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LINEAR MULTIVARIABLE CONTROL SYSTEMS – A SURVEY

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INTRODUCTION

Besides the restrictions inherent in the paper title, a further restriction is made to systems which are time-invariant, deterministic, and operate in continuous time rather than on a sampled-data basis. Space restrictions preclude any development of the ideas; accordingly, we aim more to delineate some, but obviously not all, of the areas in which results are available, offering a few details of these results. First though, we offer one or two general remarks, attempting to secure a perspective.

The claim is often heard that classical control is grounded in imprecise statements of objectives and in empirical, and therefore limited, techniques for attaining these objectives; simultaneously, it is argued that modern control, by concentrating on precise statements of objectives, and using mathematical techniques – not necessarily intuition-limited – for attaining these objectives, somehow overcomes the difficulties of classical control. The area of multivariable systems defines a laboratory in which this thesis can be tested; a preliminary look suggests it is not completely valid. For example, modern control has not really provided design procedures that exhibit tradeoffs between accuracy, transient behaviour as measured by rise time and overshoot, interaction between controls, and controller complexity. There tend only to be well-defined procedures in modern control which achieve a subset of these properties at the one time, e.g. procedures aimed at decoupling systems and achieving certain closed-loop poles; again, the heavy reliance in modern control on the concept of the state often somehow makes it correspondingly difficult to change the complexity of controllers during a design procedure. Accordingly, there is still a situation where a major gap exists between classical and modern control capabilities.

Because of the existence of this gap, the paper breaks naturally into two main parts, the first dealing with principally frequency-domain ideas, the second with principally state-space ideas.

FREQUENCY-DOMAIN IDEAS

Feedback and Series Compensated Systems

Let us consider the arrangement shown in Figure 1. Here, $W(s)$ is an $m \times r$ matrix of plant transfer functions (often $m=r$), $K(s)$ an $r \times m$ matrix of controller transfer functions, and $H(s)$ an $m \times m$ matrix of feedback-transducer transfer functions (often, $H(s) = I$). A return difference matrix can be defined as $F = I + WKH$, with the closed-loop transfer function matrix being $R(s) = F^{-1}(s)W(s)K(s)$. One can also derive the useful result, with $Q = WK$,

$$R^{-1} = Q^{-1} + H \tag{1}$$

The return difference matrix governs the relation between open-loop and closed-loop sensitivity, as, see e.g. (1)

$$R^{-1}(s)\delta R(s) = F^{-1}(s)Q^{-1}(s)\delta Q(s) \tag{2}$$

Stability

The stability of the open-loop version of Figure 1 is governed by the zeros of the open-loop characteristic polynomial, which is the product of the characteristic polynomials of $W(s)$, $K(s)$ and $H(s)$; the characteristic polynomial of, say, $W(s)$, is defined in a standard way, (2), as, e.g., the least common denominator of all minors of $W(s)$.

The closed-loop system of Figure 1 has a closed-loop characteristic polynomial given by

$$\text{closed-loop characteristic polynomial} = [\det F(s)] \times \text{open-loop characteristic polynomial} \tag{3}$$

(See e.g. (2), (3)). That the zeros alone of $\det F(s)$ do not determine stability is important – in fact, $\det F(s)$ can even be constant – and that the poles of $\det F(s)$ cancel zeros of the open-loop characteristic polynomial is nontrivial.

From this result, it can be argued, see e.g. (4) that the closed-loop system is stable if the Nyquist plot of $\det F(s)$ encircles the origin a number of times in a counterclockwise direction equal to the number of right-half-plane zeros of the open-loop characteristic polynomial. If $\rho_i(s)$ denotes an eigenvalue of $F(s)$ evaluated at the point s, and $\nu_j(s)$ an eigenvalue of the loop gain $W(s)K(s)H(s)$, then stability also follows if the net number of counterclockwise encirclements of the origin [the point $-1 + j0$] by the Nyquist plots of the eigenvalues $\rho_i(s)$ [$v_i(s)$] equals the number of right-half-plane zeros of the open-loop characteristic polynomial. (The significance of the eigenvalues of $F(s)$ will become clearer subsequently).

