STABLE-REALIZATION OF FIXED-LAG SMOOTHING EQUATIONS FOR CONTINUOUS-TIME SIGNALS*

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ABSTRACT

Novel smoother structures are introduced for the optimal fixed-lag smoothing of continuous-time signals in noise.

The smoothers have the very desirable property that they are simultaneously realizable, stable in the sense of Lyapunov and optimal. This is in contrast to those proposed to date which are either optimal and realizable but unstable, realizable and stable but suboptimal, or optimal and stable but unrealizable.

I. INTRODUCTION

An important, hitherto unsolved problem in optimal filtering theory is the realization of a stable (in the sense of Lyapunov) optimal (minimum mean square error) fixed-lag smoother for continuous-time signals in noise. Of interest in this paper is the simplest (nontrivial) such problem which is of course the case when the signal model is a finite-dimensional linear system driven by white Gaussian noise, the output of which is contaminated by additive independent white Gaussian noise.

Certainly, (bounded-input, bounded-output) optimal fixed-lag smoothing equations are available [1,2]. These express the optimal fixed-lag smoothed signal estimate as the sum of the filtered estimate, delayed by the amount of the fixed-lag, and a "correction" term. (This "correction" term is a convolution integral over the period of the fixed-lag involving either the innovations process or the states of the optimal filter.) However, the various fixed-lag smoothers (as opposed to smoothing equations) so far considered in the literature are either realizable (apart from pure time-delays) but unstable [3] or stable but unrealizable [4,5]. In fact, the only stable and realizable fixed-lag smoothers so far described are not optimal. These suboptimal smoothers are derived as causal approximations to the optimal non-causal filters (with infinite delay) arising from Weiner filtering theory [6] or as stable finite dimensional approximations to the optimal smoothers of [1,2] arising from Kalman filtering theory [4,5]. Actually there is another group of suboptimal smoothers which involves discretizing (in time) the received signal applying optimal discrete-time fixed-lag smoothing results [7,8] and reconstructing a continuous-time estimate from the discrete-time estimate [4].

Perhaps it is worth recalling from [7] that in discrete-time the optimal fixed-lag smoothing problem can be solved by imbedding it within a certain filtering problem. The filtering signal model is first augmented with delay elements which

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serve to delay the states an integer number of discrete-time instants so that the total delay is equal to the fixed-lag. It is not hard to see that the optimal filtered estimates of these delayed states are in fact the optimal fixed-lag smoothed estimates of the original signal model states.

Extension of the ideas developed for discrete-time fixed-lag smoothing to the continuous-time case, results in an infinite dimensional augmented signal model since the augmentation is a pure time delay. However, in order to apply Kalman filtering theory, finite dimensional approximations to the time delay is required. This of course results in a *suboptimal fixed-lag smoother* which approaches the desired optimal one only as its dimension approaches infinitly [4,5]. It might be added that the computations involved are extensive since not only must finite-dimensional approximations to time delays be made but high order Riccati equations need to be solved as well.

In this paper, novel fixed-lag smoother structures are introduced for the optimal fixed-lag smoothing of continuous-time signals in noise. The smoothers are simultaneously realizable (apart from pure time delays), stable* in the sense of Lyapunov, and optimal. The smoothers involve, in *loose terms two optimal "unstable" smoothers* (as in [3]) back to back, with states of each periodically set to zero to ensure stability and the output switched alternatively from one to the other. The cost of achieving the three desirable properties of realizability, stability and optimality is thus an increase in complexity on that of the unstable smoother and the introduction of the possible unpleasant feature of discontinuous signals within the smoother.

The approach taken in this paper for one of the smoother realizations can be summarized quite precisely in qualitative terms as follows.

* but not asymptotically stable

The two significant facts concerning existing unstable realizations of the optimal fixed-lag smoothing equations relevant here are first, that the realization contains an unstable finite dimensional subsystem, and second, that if the states of this subsystem are set to zero at some arbitrary time $t = t_1$ - which is certainly not normal practice - there is no error introduced subsequent to time $t \geq t_1 + \Delta$, where Δ is the fixed lag.

Consider now that the states of the unstable subsystem are reset to zero at times $2k\Delta$ for $k = 0,1, \ldots$. The smoother output is now the required fixed-lag estimate during the intervals $[\Delta, 2\Delta)$, $[3\Delta, 4\Delta)$, ... It is not difficult to see that an appropriate duplication of smoother components, with the duplicate unstable subsystem states set to zero at $(2k + 1)\Delta$, for $k = 0,1,\ldots$ allows production of a smoothed estimate in remaining intervals $[2\Delta, 3\Delta)$, $[4\Delta, 5\Delta)$, ... The two estimates together provide a fixed-lag smoothed estimate for all time. Moreover, the smoother is clearly stable since the unstable subsystems states are reset to zero at regular intervals.

