OPTIMALLY INSENSITIVE ACTIVE CIRCUIT SYNTHESIS OF SECOND ORDER TRANSFER FUNCTIONS*

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February, 1970

*Work supported by Australian Research Grants Committee, Australian-Americal Educational Foundation in Australia and USAF Contract F44620-68-C-0023 of the Air Force Office of Scientific Research.

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ABSTRACT

Using state variable techniques, the synthesis is considered of a second order transfer function using operational amplifiers, resistors and integrators. The circuit is found which minimizes the maximum of the absolute value of the sensitivity of either the ω_0 or Q of the transfer function to any component variation. An improvement by a factor of two is noted over a circuit discussed in [1].

1. Introduction

This paper is concerned with the design of circuits with prescribed second order transfer functions. The circuits considered contain operational amplifiers (two being used as integrators), together with resistors and capacitors. The major emphasis of the work is on complex-pole synthesis, and particularly the sensitivity problem, i.e., the problem of ensuring that component variations have as little as possible effect on the overall circuit performance.

We argue, with the aid of state-variable techniques, that only a portion of a circuit synthesizing a transfer function is associated with producing the denominator, or poles, of the function. We isolate this portion, and consider all possible variants on it. We end up deriving that particular variant which minimizes the sensitivity (in a certain sense) of the circuit performance to component variations.

In [1], a specific circuit offering good sensitivity performance is studied. We improve by a factor of two the sensitivity performance of the circuit in [1] by our choice of the best possible circuit.

There is little or no additional complexity in the circuit presented here over that given in [1]. Therefore, it would seem likely that the applications of the circuit of [1] described in reference [2] should all be possible with the circuit given here.

2. State-Space Realizations and Second-Order Transfer Functions

Suppose we are interested in realizing a voltage-to-voltage transfer function of the form

$$T(s) = \frac{\alpha_2 s^2 + \alpha_1 s + \alpha_0}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \qquad (2\text{-}1)$$

Any (but not all) of α_0, α_1 and α_2 may be zero. We shall be primarily concerned with the denominator of T(s) and its realization.

Our mode of realization will be via circuits composed of resistors, capacitors and operational amplifier elements.

One of the most straightforward ways to describe such a circuit is in terms of state-variables. We suppose that two of the operational amplifiers are used with resistors and capacitors as integrators; then, with x_1 and x_2 denoting voltages on the capacitors, the equations of the circuit realizing (2-1) will take the form

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_1e_{in}$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_2e_{in}$$

$$e_{out} = c_1x_1 + c_2x_2 + de_{in} \qquad (2\text{-}2)$$

Figure 1 shows a signal flowgraph of these equations.

By taking the Laplace transform of (2-2), and eliminating x_1 and x_2, it is a simple matter to show that

$$\frac{\mathscr{L}[e_{out}]}{\mathscr{L}[e_{in}]} = \frac{[c_1 \quad c_2]\begin{bmatrix} s-a_{22} & a_{12} \\ a_{21} & s-a_{11} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}{s^2-(a_{11}+a_{22})s + a_{11}a_{22} - a_{12}a_{21}} + d \qquad (2\text{-}3)$$

Therefore, for (2-2) to "realize" (2-1), the right hand sides of (2-1) and (2-3) must be the same. This, of course, restricts, but does not uniquely identify, the parameters $a_{11}, \ldots a_{22}, b_1, \ldots c_2$, although d is uniquely defined; put another way, there will be numerous, actually an infinity, of possible choices of $a_{11}, \ldots, c_2$ so that the right side of (2-3) is identical with the prescribed transfer function of (2-1).

We observe from (2-1) and (2-3) that the parameters $b_1, \ldots, c_2$ and d of (2-2) are independent of ω_0 and Q; in fact

$$-\frac{\omega_0}{Q} = a_{11} + a_{22} \qquad \omega_0^2 = a_{11}a_{22} - a_{12}a_{21} \tag{2-4}$$

For prescribed ω_0 and Q, any choice of the four quantities $a_{11}, \ldots, a_{22}$ such that (2-4) is satisfied will guarantee that (2-2) correctly realizes the denominator of the transfer function (2-1).

Put another way, the denominator of (2-1) is effectively determined by the coefficients in the homogeneous equations

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{2-5}$$

These equations have the signal flowgraph shown in Figure 2.

Questions associated with the sensitivity of Q and ω_0 in (2-1) to component variation in any particular circuit realizing (2-1) which has a flowgraph in the form of Figure 1 may be answered by studying the flowgraph of Figure 2. Since the coefficients $b_1, \ldots c_2$ and d do not affect the denominator of (2-1), the corresponding parts of the flowgraph do not affect the calculation of sensitivity and may be disregarded.

Let us now illustrate a specific choice of $a_{11}, \ldots, a_{22}$ to realize a prescribed ω_0 and Q, together with the associated flowgraph and actual circuit realization.

