaussian white noise, both the equations for the optimal fixed-point smoother and those for a sub-optimal fixed-lag smoother are presented. Simulation results for the random telegraph wave are discussed.

## 1. INTRODUCTION

It has been shown in the literature[1] that fixed-lag smoothers for both the continuous and discrete-time linear-gaussian problems exhibit the following desirable properties: (1) the expected value of the mean square error is a non-increasing function of the lag and (2) for a value of lag of the order of the dominant time constant of the associated (zero-lag) filter, practically all of the possible improvement that smoothing offers over filtering is attained. Further, for stationary time-invariant processes[2], the ratio of the fixed-lag error to the filtering error decreases as the signal-to-noise ratio increases. Hence, for those applications where a time delay between the estimate of the state of a process and the state itself is acceptable, it is worthwhile looking at the possibility of using fixed-lag smoothing rather than filtering.

Intuitively, it seems plausible that these properties should in fact hold for a larger class of processes than that above. In this paper, we derive and examine properties of a sub-optimal fixed-lag smoothed estimate of the states of a class of Markov processes in the presence of additive gaussian white noise. Simulation of the fixed-lag smoother for the special case of the random telegraph wave suggests that the above properties do in fact hold for the class of processes under study.

## 2. A NONLINEAR FILTERING RESULT

Suppose $\{x(t),\ t \geq 0\}$ is a Markov process with $n$ distinct states $a_1, \ldots, a_n$ and stationary transition probabilities $p_{ij}(h) \triangleq \Pr\{x(t+h) = a_j | x(t) = a_i\}$ satisfying

$$p_{ij}(h) = \begin{cases} 1 - \nu_i h + o(h) & i=j \\ \nu_{ij} h + o(h) & i \neq j \end{cases} \tag{1}$$

where $h \geq 0$ and $\nu_i$, $\nu_{ij}$ are nonnegative constants such that $\sum_{\substack{j=1 \\ j\neq i}}^{n} \nu_{ij} = \nu_i$. Let a measurement process $\{z(t),\ t \geq 0\}$ be defined by

$$dz(t) = h(x(t))dt + \beta(t)dw(t) \tag{2}$$

Here $dw(t)$ is an increment in a Wiener process independent of $x$ with a derivative of covariance $\delta(t-\tau)$, $\beta(t)$ is a continuously differentiable function bounded away from zero and $h$ is a real-

* Work supported by Radio Research Board and Australian Research Grants Committee.

real-valued function with values $h_1, \ldots, h_n$; not necessarily distinct. Denote the $z$ process over the interval $[0,t)$ by $Z_t$ and denote the filtered probabilities $\Pr\{x(t) = a_i | Z_t\}$ by $p_i(t)$. Then, following Wonham[3], it can be shown that these probabilities satisfy

$$dp_i(t) = [-\nu_i p_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \nu_{ji} p_j(t)]dt + \beta(t)^{-2} p_i(t)[h_i - \bar{h}(t)][dz(t) - \bar{h}(t)dt] \quad (3)$$

with $\bar{h}(t) = \sum_{i=1}^{n} h_i p_i(t)$, the conditional expectation of $h(x(t))$. The initial condition for (3) is just the initial probability distribution for $x(0)$.

## 3. FIXED-POINT SMOOTHING

For each fixed $t$, define the augmented process $\{X(t|\tau), \tau \geq t\}$ by $X(t|\tau) = [x(\tau) \quad x(t)]'$. This process has $N = n^2$ states $A_I$, $I=1, \ldots, N$; or in component form $(a_i, a_j)$ with $I = (i,j)$ and $i, j=1, \ldots, n$. From the properties of $x$, it also follows that this new process is Markov and has stationary transition probabilities $P_{IJ}(h)$ satisfying

$$P_{IJ}(h) = \begin{cases} 1 - V_I h + o(h) & I=J \\ V_{IJ} h + o(h) & I\neq J \end{cases} \quad (4)$$

for $I = (i_1, j_1)$ and $J = (i_2, j_2)$. The nonnegative constants $V_I$, $V_{IJ}$ are defined by $V_I = \nu_{i_1}$, $V_{IJ} = \delta_{j_1 j_2} \nu_{i_1 i_2}$ and satisfy $\sum_{\substack{J=1 \\ J\neq I}}^{N} V_{IJ} = V_I$.