In case one or more of $W(s)$, $K(s)$ and $H(s)$ becomes irrational, the stability problem becomes much harder indeed. The sources of the difficulties are several; first, there exist Laplace transforms of non-L_1 time functions which have no right-half-plane poles, implying that stability is not simply equivalent to the absence of such poles in a transfer function (5); second, (although this difficulty has been overcome in the single-input, single-output case), open-loop unstable systems have proved very awkward to handle; third it proves awkward to use $j\omega$-axis behaviour (as plotted in a Nyquist plot) to infer right-half-plane behaviour. The present situation is that results are avialable for systems of the type of Figure 2; the systems are open-loop stable (although some unstable systems appear to be easily tolerated), with fairly restricted forms of impulse responses. Closed-loop stability is characterized by the necessary (6) and sufficient (7) condition that

$$\inf_{Re[s] > 0} |\det[I + W(s)]| \geq 0 \qquad (4)$$

Equivalent $j\omega$-axis conditions are given in (8), and extend beyond the usual Nyquist-type conditions.

Design based on eigenvalues of transfer function matrices

Computers make it feasible to obtain the eigenvalues of a transfer function matrix $W(s)$ for many values of $s = j\omega$. This had led to at least two design approaches, (4), as well as providing insights into others. In the commutative controller method, one assumes $W(j\omega)$ is square, and writes it as $V^{-1}(j\omega)\,\mathrm{diag}\,\gamma_i(j\omega)V(j\omega)$. (Of course, in general $\gamma_i(j\omega)$ and the entries of $V(j\omega)$ are irrational). One then arbitrarily selects the matrices $K(j\omega)$ and $H(j\omega)$ to be of the same form; i.e. $V^{-1}(j\omega)\,\mathrm{diag}\,\kappa_i(j\omega)V(j\omega)$ and $V^{-1}(j\omega)\,\mathrm{diag}\,\xi_i(j\omega)V(j\omega)$. This results in

$$F(j\omega) = V^{-1}(j\omega)\,\mathrm{diag}[1+\gamma_i(j\omega)\kappa_i(j\omega)\xi_i(j\omega)]V(j\omega) \qquad (5)$$

so that the eigenvalues of $F(j\omega)$ are $1 + \gamma_i(j\omega)\kappa_i(j\omega)\xi_i(j\omega)$. One then selects the free quantities $\kappa_i(j\omega)$ and $\xi_i(j\omega)$ to ensure that the Nyquist plots of the eigenvalues of $F(j\omega)$ have the correct encirclement properties. The difficulties with this approach are (a) the resultant complexity of $K(s)$ and $H(s)$, (b) the likely intolerance of the system to failure in actuators (4), (c) the likelihood of the system being prone to high frequency cross-talk (d) the computational load, particularly in obtaining rational $K(s)$ and $H(s)$.

In the characteristic locus method (4), it is again assumed that $W(s)$ is square. Also, one takes $H(s)$ to be the identity matrix. One builds up $K(s)$ as a product of matrices $K_i(s)$, with each $K_i(s)$ of particularly simple form - either $W^{-1}(0)$, $k(s)I$, or equal to the identity matrix save for variation of one entry by an arbitrary function of s. The aim is to choose $K(s)$ so that the Nyquist plots of the eigenvalues of $W(s)K(s)$ encircle the point $-1 + j0$ a certain number of times; the point of writing $K(s)$ as $K_1(s)K_2(s) \ldots K_m(s)$ with the $K_i(s)$ of restricted form is that the relation between the Nyquist plots of the eigenvalues of $W(s)K_1(s) \ldots K_i(s)$ and of $W(s)K_1(s) \ldots K_{i-1}(s)$ is easily visually appreciated, so that selection of $K_i(s)$ to modify the eigenvalue plots of $W(s)K_1(s) \ldots K_{i-1}(s)$ toward desired plots for $W(s)K(s)$ is more or less straightforward. The method appears to be an ingenious, nontrivial extension of scalar plant graphical design procedures.

As a general rule of thumb, it appears that when each characteristic locus of the loop gain has a good shape (i.e. exhibits good phase and gain margin, high gain at least at lower frequencies), the resultant closed-loop system will have good properties. In fact, one can rigorously establish a number of such ideas, see e.g. (1).

Inverse Nyquist Array

Yet another design procedure which can be thought of in terms of eigenvalues is the Inverse Nyquist Array procedure of Rosenbrock (9). The eigenvalue interpretation (1) is not however basic to the procedure, which rests on two important preliminary ideas. First, because $R = F^{-1}Q$, and because of (3), it follows that the system is closed-loop stable if and only if the number of counterclockwise encirclements of the origin by the Nyquist plot of $\det R^{-1}$ less the number of counterclockwise encirclements of the origin by the Nyquist plot of $\det Q^{-1}$ equals the number of right-half-plane zeros of the open-loop system (10). Second, for an arbitrary $m \times m$ matrix $X(s)$ with the diagonal dominance property