3. A Specific Realization of a Second-Order Denominator

Suppose ω_0 and Q are prescribed. By taking

$$\begin{aligned} a_{11} &= 0 \qquad & a_{12} &= 1 \\ a_{21} &= -\omega_0^2 \qquad & a_{22} &= -\frac{\omega_0}{Q} \end{aligned} \tag{3-1}$$

we ensure that equations (2-4) are satisfied. The flowgraph corresponding to Figure 2 becomes that of Figure 3. This is readily realizable with circuit components, as shown in Figure 4. As is easy to check, Figure 4 is described by

$$\dot{x}_1 = \frac{R_5}{R_4 R_1 C_1} x_2$$
$$\dot{x}_2 = -\frac{1}{R_2 C_2} x_1 - \frac{1}{R_3 C_2} x_2 \tag{3-2}$$

By taking $1 = \frac{R_5}{R_4 R_1 C_1}$, $\omega_0^2 = \frac{1}{R_2 C_2}$ and $\frac{\omega_0}{Q} = \frac{1}{R_3 C_2}$, the circuit of Figure 4 will realize the denominator of the transfer function (2-1). Connections, perhaps via summers, scalors and inverting elements of input and output to appropriate points in the circuit will provide T(s) with any desired numerator. But the denominator will remain unaltered, and irrespective of the numerator of T(s), the circuit of Figure 4 will still be an integral part of the full circuit realizing a prescribed T(s).

The choice of a_{ij} in (3-1) is the choice made in [1] in a study of transfer function realizations. In the next section, we discuss other choices of a_{ij}.

For any set of a_{ij} giving the correct transfer function denominator, there is, as we shall see, an associated circuit realization. Again, with the aid of summers, scalors and inverting elements, we can simulate a transfer function (i.e., achieve a desired numerator, as well as denominator). But again the insertion of components to provide the numerator will not affect the realization of the denominator.

4. The Sensitivity Problem

In the previous section, we exhibited one particular scheme for obtaining the denominator of a prescribed transfer function. Essentially though, the equation

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2$$
$$\dot{x}_2 = a_{21} x_1 + a_{22} x_2 \tag{4-1}$$

for arbitrary a_{ij} can be realized with little or no more conceptual or physical difficulty. (We shall indicate explicit circuits subsequently.) The question then arises: What is the best set of a_{ij} to use in realizing the denominator of a prescribed transfer function. Since we only require two relations to be satisfied by the a_{ij}, viz.

$$\omega_0^2 = a_{11}a_{22} - a_{12}a_{21} \qquad \frac{\omega_0}{Q} = -(a_{11} + a_{22}) \tag{4-2}$$

the a_{ij} essentially have two degrees of freedom. We ask how this might best be exploited.

In this paper, we adopt one particular goal in the selection of the a_{ij} -- that of minimizing certain sensitivity coefficients. In any physical circuit, variation of components away from normal values will cause variation in system performance; we wish to minimize, in an appropriate sense, this latter variation.

The key quantities in the transfer function denominator are ω_0 and Q, and therefore, we make these the subject of one sensitivity analysis. (It is possible to consider related quantities, e.g., transfer function pole positions and study the sensitivity of these; however, from the application point of view, ω_0 and Q are perhaps more basic, and so we prefer to work with them.)

We define the sensitivity coefficients of ω_0 and Q with respect to variation of a circuit parameter x by

$$S_x^{\omega_0} = \frac{\partial \omega_0}{\partial x} \frac{x}{\omega_0} \qquad S_x^Q = \frac{\partial Q}{\partial x} \frac{x}{Q} \tag{4-3}$$

Neglecting second order effects, a 1% change in x will produce a $S_x^{\omega_0}$ % change in ω_0.

To determine the best circuit from the sensitivity point of view, we shall proceed as follows.

(a) We shall evaluate the sensitivity coefficients $S_{a_{ij}}^{\omega_0}$ and $S_{a_{ij}}^{Q}$

(b) We shall evaluate the sensitivity coefficient $S_x^{a_{ij}}$, where x denotes a component value in a circuit realizing (4-1).

(c) We shall evaluate $S_x^{\omega_0}$ and S_x^Q from the sensitivities computed in (a) and (b), and we shall find the circuit which minimizes the maximum of the absolute value of these sensitivities. This will be the optimum circuit.

Step (a) is straightforward; we use the definition of sensitivity coefficient, and the relations (4-2). To carry out Step (b), we must indicate the explicit circuit simulating (4-1). The first part of Step (c) is carried out by using the easily verified formula

$$S_x^{\omega_0} = \sum_{i,j} S_{a_{ij}}^{\omega_0} \; S_x^{a_{ij}} \tag{4-4}$$

(and likewise for S_x^Q).

The detailed calculations required for Step (a) are set out in the appendix. The end result of these calculations is as follows:

$$S_{a_{11}}^{\omega_0} = S_{a_{22}}^{\omega_0} = \frac{a_{11}a_{22}}{2\omega_0^2} \qquad S_{a_{12}}^{\omega_0} = S_{a_{21}}^{\omega_0} = -\frac{a_{12}a_{21}}{2\omega_0^2}$$

$$S_{a_{11}}^{Q} = \frac{a_{11}a_{22}}{2\omega_0^2} + \frac{a_{11}Q}{\omega_0} \qquad S_{a_{22}}^{Q} = \frac{a_{11}a_{22}}{2\omega_0^2} + \frac{a_{22}Q}{\omega_0} \tag{4-5}$$

$$S_{a_{12}}^{Q} = S_{a_{21}}^{Q} = -\frac{a_{21}a_{12}}{2\omega_0^2}$$

Let us set $S_{a_{11}}^{\omega_0} = k$. From this definition and from the equation $(a_{11} + a_{22})^2 = \frac{\omega_0^2}{Q^2}$, we obtain $(a_{11} - a_{22})^2 = \frac{\omega_0^2}{Q^2} - 8k\omega_0^2$. So