The following is an outline of the subsequent sections of this paper. In section II, we review the fixed-lag smoothing equations. In section III, one stable realization of the optimal fixed-lag smoothing equations is derived. The one chosen follows on naturally from the familiar unstable realizations. In section IV, a number of alternative stable realizations of the optimal fixed-lag smoother equations are derived, each with its own advantages and disadvantages. In section V we consider suboptimal smoothers resulted from the delay being approximated by a finite dimensional network. Section VI contains the conclusions.

II. REVIEW OF OPTIMAL FIXED-LAG SMOOTHING EQUATIONS

We have in mind the problem of estimating the state at time t of a lumped linear system of the form

$$\dot{x} = F(t)x + G(t)\,u \tag{1}$$

where u is Gaussian white noise of mean zero and covariance $Q(t)\delta(t-\tau)$ and the initial state of (1), namely $x(t_0)$ is a Gaussian random variable, independent of u, zero mean, and having a covariance P_0. The matrices $Q(t)$ and P_0 are symmetric and nonnegative definite.

The received measurements are

$$z = H(t)x + v(t) \tag{2}$$

where v is Gaussian white noise, independent of u and $x(t_0)$, of mean zero and having a covariance $R(t)\delta(t-\tau)$ where $R(t)$ is symmetric positive definite

Given the measurements z over $[t_0,t]$ the minimum variance linear unbiased estimate $\hat{x}(t|t) = E[x(t)|z(\tau),\ t_0 \le \tau \le t]$ is given [7] by the following equations

$$\dot{\hat{x}}(t|t) = F(t)\hat{x}(t|t) + K_f(t)\nu(t) \tag{3}$$

$$K_f(t) = P(t|t)H(t)R^{-1}(t) \tag{4}$$

$$\dot{P}(t|t) = P(t|t)F'(t) + F(t)P(t|t) - P(t|t)H'(t)R^{-1}(t) \times H(t)P(t|t) + G(t)Q(t)G'(t) \tag{5}$$

$$P(t_0|t_0) = P_0$$

where

$$\nu(t) = z(t) - H(t)\hat{x}(t|t) \tag{6}$$

is known as the innovations process, and

$$P(t|t) = E\{[x(t) - \hat{x}(t|t)]\,[x(t) - \hat{x}(t|t)]'\} \tag{7}$$

is the covariance of the error in state estimation. e.g. (3) can be rewritten as

$$\dot{\hat{x}}(t|t) = F_f(t)\hat{x}(t|t) + K_f(t)z(t) \tag{8}$$

where

$$F_f(t) = F(t) - K_f(t)H'(t) \tag{9}$$

The fixed-lag smoothed estimate $\hat{x}(t|t+\Delta) = E[x(t)|z(\tau),\ t_0 \le \tau \le t+\Delta]$ can be expressed as [2]

$$\hat{x}(t-\Delta|t) = \hat{x}(t-\Delta|t-\Delta) + K(t)\int_{t-\Delta}^{t} \phi'(\sigma,t)\bar{H}'(\sigma)\nu(\sigma)d\sigma \tag{10}$$

where

$$K(t) = P(t-\Delta|t-\Delta)\phi'(t,t-\Delta) \tag{11}$$

$$\bar{H}'(\sigma) = H'(\sigma)R^{-1}(\sigma) \tag{12}$$

and $\phi(\sigma,t)$ is the transition matrix associated with the filtering equation (8).

III. REALIZATIONS OF THE FIXED-LAG SMOOTHING EQUATIONS

We first recall the usual, albeit unstable, realization of the fixed-lag smoothing equation. This is obtained from an expanded version of (10) as follows

$$\hat{x}(t-\Delta|t) = \hat{x}(t-\Delta|t-\Delta) + K(t)\int_{t_0}^{t} \phi'(\sigma,t)\bar{H}'(\sigma)\nu(\sigma)d\sigma - P(t-\Delta|t-\Delta)\int_{t_0}^{t-\Delta} \phi'(\sigma,t-\Delta)\bar{H}'(\sigma)\nu(\sigma)d\sigma \tag{13}$$

The smoothed estimate $\hat{x}(t-\Delta|t)$ can now be obtained from the output of the time delay dynamical system having state equations derived from (13) as

$$\dot{I}(t) = -F_f'(t)I(t) + \bar{H}'(t)\nu(t), \quad I(t_0-\Delta) = 0 \tag{14}$$

$$\xi(t) = K(t)I(t) - P(t-\Delta|t-\Delta)I(t-\Delta) + \hat{x}(t-\Delta|t-\Delta) \tag{15}$$

where $\hat{x}(t-\Delta|t) = \xi(t)$ for $t \ge t_0$. Notice that the state $I(t)$ is simply the integral term

$$I(t) = \int_{t_0}^{t} \phi'(\sigma,t)\bar{H}'(\sigma)\nu(\sigma) \tag{16}$$

from (13). This dynamical system, known as a fixed-lag smoother, see also Fig. 1, is one realization of the fixed-lag smoothing equations. Observe that this realization is in fact unstable in the sense of Lyapunov when the optimal filter is stable in the sense of Lyapunov (the usual case) even though the smoothing equation (13) represents a bounded-input bounded-output mapping.