$$|x_{ii}(j\omega)| > \min\{\sum_{\substack{j=1 \\ j\neq i}}^{m} |x_{ij}(j\omega)|, \sum_{\substack{j=1 \\ j\neq i}}^{m} |x_{ji}(j\omega)|\} \qquad (6)$$

the number of origin encirclements by the Nyquist plot of $\det X$ equals the sum of the number of origin encirclements by the plots of $x_{ii}(j\omega)$, $i = 1, \ldots m$. The design procedure then amounts to choosing the series controller transfer matrix K, or rather its inverse K^{-1}, so that $Q^{-1} = K^{-1}W^{-1}$ has the primary property of diagonal dominance, and other secondary properties relating to gain and phase margin, noninteraction and the like, and then one chooses the feedback controller H, so as to ensure that R^{-1} too is diagonal dominant. This normally straightforward and done with a diagonal H, in view of the relation $R^{-1} = Q^{-1} + H$. The procedure is simplified if the plant is open-loop stable, and even simpler if the $\det W$ has no right-half-plane zeros.

The diagonal dominance amounts to having a degree of non-interaction between input-output pairs; this can serve as both a design objective, and when achieved makes simpler the design of further compensation, which to a certain extent can proceed on a single-loop basis precisely because of the diagonal dominance.

On the basis of these remarks, one might imagine that the controller matrix $K(s)$ should always be chosen so that $W(s)K(s)$ is diagonal; subsequently, $H(s)$ would be chosen to be diagonal, implying $R(s)$ is diagonal. As noted in [1] and [11], a great deal of design freedom is used up in making WK diagonal; this makes it impossible to achieve some of the extra objectives achievable by simply demanding $(WK)^{-1}$ to be diagonal dominant, and is particularly unsuitable if $\det W(s)$ has right-half-plane zeros.

Another Design Concept - Integrity

A system is of high integrity [4] if it remains stable under many likely failure conditions; the failures considered are those which are equivalent to the opening of one or more paths in the loop. Integrity against various classes of failures (e.g. actuators or transducers) is guaranteed if and only if all principal submatrices of the loop gain matrix satisfy certain Nyquist criteria. (The start and finish of the loop coincide with the failure point). For high integrity, it is helpful to take $H(s) = I$, but generally speaking the securing of high integrity may conflict with other design goals.

Other Frequency Domain Methods

Before passing to design methods based more on state-space ideas, we mention that Chen [(2), (12)] has discussed the use of feedback compensators to achieve arbitrary pole positions (with no concern for zero positions).

STATE-VARIABLE IDEAS

Instead of representing the plant by a transfer function matrix $W(s)$, one represents it by the equations

$$\dot{x} = Fx + Gu \qquad y = H'x \tag{7}$$

with $u(\cdot)$, $x(\cdot)$, and $y(\cdot)$ the input, state and output respectively, and F, G and H real constant matrices. The representation is normally minimal, i.e. it uses an F matrix of minimal dimension. As shown in for example [2], TFT^{-1}, TG, $(T^{-1})'H$ for any nonsingular real T also provides a minimal representation, and all minimal representations are obtainable in this way. Minimality of the representation is equivalent to the properties of controllability and observability, in this context, respectively, the ability to find a control taking any one state at any one time to any other state at any other time, and the ability to deduce the state at any time by input and output measurements up to that time. Also, with F an $n \times n$ matrix, there exist minimal integers α and β, termed respectively the controllability and observability index such that $\operatorname{rank}[G \;\; FG \;\; \ldots \;\; F^{\alpha-1}G] = \operatorname{rank}[H \;\; F'H \;\; \ldots \;\; (F')^{\beta-1}H] = n$. The transfer function matrix mapping $L[u]$ into $L[y]$ is $H'(sI - F)^{-1}G$.

State Feedback and Modal Control

Application of the control $u = v_{ext} + K'x$ corresponds to applying state feedback to (7). The resulting closed-loop transfer function is $H'(sI - \overline{F+GK'})^{-1}G$. If the original representation is minimal, K can always be chosen so that the eigenvalues of $F+GK'$ are arbitrary (this procedure is known as modal control); these eigenvalues are more or less the closed loop system poles. This result has been proved independently by a number of authors; for a group of references, together with procedures for selecting K, see [2]. Although for a scalar input plant, specification of the closed-loop poles uniquely determines K, this is not so for a vector-input, and there is often substantial freedom in its selection. In [13], an attempt is made to exploit this freedom in locating the zeros of the scalar transfer functions arising as elements of the closed-loop transfer function matrix. A good parametric characterization of all K producing a prescribed set of poles has yet to be found, and procedures for selecting K often restrict a priori the possible freedom.