$$S_{a_{11}}^{\omega_0} = k \leq \frac{1}{8Q^2} \tag{4-6}$$

The other sensitivities can all be expressed in terms of k and Q, (see the appendix) and we can write

$$S_{a_{11}}^{\omega_0} = S_{a_{22}}^{\omega_0} = k$$

$$S_{a_{12}}^{\omega_0} = S_{a_{21}}^{\omega_0} = \frac{1}{2} - k$$

$$S_{a_{11}}^{Q} = k - \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 8kQ^2}$$

$$S_{a_{22}}^{Q} = k - \frac{1}{2} \mp \frac{1}{2} \sqrt{1 - 8kQ^2}$$

$$S_{a_{12}}^{Q} = S_{a_{21}}^{Q} = \frac{1}{2} - k \qquad (4\text{-}7)$$

Observe in (4-5) that if any a_{ij} is zero, the two associated sensitivity coefficients are zero. If this were not the case, we would have had to force them to be zero for a zero a_{ij}, since physically a zero gain is something we can realize exactly, in contrast to a nonzero gain. Thus, from the physical point of view, it would make sense to postulate ideal zero gain components with associated zero sensitivity coefficients.

In the next section, we discuss circuit realizations of (4-1) in order to evaluate component sensitivity coefficients.

5. Circuit Realizations and $S_x^{a_{ij}}$ -type Sensitivity Coefficients

There are a small number of different circuit configurations required to realize

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 \qquad (5\text{-}1)$$

with $\omega_0^2 = a_{11}a_{22} - a_{12}a_{21}$ $\qquad \frac{\omega_0}{Q} = -(a_{11} + a_{22})$ $\qquad$ (5-2) and the variation is caused by different sign patterns among the a_{ij}. First, suppose all a_{ij} are nonzero. Then sign patterns consistent with the equations relating the a_{ij} to ω_0 and Q are as follows

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} + & - \\ + & - \end{bmatrix}, \begin{bmatrix} - & + \\ - & + \end{bmatrix}, \begin{bmatrix} - & - \\ + & + \end{bmatrix}$$

$$\begin{bmatrix} - & + \\ + & - \end{bmatrix}, \begin{bmatrix} - & - \\ - & - \end{bmatrix}, \begin{bmatrix} - & + \\ - & - \end{bmatrix}, \begin{bmatrix} - & - \\ + & - \end{bmatrix}$$

The third and fourth are the same as the first and second, aside from a reordering of the two voltages x_1, x_2. If we restrict interest to $Q > \frac{1}{2}$ (as is entirely reasonable), the fifth and sixth sign patterns become impossible, as some simple algebra shows. The seventh and eighth patterns are the same, other than for a reordering of x_1 and x_2. Thus we are left with the following three patterns

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} + & - \\ + & - \end{bmatrix}, \begin{bmatrix} - & + \\ - & - \end{bmatrix} \qquad (5\text{-}3)$$

Now let us study what happens if one or more of the a_{ij} are zero. Examination of (5-2) shows that $a_{12} = 0$ or $a_{21} = 0$ is not possible if $Q > \frac{1}{2}$. Also, $a_{11} = a_{22} = 0$ is impossible. Hence either $a_{11} = 0$ or $a_{22} = 0$ (but not both). The possible sign patterns, disregarding variants caused by reordering of x_1 and x_2, are

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & - \\ + & - \end{bmatrix}, \begin{bmatrix} 0 & + \\ - & - \end{bmatrix}$$

These are limiting versions of the last two sign patterns in (5-3). The second of these we have already struck, in Section 3.

Circuits realizing the sign pattern of (5-3) are shown in Figures 5a - 5c. The equations of Figure 5a are

$$\dot{x}_1 = \frac{R_4}{R_1R_3C_1} x_1 + \frac{R_4}{R_1R_5C_1} x_2$$

$$\dot{x}_2 = -\frac{1}{R_2C_2} x_1 - \frac{1}{R_6C_2} x_2 \tag{5-5}$$

Figure 5b has equations

$$\dot{x}_1 = \frac{R_4}{R_3R_5C_1} x_1 - \frac{1}{R_1C_1} x_2$$

$$\dot{x}_2 = \frac{R_4}{R_2R_3C_2} x_1 - \frac{1}{R_6C_2} x_2 \tag{5-6}$$

and Figure 5c has equations

$$\dot{x}_1 = -\frac{1}{R_3C_1} x_1 + \frac{R_4}{R_1R_5C_1} x_2$$

$$\dot{x}_2 = -\frac{1}{R_2C_2} x_1 - \frac{1}{R_6C_2} x_2 \tag{5-7}$$

The sign patterns of (5-4) are achieved by removing R_5 (replacing it by an open circuit) in Figure 5b, corresponding to $R_5 \to \infty$ in (5-6), and by removing R_3 (replacing it by an open circuit) in Figure 5c, corresponding to $R_3 \to \infty$ in (5-7).

The computation of $S_x^{a_{ij}}$ for $x = R_1$, R_2, etc. for any of (5-5) through (5-7) is straightforward. The following three tables contain the requisite coefficients.