Of course, there are realizations of (10) organized a little differently from the one shown in Fig. 1, but all conventional realizations of (10) result in unstable systems [3]. The realization as shown in Fig. 1 is chosen here because firstly it shows vividly the source of instability and secondly it can be used as a starting point for the design of a stable smoother, as we now go on to show.

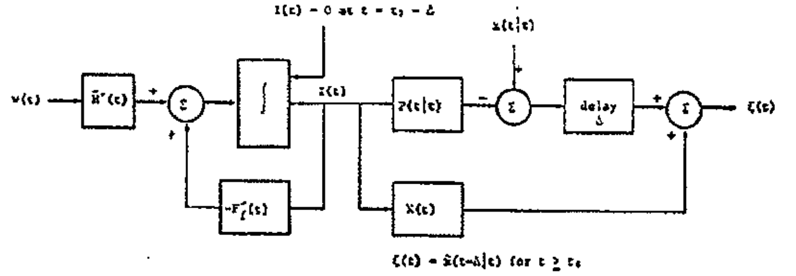


Figure 1. Unstable fixed-lag smoother.

In deriving a stable realization of the smoothing equations (10), the first step is to expand them as follows

$$\hat{x}(t-\Delta|t) = \hat{x}(t-\Delta|t-\Delta) + K(t)\int_{\alpha(t)}^{t} \phi'(\sigma,t)\bar{H}'(\sigma)\nu(\sigma)d\sigma$$
$$- P(t-\Delta|t-\Delta)\int_{\alpha(t)}^{t-\Delta} \phi'(\sigma,t-\Delta)\bar{H}'(\sigma)\nu(\sigma)d\sigma \qquad (17)$$

where $\alpha(t)$ is a discontinuous (staircase) function of t defined from

$$\alpha(t) = t_0 + 2(k-1)\Delta;\ t_0 + (2k)\Delta \leq t < t_0 + (2k+1)\Delta$$
$$= t_0 + 2k\Delta;\ t_0 + (2k+1)\Delta \leq t < t_0 + (2k+2)\Delta \qquad (18)$$

for $k = 0, 1, 2 \ldots$ (Note that $(t-\Delta) \geq \alpha(t)$ for all t). These equations appear complicated because of the lower limit of the integrations depending in a discontinuous manner on the value of t. However, it is not too difficult to see that an appropriate duplication of the smoother of Fig. 1 with appropriate switching and resetting to zero of the integrators, as described in the introduction of this paper and indicated in Fig. 2a, is in fact a realization of these equations. Observe that the periodic resetting to zero of the integrators ensures Lyapunov stability of the smoother, (but not asymptotic stability of the smoother). Actually not all the duplications indicated in Fig. 2a are necessary and an obvious simplification yields the realization of Fig. 2b.

So far we have indicated one novel fixed-lag smoothing structure which is simultaneously realizable (apart from the pure time delays), stable and optimal. Before proceeding with other realizations, an alternative approach to the derivation of a smoother very like that of Fig. 2b is now discussed.

In this alternative approach the smoothing equation (10) is expanded as

$$\hat{x}(t-\Delta|t) = \hat{x}(t-\Delta|t-\Delta) + K(t)\int_{k\Delta}^{t} \phi'(\sigma,t)\bar{H}'(\sigma)\nu(\sigma)d\sigma$$
$$+ K(t)\int_{(k-1)\Delta}^{k\Delta} \phi'(\sigma,t)\bar{H}'(\sigma)\nu(\sigma)d\sigma$$
$$- P(t-\Delta|t-\Delta)\int_{(k-1)\Delta}^{t-\Delta} \phi'(\sigma;t-\Delta)\bar{H}'(\sigma)\nu(\sigma)d\sigma \qquad (19)$$

where $k\Delta \leq t < (k+1)\Delta$. A dynamical system for the realization of (10) is indicated in Fig. 3. Observe that it differs only in switching arrangements to that of the previous figure.

IV. ALTERNATIVE REALIZATIONS

The system in Fig. 2b requires n scalar delay elements where n is the dimension of the state