With $h(X(t|\tau)) = x(\tau)$, the measurements are given by

$$dz(\tau) = x(\tau)d\tau + \beta(\tau)dw(\tau), \quad \tau \geq t \quad (5)$$

Then denote the filtered probabilities of $X$, $\Pr\{X(t|\tau) = A_I | Z_\tau\}$ by $P_I(t|\tau)$ and the fixed-point smoothing probabilities of $x$, $\Pr\{x(t) = a_j | Z_\tau\}$ by $p_j(t|\tau)$. Since $p_j(t|\tau) = \sum_{i=1}^{n} P_I(t|\tau)$ with $I = (i,j)$ we have the filtered probabilities of $X$ completely determining the fixed-point probabilities of $x$. From Section 2 the former are easily found as the solution of the equations

$$dP_I(t|\tau) = [-V_I P_I(t|\tau) + \sum_{\substack{J=1 \\ J\neq I}}^{N} V_{JI} P_J(t|\tau)]d\tau + \beta(\tau)^{-2} P_I(t|\tau)[a_i - \bar{x}(\tau)][dz(\tau) - \bar{x}(\tau)d\tau] \quad (6)$$

with the initial condition $P_I(t|t) = \delta_{ij} p_i(t)$, $I = (i,j)$ and with $\bar{x}(\tau) = \sum_{I=1}^{N} a_i P_I(t|\tau)$, the conditional expectation of $x(\tau)$.

The above approach to the continuous-time fixed-point smoothing problem has been used previously for both the linear[4] and nonlinear[5] cases.

### RANDOM TELEGRAPH WAVE EXAMPLE

For a process with states $+1$, $-1$ and switching parameter $\nu$ with $\nu_i = \nu$ and $\nu_{ij} = \nu$ for $i,j = 1, 2$, the fixed-point equations (6) are, with independent variables $q$, $Z$ and $D$,

$$dq(\tau) = -2\nu q(\tau) + \beta^{-2}[1-q(\tau)^2]dI(\tau), \quad q(0) = 0 \quad (7)$$

$$dZ(t|\tau) = \beta^{-2}[2D(t|\tau)-1-Z(t|\tau)q(\tau)]dI(\tau), \quad Z(t|t) = q(t) \quad (8)$$

$$dD(t|\tau) = \nu[1-2D(t|\tau)]d\tau + \beta^{-2}[\tfrac{1}{2}q(\tau) + \tfrac{1}{2}Z(t|\tau) - D(t|\tau)q(\tau)]dI(\tau), \quad D(t|t) = 1 \quad (9)$$

where $q$ is the conditional expectation of $x$, $Z$ is the fixed-point estimate of $x$, $D$ is the probability that both components of the X-process are the same; and $I$ is the innovations process with $dI(\tau) = dz(\tau) - q(\tau)d\tau$.

## 4. FIXED-LAG SMOOTHING (DISCRETE-TIME)

For the fixed-lag smoothing problem we need to know the evolution of the quantities $P_I(t|t+L)$ with $L$ fixed and $t$ varying. One approach which suggests itself is to take equation (6) for $P_I(t|\tau)$, to express $P_I(t|t+L)$ as an integral using this equation, and then to compute the differential now letting $t$ rather than $\tau$ vary. Although this method works for the linear-gaussian problem[6] it fails here because of the conditional rather than unconditional expectations in the integrand. We thus treat only the discrete-time case and derive a suboptimal fixed-lag estimate as a linear combination of the states of a discrete-time nonlinear system driven by the measurements.