Table 1: $S_x^{a_{ij}}$ for Equation 5-5

a_{ij} \ x	R_1	R_2	R_3	R_4	R_5	R_6	C_1	C_2
a_{11}	-1	0	-1	1	0	0	-1	0
a_{12}	-1	0	0	1	-1	0	-1	0
a_{21}	0	-1	0	0	0	0	0	-1
a_{22}	0	0	0	0	0	-1	0	-1

Table 2: $S_x^{a_{ij}}$ for Equation 5-6

a_{ij} \ x	R_1	R_2	R_3	R_4	R_5	R_6	C_1	C_2
a_{11}	0	0	-1	1	-1	0	-1	0
a_{12}	-1	0	0	0	0	0	-1	0
a_{21}	0	-1	-1	1	0	0	0	-1
a_{22}	0	0	0	0	0	-1	0	-1

Table 3: $S_x^{a_{ij}}$ for Equation 5-7

a_{ij} \ x	R_1	R_2	R_3	R_4	R_5	R_6	C_1	C_2
a_{11}	0	0	-1	0	0	0	-1	0
a_{12}	-1	0	0	1	-1	0	-1	0
a_{21}	0	-1	0	0	0	0	0	-1
a_{22}	0	0	0	0	0	-1	0	-1

6. The Minimum Sensitivity Circuit

In this section we compute the sensitivities $S_x^{\omega_0}$ and S_x^Q for x a circuit component, and we find the circuit which minimizes the maximum of any one of these coefficients.

For our data, we take equation (4-7), giving $S_{a_{ij}}^{\omega_0}$, $S_{a_{ij}}^{Q}$ and Tables 1, 2 and 3 of section 5, giving $S_x^{a_{ij}}$ for the three different circuits. To compute $S_x^{\omega_0}$, we use

$$S_x^{\omega_0} = \sum_{i,j} S_{a_{ij}}^{\omega_0} S_x^{a_{ij}} \qquad (6\text{-}1)$$

and similarly for S_x^Q.

For each of Tables 1 to 3, we do the following calculation: compute $S_x^{\omega_0}$ for each x, and then list the absolute values of those coefficients which are different, rather than all 8 coefficients. Then do likewise for S_x^Q, and also eliminate any coefficients whose absolute values agree with the absolute value of some $S_x^{\omega_0}$. The distinct coefficient values which result turn out to be the same for all three tables, and are as follows:

$$\frac{1}{2},\ |k|,\ |\tfrac{1}{2} - k|,\ |k - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 - 8kQ^2}|,$$

$$|k - \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 8kQ^2}|,\ \ \tfrac{1}{2}\sqrt{1 - 8kQ^2}$$

Now we ask the question: how do we minimize the maximum value of the above list of quantities (i.e., what value of k achieves the minimum), <u>and</u> what is the associated circuit realization?

Recalling that $k \leq \dfrac{1}{8Q^2}$, it turns out that

$$k = \frac{1}{8Q^2} \qquad (6\text{-}2)$$

will achieve the minimum. Examination of the list of coefficients shows that with this value of k, all are less than $\frac{1}{2}$, except the first one. In fact, values of k near $\frac{1}{8Q^2}$ will still keep all coefficients except the first less than $\frac{1}{2}$, but note that indeed a high degree of nearness is required for if $k = 0$ say, (corresponding to the circuit in [1]), the maximum becomes 1 rather than $\frac{1}{2}$! (Thus we have improved on the sensitivity of the circuit in [1] by a factor of 2.)

This value of k uniquely specifies a_{11} and a_{22} in the following way. From (4-5) and the definition of k,

$$\frac{a_{11}a_{22}}{2\omega_0^2} = k$$

$$= \frac{1}{8Q^2} \quad \text{from (6-2)}$$

Since also $a_{11} + a_{22} = -\frac{\omega_0}{Q}$, we obtain

$$a_{11} = a_{22} = -\frac{\omega_0}{2Q} \tag{6-3}$$

On the other hand, although the product $a_{12}a_{21}$ is specified, neither a_{12} nor a_{21} is separately specified by the above minimization. Thus $\omega_0^2 = a_{11}a_{22} - a_{12}a_{21}$ implies

$$a_{12}a_{21} = -\omega_0^2\left(1 - \frac{1}{4Q^2}\right) \tag{6-4}$$

and for any a_{12} and a_{21} satisfying (6-4), the desired transfer function denominator will be achieved, together with the minimum sensitivity.

Comparing the signs of a_{11} and a_{22} in (6-3) with the allowable sign patterns of (5-3), we see that the circuit of Figure 5c is the one which yields minimum sensitivity in the sense we have described. Moreover, from (5-7), it follows that $R_1, \ldots, R_6$, C_1 and C_2 must be chosen to satisfy

$$\frac{1}{R_3C_1} = \frac{1}{R_6C_2} = \frac{\omega_0}{2Q}$$

and

$$\frac{R_4}{R_1R_5C_1} \cdot \frac{1}{R_2C_2} = \omega_0^2\left(1 - \frac{1}{4Q^2}\right)$$

To minimize the dynamic range of signals present when a complete filter is built, it is probably appropriate to take $-a_{12} = a_{21}$, or

$$\frac{R_4}{R_1R_5C_1} = \frac{1}{R_2C_2} = \omega_0\sqrt{1-\frac{1}{4Q^2}}$$

Then the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ takes the form $\begin{bmatrix} -\lambda & \mu \\ -\mu & -\lambda \end{bmatrix}$

7. Sensitivity to Amplifier Gain Changes

In this section, we compare the sensitivity to amplifier gain changes of the circuit we have derived to the circuit of [1].