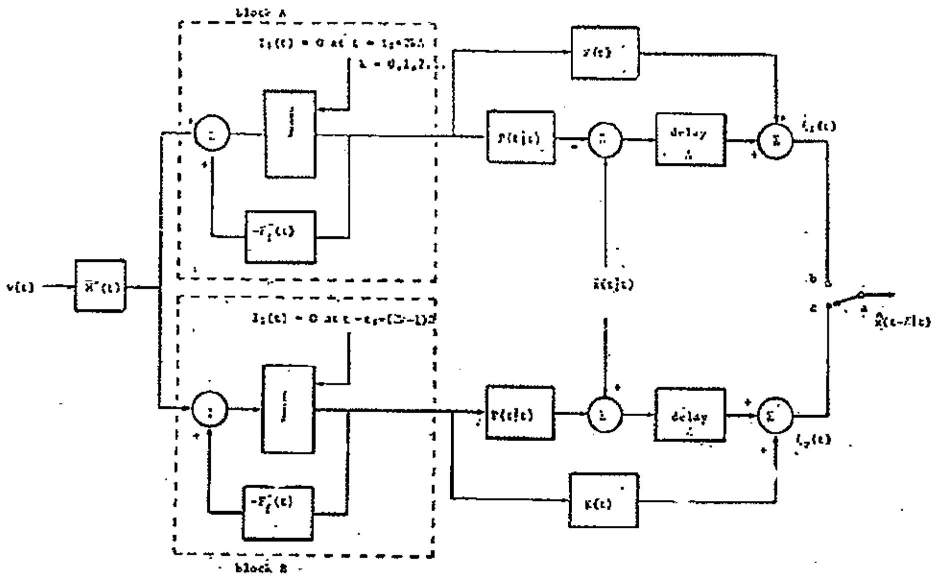


Figure 3a.

$\hat{\xi}_1(t)$ gives $\hat{x}(t;t+\Delta)$ during $[t_0 + 2k\Delta, t_0 + (2k+1)\Delta]$

$\hat{\xi}_2(t)$ " " " $[t_0 + (2k+1)\Delta, t_0 + (2k+2)\Delta]$

a b closed at $t = t_0 + (2k-1)\Delta$

a c " " $t = t_0 + 2k\Delta$

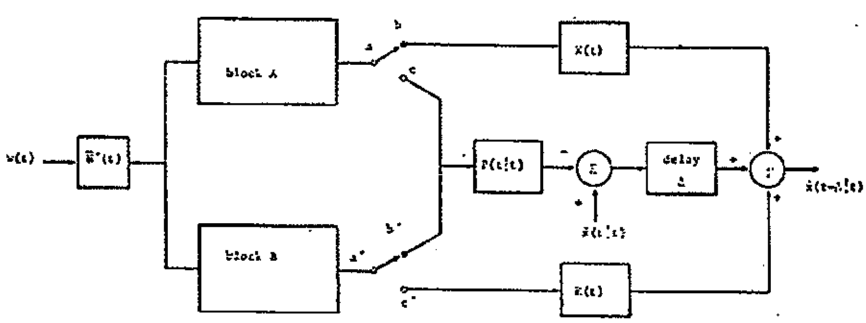


Figure 3b.

ab, a'b' closed at $t = t_0 + (2k-1)\Delta$

ac, a'c' " " $t = t_0 + 2k\Delta$

$x(t)$. These delay elements are required to transmit discontinuous signals due to the periodic resettings of the integrators and the periodic switchovers from one set of integrators to the other.

In this section, we present alternative fixed-lag smoother realizations which reduce the number of delay elements and/or avoid the need to delay discontinuous signals. Also, what may be termed a non-minimal realization is presented which reduces computational errors at the cost of more frequent switching and additional dynamics.

Smoothers Driven by the Filtered State Estimates

It is possible to avoid the use of the innovation process as a driving term in the realization of a fixed-lag smoother as follows, see also [9]. In view of (3) and (12) the fixed-lag smoothing equations (10) may be rewritten as

$$\hat{x}(t-\Delta|t) = \hat{x}(t-\Delta|t-\Delta) + K(t)\int_{t-\Delta}^{t}\phi'(\sigma,t)P^{-1}(\sigma|\sigma) \times[\dot{\hat{x}}(\sigma|\sigma)-F(\sigma)\hat{x}(\sigma|\sigma)]d\sigma$$

An integration by parts allows the following re-arrangement

$$\hat{x}(t-\Delta|t) = K_0(t)\hat{x}(t|t) + K(t)\int_{t-\Delta}^{t}\phi'(\sigma,t)K_1(\sigma)\hat{x}(\sigma|\sigma)d\sigma \qquad (20)$$

where

$$K_0(t) = K(t)P^{-1}(t|t)$$

$$K_1(\sigma) = P^{-1}(\sigma|\sigma)G(\sigma)Q(\sigma)G'(\sigma)P^{-1}(\sigma|\sigma)$$

(The existence of $P^{-1}(\sigma|\sigma)$ is guaranteed in the case when $[F,\bar{G}]$ is completely controllable for any $\bar{G}$ such that $\bar{G}'\bar{G} = Q$.)

The ideas of the previous sections suggest the following expansion for (20)

$$\hat{x}(t-\Delta|t) = K_0(t)\hat{x}(t|t) + K(t)\int_{\alpha(t)}^{t}\phi'(\sigma,t)K_1(\sigma)\hat{x}(\sigma|\sigma)d\sigma - P(t-\Delta|t-\Delta)\int_{\alpha(t)}^{t-\Delta}\phi'(\sigma,t-\Delta)K_1(\sigma)\hat{x}(\sigma|\sigma)d\sigma \qquad (21)$$

where $\alpha(t)$ is the discontinuous quantity defined in (18). The realization of (21), indicated in Fig. 4, is thus an alternative one to those indicated previously. It has the advantage that it is driven by the states of the filter $\hat{x}(t|t)$ rather than this together with the innovation process $\nu(t)$.