With obvious notation we have, discretizing (6)

$$P_I(k|\ell+1) = P_I(k|\ell) + [-V_I P_I(k|\ell) + \sum_{\substack{J=1 \\ J\neq I}}^{N} V_{JI} P_J(k|\ell)]T + \beta(\ell)^{-2} P_I(k|\ell)[a_i - \bar{x}(\ell)]I(\ell) \qquad (10)$$

where $k$ is fixed, $\ell$ takes the values $k$, $k+1$, $k+2$, ..., and $I(\ell) = z(\ell+1) - z(\ell) - \bar{x}(\ell)T$. Now, fix $\ell$ in these equations, and in writing down (10) for each $k$ in the range $\ell$ to $\ell-L+1$, define the new variables $F_{Ij}(\ell) = P_I(\ell-j+1|\ell)$ for $I=1$, ..., $N$ and $j=1$, ..., $L+1$. We then obtain

$$F_{I,j+1}(\ell) = F_{Ij}(\ell) + [-V_I F_{Ij}(\ell) + \sum_{\substack{J=1 \\ J\neq I}}^{N} V_{JI} F_{Jj}(\ell)]T + \beta(\ell)^{-2} F_{Ij}(\ell)[a_i - \bar{x}(\ell)]I(\ell) \qquad (11)$$

for $j=1, \ldots, L$

while the quantities $F_{I1}(\ell)$ are none other than the filtered probabilities associated with the original x-process, and hence are updated by the discrete-time version of the filter equations (3). Finally, the suboptimal fixed-lag estimate of $x(\ell-L)$ given $Z_\ell$ is $\sum_{I=1}^{N} a_i F_{I,L+1}(\ell)$.

ALTERNATIVE DERIVATION

We now outline an alternative derivation of (11) which is based on a discrete-time approximation[7] to the continuous-time processes involved and the idea[8] that the discrete-time fixed-lag smoothing problem can be posed as one of filtering a related process. Consider first a discrete-time Markov process $\{s(k), k=0, 1, \ldots\}$ for which the transition probabilities $\Pr\{s(k+1) = a_j | s(k) = a_i\}$ are denoted by $p_{ij}$ with $p_{ij} = 1-\nu_i T$ for $i=j$ and $p_{ij} = \nu_{ij} T$ for $i\neq j$, $T$ being the discretization interval; and second, the observation process defined as

$$(k) = s(k)T + \beta(k)w(k)\sqrt{T}$$

where $\{w(k), k=0, 1, \ldots\}$ is gaussian white noise with unit variance.

Suppose we define a new Markov process $\{S(k), k=L, L+1, \ldots\}$ with $S(k) = [s(k) \;\; s(k-1) \; \ldots \; s(k-L)]'$, then the fixed-lag probabilities of the s-process are simply sums of the filtered probabilities of the S-process. The latter can be derived using a well-known recursive formula[9, p174] and approximating[7] for small $T$.

## 5. SIMULATION - DISCUSSION AND RESULTS

For our example, the random telegraph wave, we generated the approximated discrete-time processes mentioned in the final part of section 4. It was found that for reliable results the following points need to be considered.

(a) Consider the discrete-time filter equation (see (7))

$$q(k+1) - q(k) = -2\nu q(k)T + \beta^{-2}[1-q(k)^2][x(k)-q(k)]T + \beta^{-1}[1-q(k)^2]w(k)\sqrt{T}, \quad q(0) = 0 \qquad (12)$$

The definition of $q$ implies $|q(k)| \leq 1$ for all $k$. However, from (12) it is clear that for a sufficiently large value of $w$, this bound will be violated. Then, the term $[1-q(k)^2][x(k)-q(k)]$ is destabilizing in the sense that its effect on the increment is to cause $|q(k+1)| > |q(k)|$. Hence, for a sequence of values of $w$ with suitable sign, $q$ may become arbitrarily large. An ad hoc solution, which was found to be satisfactory, was to redefine $q(k)$ as $\pm 1$ at each iteration whenever $|q(k)| > 1$. An alternative method is to bound the noise samples and choose a sufficiently small sampling interval $T$.