Although state-variable feedback moves the system poles, the system zeros defined with $r = m$ by $|sI - F| \times \det H'(sI - F)^{-1}G$ are unaltered by state variable feedback; other quantities important in studying the decoupling of systems are also left invariant, see e.g. [14].

Disadvantages of modal control are the inability to design for a desired closed-loop transient response - this involves zero positions - as well as the need to have all the states available. [In case the full n-dimensional state is not available, some pole-positioning is still possible. In fact, the number of assignable poles is the maximum of the number of nontrivially different inputs and nontrivially different scalar functions of the state available [15]].

State Estimation

As noted above, one requires availability of the states to implement state feedback laws. If they are not directly measurable, they may be derived from u and y by a linear system termed a state estimator, the output vector of which is an estimate $\hat{x}$ of the plant state x, such that the error $\hat{x}-x$ asymptotically approaches zero. The estimate is used in place of the true state in implementing a feedback law. If the plant F matrix has dimension n and there are p outputs, then the estimator can be of dimension $n-p$, although if optimization in respect of noise elimination is required, it will normally be of dimension n. The rate at which the error decays to zero can be set arbitrarily fast (although very fast decay may generate difficult magnitude and stability specifications on the physical components of the estimator). For estimator theory, see e.g. [2] which contains original references.

If the input dimension m is less than n, then the whole state vector need not be estimated if an estimate of $K'x$ is all that is desired. Estimation of $K'x$ can then sometimes take place with an estimator of lower dimension than $n-p$; in particular, if $m=1$, estimation can take place with an estimator of dimension one less than β = observability index, which can be as low as n/p. The general problem of how to minimize estimator dimension is not fully understood; for some results, see [16].

If one seeks instead of implementation of a particular feedback law $K'x$, implementation of any feedback law which will produce a set of prescribed closed-loop poles, then the dimension of the compensator (comprising in essence an estimator of $K_1'x$ for some suitable K_1) need be no higher than min. $(\alpha-1, \beta-1)$ see [17]. An easy way to see this result is to note [18] that a minimal realization may be made controllable from any input by output static feedback (except in trivial situations). Then one uses a scalar input which can be generated by an estimator of dimension $\beta-1$, to position the poles. A dual argument gives the second bound. [Extensions of the result of [18], of interest in their own right, are that output static feedback can make a system controllable for each of any set of inputs and observable from each of any set of outputs, all at the one time].

Finally, we remark that the dynamics associated with the estimator itself do not show up in the closed-loop transfer function, which is the same whether feedback of the true state or a state estimate is used [2].

Linear Optimal Control

As remarked earlier, one of the disadvantages of the pole-positioning approach is that there is little direct control over the transient response of the closed-loop system. Somewhat more control over this response is provided by selecting the control law to minimize a performance index

$$V(x(0), u(\cdot)) = \int_0^\infty (x'Qx + u'Ru)dt \tag{8}$$

Here, R is positive definite, Q is nonnegative definite and the index depends on the initial state of (7). With minor and reasonable precautions, one finds [19] that (8) is minimized by a control law $u = K'x$ that ensures stability of the closed-loop system. Also, as noted in [19], there tend to be a number of other payoffs: tolerance of nonlinearities, insensitivity to plant parameter variations, and good phase and gain margin (eigenvalues of the loop gain matrix all have a phase margin of at least 60°, and gain margin of infinity). Also, it is possible by careful choice of Q and R to perhaps meet transient performance objectives, or to partly position the closed-loop poles. One is still some distance though with meeting via linear optimal control the objectives of low interaction, good following by the output of a reference input, and so on. Much of the difficulty is associated with the insistence on using nondynamic state feedback, which is inherent in the problem statement. By changing the performance index to include derivative-of-u terms, one can obtain dynamic optimal controllers however, which may prove more satisfactory [15].

As with modal control, linear optimal control demands measurability of the entire state. Some progress has been made with output-only feedback [20], or with feedback derivable from limited dimension compensation [21], but the fundamental question of how many independent variables one needs to feed back to stabilize a prescribed unstable open-loop system is unresolved. This question needs to be answered before success can be claimed for the suboptimal methods.