The disadvantage of all the smoothers mentioned so far is that the delay elements are required to pass discontinuous (continuous-time) signals. With a view to avoiding this requirement a change of variable $(\tau = \sigma + \Delta)$ in the last term of (21) allows the alternative expansion of (20)

$$\hat{x}(t-\Delta|t) = K_0(t)\hat{x}(t|t) + K(t)\int_{\alpha(t)}^{t}\phi'(\sigma,t)K_1(\sigma)\hat{x}(\sigma|\sigma)d\sigma - K(t)\int_{\alpha(t)+\Delta}^{t}\phi'(\tau,t)\phi'(\tau-\Delta,\tau)K_1(\tau-\Delta)\hat{x}(\tau-\Delta|\tau-\Delta)d\tau \qquad (22)$$

A convenient realization of (22) is given in Fig.5 where $K_1(\sigma)$ is expanded as $K_1 = K_{1a}K_{1b}$ with

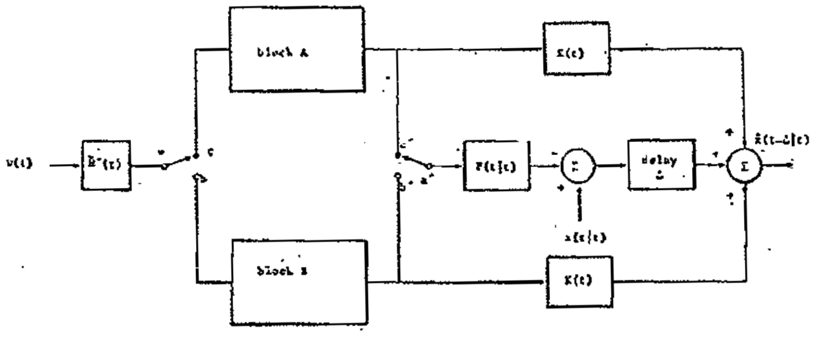
$$K_{1a}(\sigma) = P^{-1}(\sigma|\sigma)G(\sigma) \ ; \ K_{1b}(\sigma) = Q(\sigma)G'(\sigma)P^{-1}(\sigma|\sigma) \qquad (23)$$

Note that $K_{1b}(t)\hat{x}(t-\Delta|t-\Delta)$ is of the same dimension as the input u of (1), which is usually of lower dimension than the state x. Thus, if m is the dimension of the input u, we need only m scalar delay elements. In particular, for single-input systems, only one scalar delay element is required.

Thus the system of Fig. 5 has the important advantages over the other systems in Figs. 2-4 in that we can achieve a reduced order realization (in the sense of requiring less number of delay elements) and less stringent requirement on the delay elements at the same time. On this latter point, notice that if $x(t)$ and thus $\hat{x}(t|t)$ is a nominally band-linked signal over a pass band B the delay elements can be designed to be networks which are phase linear with a flat response over the pass band B.

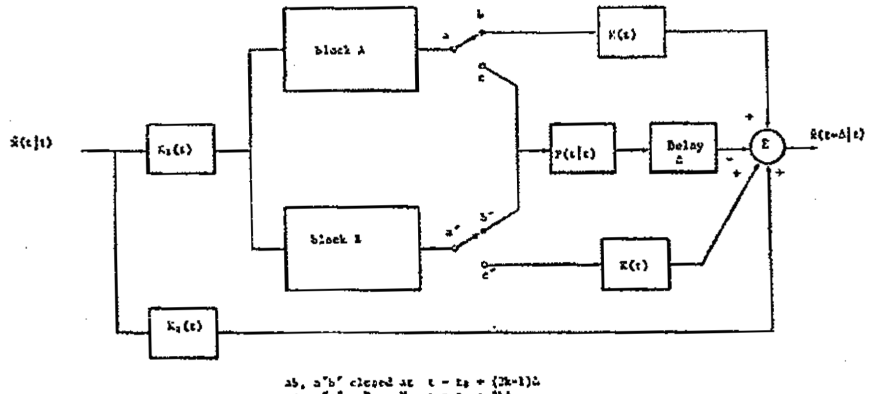
Signal Smoothing

In many applications, smoothed estimates of the quantity $y(t) = H(t)x(t)$, rather than the full state $x(t)$ are sought. In such cases, it is



ab, a″b″ closed at $t = t_0 + (2k-1)\Delta$
ac, a″c″ " " $t = t_0 + 2k\Delta$

Figure 3. Alternative fixed-lag smoother.



ab, a″b″ closed at $t = t_0 + (2k-1)\Delta$
ac, a″c″ " " $t = t_0 + 2k\Delta$

Figure 4. Fixed-lag smoother driven by filter state.

possible to construct signal smoothers having p delay elements, where p is the dimension of the signal $y(t)$. Two approaches are described.

1) Premultiply both sides of (17) by $H(t)$. Thus, the system in Fig. 2b is modified to that of Fig. 6, which requires p delay elements. It is obvious that Figs. 3,4 also have their counterparts in which the signal smoothers need only p delay elements.