(b) It is also necessary to ensure that the discrete-time processes are, in fact, approximations to their continuous-time counterparts. Each term on the right side of (12) is small if we choose $T$ such that $T < \beta/20\sqrt{\nu}$ and $\nu T$ is small. Although ad hoc, this bounding proves to be a successful guideline in the choice of parameters in the simulation. This bound on $T$ implies that the larger is the signal-to-noise ratio the greater is the amount of computation per unit of time.

An example of the results of simulation is shown in Fig. 1, with $\nu = 50$, $\beta = 0.07$ and $T = 0.0005$. Then $\nu T = 0.025$ which is "small" and $\beta/20\sqrt{\nu} \simeq 0.0005$. Thus the conditions on $T$ are satisfied. It is apparent from Fig. 1 that there is an initial rapid improvement in estimation with lag, followed by an evening out to a steady-state value. For the filter eq.(12), we can associate a time constant of $1/2\nu T$ sampling intervals. For our example, $1/2\nu T = 20$. Compare this with Fig. 1 where the lag for which practically all improvement is obtained is about 15. We found that every example for which the conditions on $\beta$, $\nu$ and $T$ were satisfied, this type of behaviour was observed.

Fig. 2 shows a plot of improvement of smoothing over filtering versus a noise measure $\mu = \nu\beta^2$. The similarity with the linear-gaussian case can be seen.

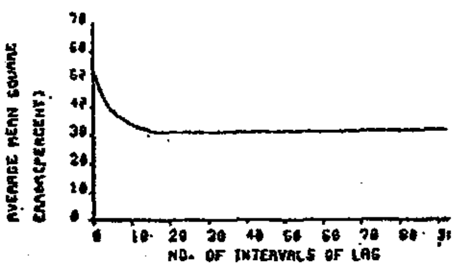


Figure 1

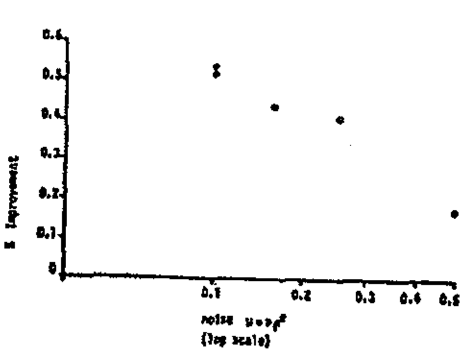


Figure 2

REFERENCES

REFERENCES


[1] B.D.O. Anderson, "Properties of Optimal Linear Smoothing" IEEE Transactions on Automatic Control, Vol. AC-14, pp.114-115, 1969.

[2] B.D.O. Anderson and S. Chirarattananon, "Smoothing as an Improvement on Filtering: a Universal Bound", Electronics Letters, Vol. 7, No. 18, 1971.

[3] W. M. Wonham, "Some Applications of Stochastic Differential Equations to Optimal Nonlinear Filtering", SIAM J. Control, Vol.2 pp.347-369, 1965.

[4] L. E. Zachrisson, "On Optimal Smoothing of Continuous-Time Kalman Processes," Information Sciences, Vol. 1, pp. 143-172, 1969.

[5] J. B. Moore and B.D.O. Anderson, "Fixed-Point Smoothing as a Filtering Problem", Proceedings 2nd Symposium on Nonlinear Estimation Theory and its Applications, San Diego, 1971.

[6] T. Kailath and P. Frost, "An Innovations Approach to Least-Squares Estimation - Part II: Linear Smoothing in Additive White Noise", IEEE Trans. Auto. Control, Vol. AC-13, pp. 655-660, 1968.

[7] H. J. Kushner, "On the Differential Equations Satisfied by Conditional Probability Densities of Markov Processes, with Applications", SIAM J. Control, Vol. 2, pp. 106-119, 1964.

[8] J. B. Moore, "Discrete-Time Fixed-Lag Smoothing Algorithms", Automatica, Vol. 9, No. 2, pp. 163-174, 1973.

[9] A. H. Jazwinski, "Stochastic Processes and Filtering Theory", Academic Press, Inc., New York, 1970.