Realization Problem

Given a state space equation description of a plant, as in (7), it follows that the plant transfer function matrix is $H'(sI - F)^{-1}G$. The realization problem is the problem requiring reversal of this idea: one is given the plant transfer function matrix $W(s)$, and one is required to form a realization, preferably a minimal one. One method, requiring the break up into partial fractions of the transfer function matrix, appeared in [22]. Another quite efficient method, see e.g. [23] involves the construction of a nonminimal, but controllable realization, and then the elimination of the unobservable part by procedures set out in e.g. [24]. Many other variants on these ideas can be found, see [2] and [25] for further references. There is however a major second group of procedures based on use of the Hankel matrix. Suppose $W(s)$ is expanded as

$$W(s) = \frac{W_0}{s} + \frac{W_1}{s^2} + \frac{W_2}{s^3} + \dots \tag{9}$$

and define the Hankel matrix $H_{k,\ell}$ by

$$H_{k,\ell} = \begin{bmatrix} W_0 & W_1 & \dots & W_{\ell-1} \\ W_1 & W_2 & \dots & W_\ell \\ \vdots & & & \\ W_{k-1} & \dots & \dots & W_{k+\ell-2} \end{bmatrix} \tag{10}$$

It is possible to show that $W(s)$ has a minimal realization of dimension n with controllability and observability indices α and β if and only if rank $H_{\beta,\alpha}$ = rank $H_{\beta+1,\alpha+j} = n$ for $j=1,2,\dots$ and then

$$H_{\beta,\alpha} = [H \;\; F'H \dots (F')^{\beta-1}H]'[G \;\; FG \dots F^{\alpha-1}G] \tag{11}$$

From $H_{\beta,\alpha}$ and $H_{\beta+1,\alpha}$ or $H_{\beta,\alpha+1}$, it is then possible to compute F, G and H satisfying (11). One of the first techniques for doing this was due to Ho, see [26]. Since the original procedures, there have been a number of developments. One of the most interesting is that due to Rissanen [27]; he obtains F, G and H via a recursive procedure that examines $H_{k,\ell}$ with k and ℓ increasing at each step in the recursion. Another procedure, noteworthy because of its computational simplicity, obtains F, G and H directly from $H_{\beta,\alpha+1}$ and from this same matrix subjected to a Gauss elimination procedure, [25].

Silverman Structure Algorithm

An important algorithm for defining the structure of multivariable linear systems has been developed by Silverman (28) (29). This has been applied to the problem of determining the functional range of outputs of a prescribed system (including the inputs which will generate a prescribed output), to determining a left or right inverse for a system when these exist, to determining precompensators which, with state feedback from the prescribed system, will decouple the system, and to the exact model matching problem (30) (This problem requires the determination of precompensators and state feedback so as to generate a prescribed closed-loop transfer function matrix).

Decoupling

The aim in the basic decoupling problem is to control a system so that the closed-loop system has a diagonal transfer function matrix. Initially, for a system of the form of (7), this can be attempted by a control law of the form $u = K'x + Lv$, with v a new input. This means that $H'[sI - (F+GK')]^{-1}GL$ is required to be diagonal. Necessary and sufficient conditions for the existence of K and L, together with a procedure for finding them, were given first by Falb and Wolovich (31), following a number of early partial solutions to the problem. There is some freedom in the choice of K, and therefore some freedom, but not total freedom, over the p[illegible]tioning of closed-loop system poles. Gilbert (14) has explicitly demonstrated this freedom, and shown that it is sometimes impossible to obtain a stable closed-loop system.

When decoupling is not possible, partial decoupling can still be achieved with the closed-loop transfer function matrix block diagonal. In (32), it is shown how to obtain a 'maximally decoupled' system, with the blocks in the closed-loop matrix of smallest possible size. By allowing precompensation, it is possible to do better, and in fact if $H'(sI - F)^{-1}G$ is not singular everywhere, it is possible both to completely decouple and obtain arbitrary closed-loop system pole positions by a sufficiently high dimension precompensator, see e.g. (29), (33), (34). Minimization of the precompensator dimension cannot always be achieved, other than by trial and error.

Future Challenge

The major challenge still seems to be one of marrying the ideas of modern control with the goals of classical control, by developing design procedures that minimize their empirical content.

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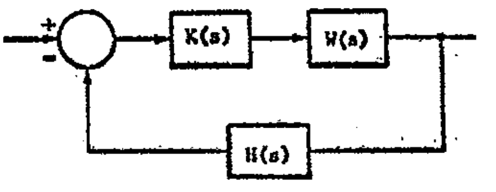

FIGURE 1

General Closed-Loop Arrangement.

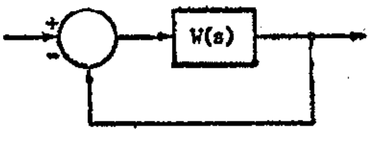

FIGURE 2

Basic Arrangement for Studying Stability.