2) A change of variable $(\tau = \sigma + \Delta)$ in the last term of (17) gives

$$\hat{x}(t-\Delta|t) = \hat{x}(t-\Delta|t-\Delta) + K(t)\int_{\alpha(t)}^{t} \phi'(\sigma,t)H^{-1}(\sigma)\nu(\sigma)d\sigma + K(t)\int_{\alpha(t)+\Delta}^{t} \phi'(\tau,t)\phi'(\tau-\Delta,\tau)H^{-1}(\tau-\Delta)\nu(\tau-\Delta)d\tau \quad (24)$$

Then in the same way that the system of Fig.5 can be obtained using (22), it is obvious that the integrators can be driven by a suitable combination of the innovations $\nu(t)$ and its delayed version $\nu(t-\Delta)$, which is of dimension p. Note that $\hat{x}(t-\Delta|t-\Delta)$ can be obtained using a delayed version of Kalman filter driven by $\nu(t-\Delta)$.

Figs. 2b, 3 or 4 require n delay elements while in Fig. 5 m delay elements are required. It seems that the above approaches will be useful in cases where p is a number smaller than either n or m. (Consider the case of a multiple input-single output system, in which $p = 1$). However, the above approaches suffer from the following disadvantages. In 1) the delay elements are required to pass discontinuous signals. In 2) the delay elements are required to transmit the innovations $\nu(t)$, which is white noise [2]. Both cases require very good delay elements. Provided these problems can be overcome, the above approaches may be attractive in situations where p is smaller than n or m.

<u>Smoothers With Reduced Computational Error</u>

Suppose the system of Fig.1 is so unstable that to run it between resetting for a time interval greater than $1\frac{1}{2}\Delta$ (say) would result in too much error build up. For this case consider the following arrangement:

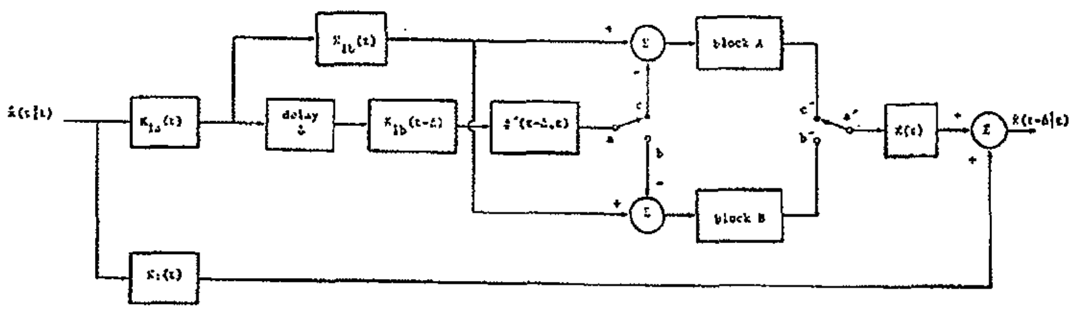
Let each of the systems A,B,C be similar in arrangement to that of Fig. 1, except the following: System A has resetting times, $t = -\Delta$, $\frac{1}{2}\Delta$, 2Δ, ... etc. and gives $\hat{x}(t|t + \Delta)$ during $[0, \frac{1}{2}\Delta)$, $[1\frac{1}{2}\Delta, 2\Delta)$; System B has resetting times, $t = -\frac{1}{2}\Delta$, Δ, $2\frac{1}{2}\Delta$, ... etc. and gives $\hat{x}(t|t + \Delta)$ during $[\frac{1}{2}\Delta, \Delta)$, $[2\Delta, 2\frac{1}{2}\Delta)$; System C has resetting times, $t = 0$, $1\frac{1}{2}$, 3Δ, ... etc. and gives $\hat{x}(t|t + \Delta)$ during $[\Delta, 1\frac{1}{2}\Delta)$, $[2\frac{1}{2}\Delta, 3\Delta)$. Thus systems A,B,C combined will give $\hat{x}(t|t + \Delta)$ for all t.

Techniques which reduce Fig. 2a to 2b, 3, 4, 5, and 6 can also be applied to our combined systems A,B,C, with the resultant reduction in half the required amount of delay elements. Also, it is obvious that the technique of (a) dealing with transient effects and the above technique can be combined in a practical design situation to achieve optimal results.

V. SUBOPTIMAL SMOOTHERS

When the delay is approximated by a finite dimensional network, the optimal smoothers discussed in the last two sections become sub-optimal smoothers. Analysis of all the suboptimal smoothers resulting from the optimal smoothers would be too tedious. Rather, we shall restrict our discussions in this section to Fig. 5 since it is readily amendable to analysis. Moreover, for this case considerable simplification in the realization of the delay is frequently possible. An important point to note is that any error produced in the delay is amplified by the instability of the subsystems over a period 2Δ.