Our strategy in looking at amplifier gain changes will be to first compute $S^{a_{ij}}_{K^{-1}}$ where K is an operational amplifier gain. Nominally of course, $K^{-1}=0$.

The circuit we consider is that of Figure 5c. If K_1 denotes the gain of the amplifier whose output is x_1, K_2 similarly, then the equations describing Figure 5c with nonzero K_1^{-1} and K_2^{-1} are

$$\dot{x}_1 = -\left[\frac{1}{R_3C_1} + \frac{K_1^{-1}}{(1+K_1^{-1})R_1C_1}\right] x_1 + \frac{R_4}{(1+K_1^{-1})R_1R_5C_1} x_2$$

$$\dot{x}_2 = -\frac{1}{(1+K_2^{-1})R_2C_2} x_1 - \left[\frac{1}{R_6C_2} + \frac{K_2^{-1}}{(1+K_2^{-1})R_2C_2}\right] x_2 \qquad (7\text{-}1)$$

It follows, on using the relations $a_{11} = -\frac{\omega_0}{2Q}$ etc, and taking $a_{12} = -a_{21}$, that

$$S^{a_{11}}_{K_1^{-1}} = \frac{R_5}{R_4}\, 2QK_1^{-1} \qquad S^{a_{12}}_{K_1^{-1}} = -K_1^{-1}$$

$$S^{a_{21}}_{K_2^{-1}} = 2QK_2^{-1} \qquad S^{a_{22}}_{K_2^{-1}} = -K_2^{-1} \qquad (7\text{-}2)$$

with higher order powers of K_1^{-1} and K_2^{-1} being neglected. Of course, other sensitivity coefficients such as $S^{a_{11}}_{K_2^{-1}}$ are equal to zero.

Next, equations (4-7) for $S_{a_{ij}}^{\omega_0}$ and $S_{a_{ij}}^{Q}$ are used. The conclusion is that (with $k = \frac{1}{8Q^2}$ in (4-7))

$$S_{K_1^{-1}}^{\omega_0} = -\frac{1}{2} K_1^{-1} + \frac{R_5}{4R_4 Q} K_1^{-1}$$

$$S_{K_2^{-1}}^{\omega_0} = -\frac{1}{2} K_2^{-1} + \frac{1}{4Q} K_2^{-1}$$

$$S_{K_1^{-1}}^{Q} = -\frac{R_5}{R_4} Q K_1^{-1} - \frac{1}{2} K_1^{-1}$$

$$S_{K_2^{-1}}^{Q} = -QK_2^{-1} - \frac{1}{2} K_2^{-1} \qquad (7\text{-}3)$$

where $\frac{1}{Q^2}$ has been neglected in comparison with 1.

Let us compare these figures with figures obtained from the circuit considered in Section 3, and examined in [1].

For this circuit, the equations become

$$\dot{x}_1 = -\frac{K_1^{-1}}{(1+K_1^{-1})R_1C_1} x_1 + \frac{R_5}{(1+K_1^{-1})R_4C_1C_1} x_2$$

$$\dot{x}_2 = -\frac{1}{(1+K_2^{-1})R_2C_2} x_1 - \left[\frac{1}{R_3C_2} + \frac{K_2^{-1}}{(1+K_2^{-1})R_2C_2}\right] x_2 \qquad (7\text{-}4)$$

It is then straightforward to compute

$$S_{K_1^{-1}}^{a_{11}} = 1 - K_1^{-1} \qquad S_{K_1^{-1}}^{a_{12}} = -K_1^{-1}$$

$$S_{K_2^{-1}}^{a_{21}} = -K_2^{-1} \qquad S_{K_2^{-1}}^{a_{22}} = K_2^{-1} Q\omega_0 \qquad (7\text{-}5)$$

These equations are to first order in K_1^{-1} or K_2^{-1}; other sensitivity coefficients are zero. In computing these relations, we have used the nominal values $a_{12} = 1$, $a_{21} = -\omega_0^2$; and $a_{22} = -\frac{\omega_0}{Q}$.

From (7-5), the expressions (4-5) for $S_{a_{ij}}^{\omega_0}$, $S_{a_{ij}}^{Q}$, and the nominal values of a_{ij}, we derive

$$S_{K_1^{-1}}^{\omega_0} = -\frac{K_1^{-1}}{2} \qquad S_{K_2^{-1}}^{\omega_0} = -\frac{K_2^{-1}}{2}$$

$$S_{K_1^{-1}}^{Q} = -\frac{K_1^{-1}}{2} \qquad S_{K_2^{-1}}^{Q} = -\frac{K_2^{-1}}{2} - K_2^{-1}Q\omega_0$$

We see that there is perhaps some advantage in the just considered circuit in that no sensitivity is proportional to Q times K_1^{-1}, in contrast to the result achieved for the optimum circuit. Otherwise, there is little significant difference.

8. *Conclusions*

In the preceding sections, we have exhibited a circuit which gives optimal performance from a very specific point of view -- minimization of the effect of R and C component variations on the Q and ω_0 of the transfer function simulated. There are, however, many other matters which should be considered in using such a design in a specific situation. Thus, although we have considered in the last section the effect of noninfinite operational amplifier gain, we have not considered the effect of nonflat frequency response of the operational amplifiers, of noninfinite input impedance and output conductance. Nor have we properly compared the dynamic range required of signals in different circuits.