Without going into details, we outline the analysis as follows.



ab, a'b' closed at $t = t_0 + (2k-1)\Delta$
ac, a'c' " " $t = t_0 + 2k\Delta$

Figure 5. Fixed-lag smoother using a minimal number of delays.

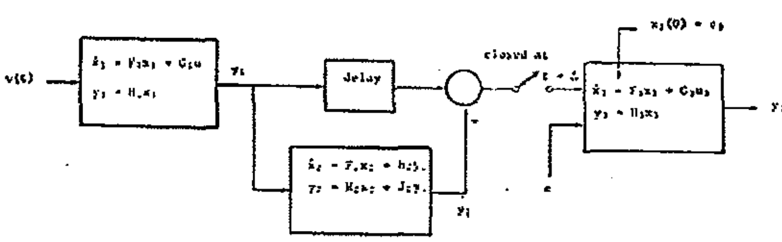


Figure 6.

Consider the system shown in Fig.6. The innovation process $\nu(\cdot)$ has covariance $E[\nu(t)\nu'(\tau)] = R(t)\delta(t-\tau)$. The variable $y_1(t)$ corresponds to $K_1(t)\hat{x}(t|t)$ of Fig.5. $y_2(t)$ is the output of the approximate delay. $\varepsilon(t)$ is the difference between the output of the non-ideal delay and the ideal delay. Imperfections appearing at the resettings of the integrators of the unstable subsystem are modelled as a random variable e_0 with covariance Σ_0. Other imperfections at the input of the unstable subsystem are modelled as white noise $e(\cdot)$ with covariance $E[e(t)e'(\tau)] = \Sigma\,\delta(t-\tau)$. We consider the error at $t = 2\Delta$ subsequent to a resetting of the unstable integrators at $t = 0$. For simplicity in presentation, the systems are taken as time-invariant (i.e. F_i, G_i etc. are constants).

Let $D = E[x_3(2\Delta)x_3'(2\Delta)]$. Then $D = D_1 + D_2 + D_3$ where D_1, D_2 and D_3 are the contribution to D due to e_0, e and ε respectively acting separately

$$D_1 = \exp(2\Delta F_3)\,\Sigma_0\,\exp(2\Delta F_3') \tag{25}$$

$$D_2 = \int_0^{2\Delta} \exp[(2\Delta-s)F_3]\,G_3\,\Sigma\,G_3'\,\exp[(2\Delta-s)F_3']ds \tag{26}$$

$$D_3 = \int_\Delta^{2\Delta}\int_\Delta^{2\Delta} \exp[(2\Delta-t)F_3]G_3[A+C-B_1-B_2'] \times G_3'\exp[(2\Delta-\tau)F_3']dt\,d\tau \tag{27}$$

where

$$A = [J_2H_1 \quad H_2]\,P_a(t-\tau)\begin{bmatrix} H_1'J_2' \\ H_2' \end{bmatrix} \tag{28}$$

$$B = [H_1 \quad 0]\,P_a(t-\Delta-\tau)\begin{bmatrix} H_1'J_2' \\ H_1' \end{bmatrix} \tag{29}$$

$$C = [H_1 \quad 0]\,P_a(t-\tau)\begin{bmatrix} H_1' \\ 0 \end{bmatrix} \tag{30}$$

$$\begin{aligned} P_a(t-\tau) &= \exp[(t-\tau)F_a]P_a \quad \text{for} \quad t-\tau \geq 0 \\ &= P_a\exp[(\tau-t)F_a'] \quad \text{for} \quad t-\tau \leq 0 \end{aligned} \tag{31}$$

$$F_a = \begin{bmatrix} F_1 & 0 \\ G_2H_1 & F_2 \end{bmatrix} \tag{32}$$

and P_a satisfies

$$F_aP_a + P_aF_a' + \begin{bmatrix} G_1RG_1' & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{33}$$

The improvement of smoothing over filtering is [10]

$$PMP = P\int_0^\Delta \exp[\sigma F_1']H_1'R^{-1}H_1\exp[\sigma F_1]d\sigma P \tag{34}$$

$$\text{where} \quad P = E[x_1(t)x_1'(t)] \tag{35}$$

In designing the suboptimal smoother, we could pose the problem of choosing F_2, G_2, H_2, J_2 with $\mathrm{Re}\lambda_i(F_2) < 0$ to minimize $tr[D]$, or the more involved problem of selecting F_2, G_2, H_2, J_2 and Δ to minimize $tr[D-PMP]$.