A further factor with which the user must be concerned is the alignment or tuning of a filter. We note that the state-variable equations are, for the optimal arrangement,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{\omega_0}{2Q} & \omega_0\sqrt{1-\frac{1}{4Q^2}} \\ -\omega_0\sqrt{1-\frac{1}{4Q^2}} & -\frac{\omega_0}{2Q} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8-1}$$

which suggest that to adjust ω_0 either a_{12} or a_{21} (or both) should be adjusted, and to adjust Q either a_{11} or a_{22} should be adjusted. These adjustments will fortunately be largely independent for reasonable sized Q. Since

$$a_{11} = -\frac{1}{R_3C_1} \qquad a_{22} = -\frac{1}{R_6C_2}$$

$$a_{12} = \frac{R_4}{R_1R_5C_1} \qquad a_{21} = -\frac{1}{R_2C_2}$$

we see that there are components which affect one only of the a_{ij}, thus R_3 adjustments affects a_{11} only, etc. Hence tuning is presumably feasible.

Finally, we would emphasize that the optimal circuit does appear superior to the circuit of [1] -- by a factor of two for component sensitivities, while it is essentially no more complex.

APPENDIX

Calculation of $S_{a_{ij}}^{\omega_0}$, $S_{a_{ij}}^{Q}$

The basic equations are

$$\omega_0^{\;2} = a_{11}a_{22} - a_{12}a_{21}$$

$$\frac{\omega_0}{Q} = -(a_{11} + a_{22})$$

or

$$Q = -\frac{\sqrt{a_{11}a_{22} - a_{12}a_{21}}}{a_{11} + a_{22}}$$

These imply that

$$2\omega_0 d\omega_0 = a_{11}da_{22} + (da_{11})a_{22} - a_{12}da_{21} - (da_{12})a_{21}$$

and

$$dQ = -\frac{1}{(a_{11} + a_{22})^2}\left[\frac{a_{11}+a_{22}}{\sqrt{a_{11}a_{22} - a_{12}a_{21}}}(a_{11}da_{22} + a_{22}da_{11} - a_{12}da_{21} - a_{21}da_{12}) - \sqrt{a_{11}a_{22} - a_{12}a_{21}}\;(da_{11} + da_{22})\right]$$

From these, it follows that

$$\frac{\partial\omega_0}{\partial a_{11}}\frac{a_{11}}{\omega_0} = \frac{\partial\omega_0}{\partial a_{22}}\frac{a_{22}}{\omega_0} = \frac{a_{11}a_{22}}{2\omega_0^{\;2}}$$

$$\frac{\partial\omega_0}{\partial a_{12}}\frac{a_{12}}{\omega_0} = \frac{\partial\omega_0}{\partial a_{21}}\frac{a_{21}}{\omega_0} = -\frac{a_{12}a_{21}}{2\omega_0^{\;2}}$$

and

$$\frac{\partial Q}{\partial a_{11}}\frac{a_{11}}{Q} = \frac{a_{11}a_{22}}{2\omega_0^{\;2}} + \frac{a_{11}Q}{\omega_0} \qquad \frac{\partial Q}{\partial a_{22}}\frac{a_{22}}{Q} = \frac{a_{11}a_{22}}{2\omega_0^{\;2}} + \frac{a_{22}Q}{\omega_0}$$

$$\frac{\partial Q}{\partial a_{12}}\frac{a_{12}}{Q} = \frac{\partial Q}{\partial a_{21}}\frac{a_{21}}{Q} = -\frac{a_{21}a_{12}}{2\omega_0^{\;2}}$$

Set now $k = \frac{a_{11}a_{22}}{2\omega_0^2}$

Then $S_{a_{11}}^{\omega_0} = S_{a_{22}}^{\omega_0} = k$. Using $\omega_0^2 = a_{11}a_{22} - a_{12}a_{21}$, it follows that $S_{a_{12}}^{\omega_0} + S_{a_{11}}^{\omega_0} = \frac{1}{2}$, or $S_{a_{12}}^{\omega_0} = \frac{1}{2} - k$. Also, $S_{a_{21}}^{\omega_0} = S_{a_{12}}^{Q} = S_{a_{21}}^{Q} = \frac{1}{2} - k$.

Now

$$a_{11}a_{22} = 2\omega_0^2 k$$

and

$$(a_{11} + a_{22})^2 = \frac{\omega_0^2}{Q^2}$$

Hence

$$(a_{11} - a_{22})^2 = \frac{\omega_0^2}{Q^2} - 8k\omega_0^2$$

and

$$a_{11} - a_{22} = \pm\, \omega_0 \sqrt{\frac{1}{Q^2} - 8k}$$

Then

$$a_{11} = \frac{\omega_0}{2} \left[-\frac{1}{Q} \pm \sqrt{\frac{1}{Q^2} - 8k} \right]$$

$$a_{22} = \frac{\omega_0}{2} \left[-\frac{1}{Q} \mp \sqrt{\frac{1}{Q^2} - 8k} \right]$$

Finally

$$S_{a_{11}}^{Q} = k - \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 8kQ^2}$$

$$S_{a_{22}}^{Q} = k - \frac{1}{2} \mp \frac{1}{2} \sqrt{1 - 8kQ^2}$$

REFERENCES

[1] W. J. Kerwin, L. P. Huelsman and R. W. Newcomb, "State-Variable Synthesis for Insensitive Circuit Transfer Functions," <u>IEEE Journal on Solid-State Circuits</u>, Vol. SC-2, No. 3, September 1967, pp. 87-92.