As illustration, we consider some simple examples

(1) Ideal delay. Single-input, single-output systems. Choose Δ^* to minimize $V = H_3[D_1 + D_2 - PMP]H_3'$. Assume the filter has dominant time constant $(^1/\lambda\text{min})$ and least dominant time constant $(^1/\lambda\text{max})$. Equation (13) shows that for the unstable system of the smoother, $(^1/\lambda\text{max})$ becomes the dominant time constant. We have

$$V = e^{4\Delta\lambda max}H_3\,\Sigma_0\,H_3' + \int_0^{2\Delta} e^{-2\lambda max(2\Delta-s)}ds\,H_3G_3\,\Sigma\,G_3'H_3' - \int_0^\Delta e^{-2\lambda min\,s}ds\,H_3PH_1'R^{-1}H_1PH_3' \tag{36}$$

which is minimized for

$$\Delta^* = \frac{1}{4\lambda max + 2\lambda min}\ln\frac{H_3PH_1'R^{-1}H_1PH_3'}{H_3(4\lambda max\,\Sigma_0 + 2G_3\Sigma G_3')H_3'} \tag{37}$$

Observe that $\Delta^* \to \infty$ as $\Sigma_0, \Sigma \to 0$. In practice a value of fixed-lag Δ would not be chosen beyond say two or three times the dominant time constant $(^1/\lambda\text{min})$ of the filter since the added improvement due to smoothing obtained by taking Δ greater than this is negligible.

(2) Fixed Δ, F_2, G_2. First-order systems (i.e. F_1, F_3, G_1, G_3, H_i are scalars). Choose

J_2, H_2 to minimize D_3

Setting $\frac{\partial D_3}{\partial J_2}$ and $\frac{\partial D_3}{\partial H_2}$ to zero, we have the optimum values J_2^*, H_2^* given as the solution of the following equation:

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} J_2^* \\ H_2^* \end{bmatrix} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} \begin{bmatrix} h_1 \\ 0 \end{bmatrix} \qquad (38)$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \int_{\Delta}^{2\Delta}\int_{\Delta}^{2\Delta} \left[\frac{P_a(t-\tau)+P_a'(t-\tau)}{2}\right] e^{-(t+\tau)F_3} dt d\tau \qquad (39)$$

$$\begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} = \int_{\Delta}^{2\Delta}\int_{\Delta}^{2\Delta} P_a'(t-\Delta-\tau) e^{-(t+\tau)F_3} dt d\tau \qquad (40)$$

P_{11}, $\bar{P}_{11}$ are scalars.

For example, with $F_3=-F_1=-F_2=G_1=G_3=H_1=P=1$ and $\Sigma=\Sigma_0=0$, relevant results are given in Table 1 below. Thus, for this suboptimal smoother with the delay approximated by a first order subsystem with fixed pole (which is indeed a very crude approximation), we can have improvement for $\Delta < 1.5$. The optimum Δ in this case is approximately 0.5 and the associated improvement due to smoothing with a first order approximation to the delay is 26.6%.

VI CONCLUSIONS

Of the different realizations shown in Figs.2-5, the most practical one is that of Fig. 5. A minimal number of non-ideal delay elements can be used for such a realization.

Further research is needed to compare the performance of our suboptimal smoothers discussed in section V with other suboptimal smoothers such as those discussed in [4], [5].

REFERENCES

[1] H. Kwakernaak, "Optimal Filtering in Linear Systems with Time Delays," IEEE, Vol.AC-12, April, pp. 169-173.

[2] T. Kailath and P. Frost, "An Innovations Approach to Least-Squares Estimation: Part II - Linear Smoothing in Additive White Noise," IEEE Trans. on Automatic Control, Vol. AC-13, No.6, December 1968.pp.646-655.

[3] C.N. Kelly and B.D.O. Anderson, "On the Stability of Fixed-Lag Smoothing Algorithms," Journal of the Franklin Institute, Vol.291, No. 4, April 1971, pp.271-281.

[4] S. Chirarattananon and B.D.O. Anderson, "Outline Designs for Stable Continuous-Time Fixed-Lag Smoother," Electronics Letter, 18th May 1972.Vol.8, No.10, pp.263-264.

[5] S. Chirarattananon, "Optimal and Suboptimal Fixed-Lag Smoothing," Ph.D Thesis, Department of Electrical Engineering, University of Newcastle (in preparation).

[6] N. Wiener, "Extrapolation, Interpolation and Smoothing of Stationary Time Series," John Wiley & Sons, New York, 1949.

[7] J.B. Moore and B.D.O. Anderson, "Discrete-Time Fixed-Lag Smoothing Algorithms, IFAC-Automatica, March 1972.

[8] J.B.Moore and P.Tam, "Fixed-Lag Smoothing of Nonlinear Systems with Discrete Measurements," Information Sciences, to appear. Also proceedings of Fifth Hawaii International Conference on System Sciences, Jan. 1972.

[9] B.D.O.Anderson and S. Chirarattananon, "New Linear Smoothing Formulas",IEEE Trans.on Auto.Control,Vol.AC-17 No.1,Feb.1972,p.160-161.

Δ	J_2^*	H_2^*	%Improvement due to Smoothing with ideal delay.	%Improvement due to Smoothing with first order approx. to delay
.1	0.176	-0.669	9%	8.4%
0.5	0.091	0.941	31.6%	26.6%
1.0	-0.676	1.664	43.2%	25.2%
1.5	-0.792	1.792	47.5%	-2.4%

TABLE 1 SUBOPTIMAL SMOOTHER PERFORMANCE