[2] J. Tow, "A Step-by-Step Active-Filter Design," <u>IEEE Spectrum</u>, Vol. 6, No. 12, December 1969, pp. 64-68.

FIGURE CAPTIONS

1. Signal flowgraph for transfer function synthesis
2. Signal flowgraph achieving prescribed denominator
3. Special signal flowgraph for prescribed denominator
4. Special circuit achieving prescribed denominator
5. General forms of circuits achieving prescribed denominator

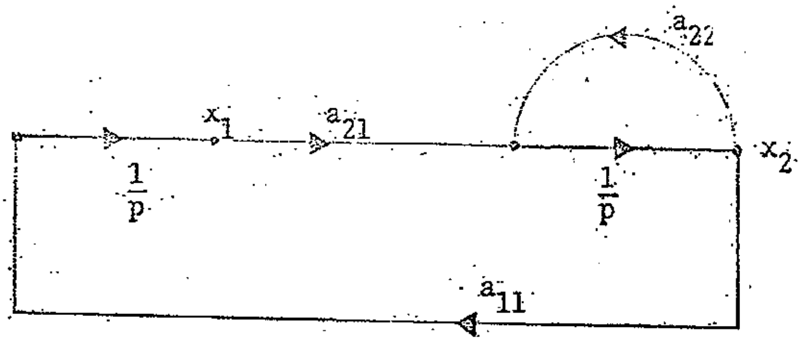

Figure 3

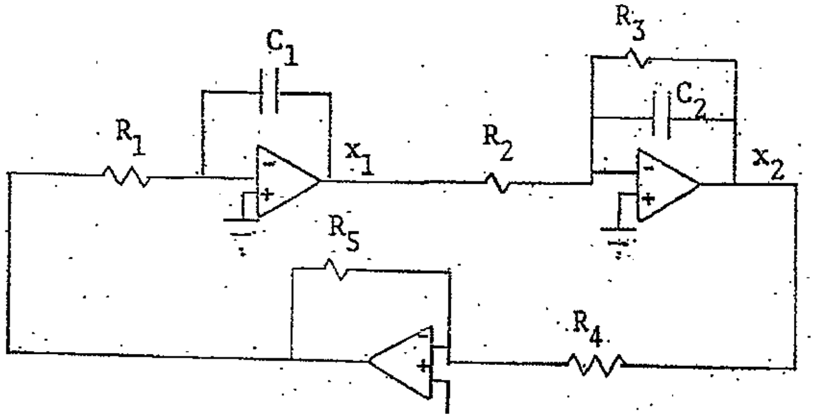

Figure 4

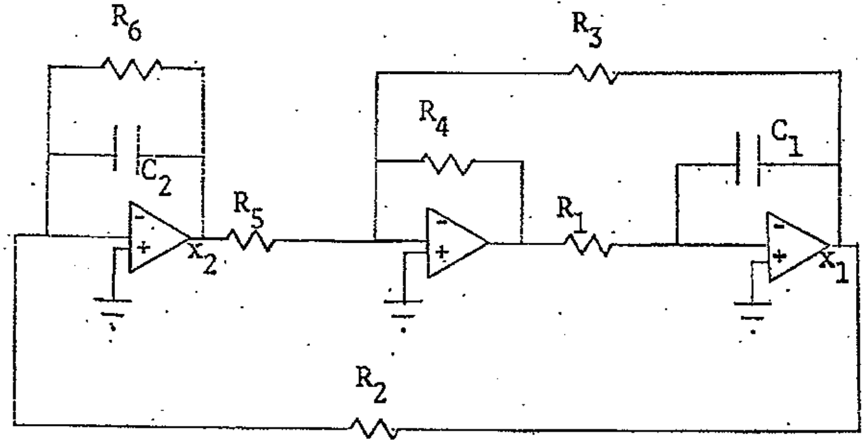

Figure 5a

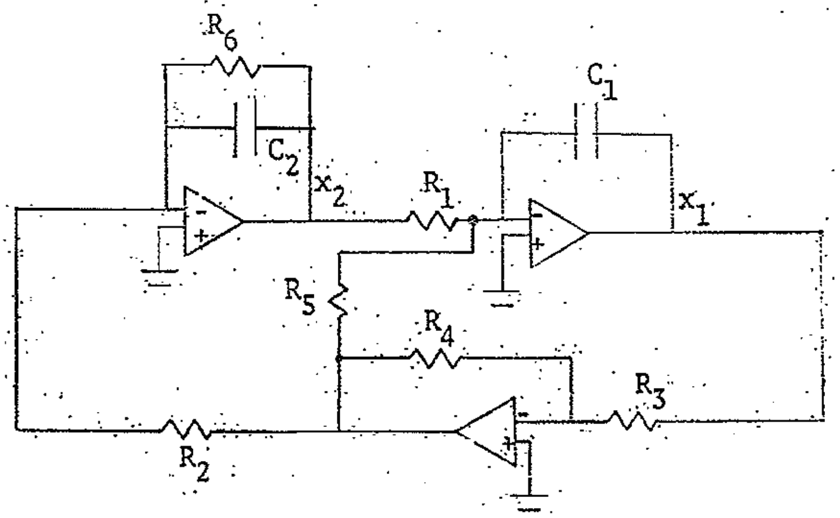

Figure 5b

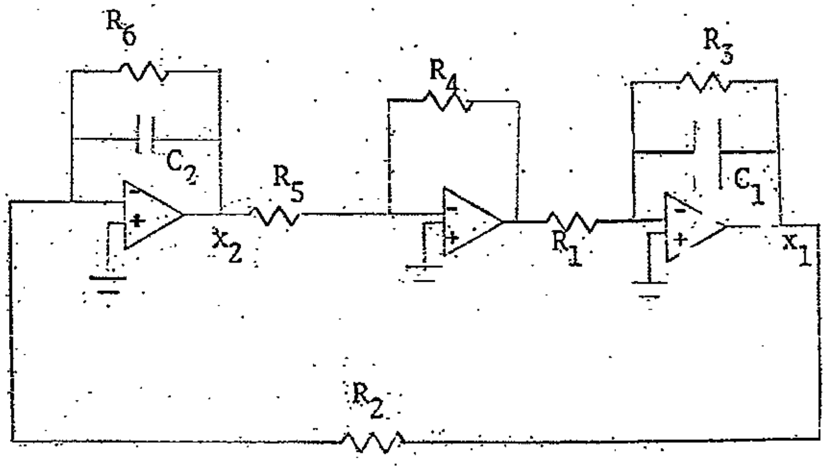

Figure 5c

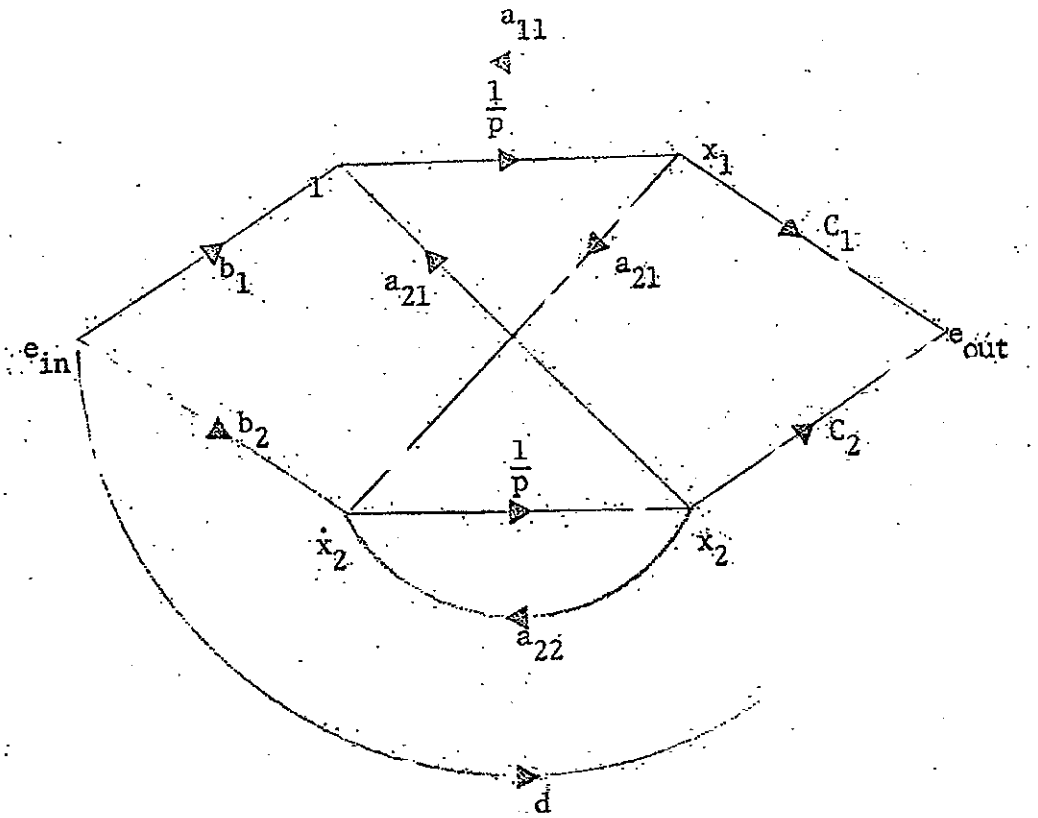

Figure 1

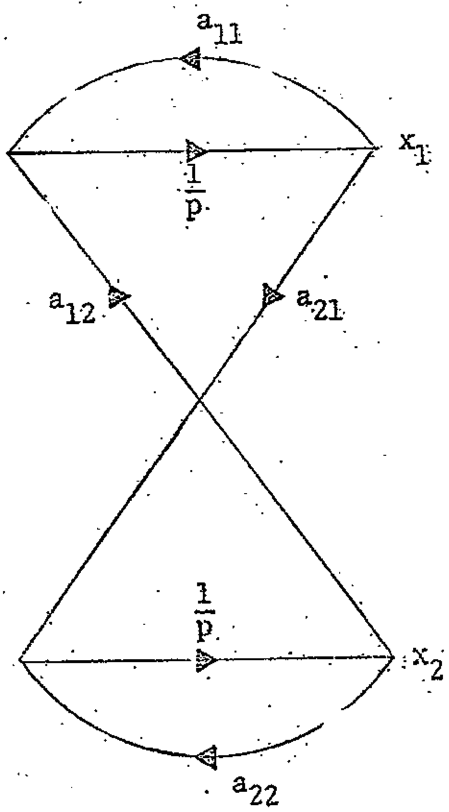

Figure 2