TRADE-OFF BETWEEN LOOP GAIN TIME-VARIATIONS AND RELATIVE STABILITY OF LINEAR SYSTEMS

by

J.B. Moore* and B.D.O. Anderson*
Department of Electrical Engineering
University of Newcastle, N.S.W. Australia.

This paper considers the system of Figure 1 where a single input linear system having a transfer junction $w(s)$ with input u and output y has a time-varying feedback gain element $k(t)$ and an external input which includes any initial condition response. Thus we have that $u = x - k(t)y$. The time-varying gain $k(t)$ is assumed to satisfy

Condition 1: $a + \varepsilon \leq k(t) \leq b - \varepsilon$ for all t, for some a and b with $b > a \geq 0$, and some ε with

$$\frac{b-a}{2} > \varepsilon > 0.$$

The transfer function $w(s)$ is assumed to satisfy.

Condition 2: The closed loop system $w(s)/(1+kw(s))$ has a degree of stability σ_0 for all constant gains k in the interval $[a, b]$.

Let us introduce the quantity $\hat{k}$, defined for fixed but arbitrary T by

$$\hat{k} = \sup_{t \geq 0} \frac{1}{4T} \int_t^{t+T} \left| \frac{\dot{k}(\tau)(b-a)}{[k(\tau)-b][k(\tau)-a]} \right| dt$$

Freedman and Zames [1] have presented sufficient conditions for

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asymptotic stability of the closed-loop system. These are that conditions 1 and 2 hold, that the system input, denoted by x, be square intergrable and that $\hat{k} < \sigma_0$.

Plainly, Freedman and Zames are sacrificing the degree of stability σ_0 of the system with constant gains for the freedom to tolerate time-variation in the feedback. However for engineering purposes, what is required is a statement of the exact trade-off possible between degree of stability and tolerance of time-variation; such a statement is lacking in the Freedman and Zames paper.

In this paper we present sufficient conditions for the closed-loop system to have a degree of stability α, that is for both $ue^{\alpha t}$ and $ye^{\alpha t}$ to be L_2 functions. These are that conditions 1 and 2 hold, that $xe^{\alpha t} \varepsilon L_2$ and that $\hat{k} < (\sigma_0 + \alpha) \leq \sigma_0$. This result provides a direct and continuous trade-off between degree of stability and tolerance of time variation.

We now give an outline proof based on "energy balance" arguments for the above stability result. Without loss of generality we consider the case when $a = 0$ and $b = \infty$. The proof follows basically that given in [1] for the special case when $\alpha = 0$. All we do here is indicate how the proof given in [1] may be modified for the case when α is nonzero. We first give specializations of two lemmas given in [1]. The second is restated so as to cover the case $\alpha \neq 0$.

<u>Lemma 1</u>: Condition 2 implies that there is a positive real function $m(s - \sigma_0 - \varepsilon)$ such that $m(s - \sigma_0 - \varepsilon)w(s - \sigma_0 - \varepsilon)$ is positive real for some $\varepsilon > 0$.

<u>Lemma 2</u>: Condition 1 (for the case $a = o$, $b = \infty$) together with the inequality $\hat{k} < (\sigma_0 - \alpha) \leq \sigma_0$ implies that there is a constant $\underline{f}$ and a junction $f(\cdot)$ such that $0 < \underline{f} \leq f(t) \leq f < \infty$ for all t and such that $f(t)e^{-2(\sigma_0-\alpha)t}$, $f(t)k(t)e^{-2(\sigma_0-\alpha)t}$, and $[f(t) - \underline{f}]e^{-2(\sigma_0-\alpha)t}$ are decreasing functions of time t.

Proceding with the "energy balance" argument we have

$$x(t) = u(t) + k(t)y(t)$$

$$e^{\alpha t}x(t)f(t) = e^{\alpha t}u(t)\underline{f} + e^{\alpha t}u(t)[f(t) - \underline{f}] + e^{\alpha t}k(t)y(t)f(t)$$

Using $< \; >_{t_1}$ to denote inner product, $*$ to denote convolution and $w(t-\tau)\,[w(t-\tau) = e^{\alpha t}\hat{w}(t-\tau)e^{-\alpha\tau}]$ to denote the impulse responses associated with $w(s)\,[w(s-\alpha)]$, we have that

$$< e^{\alpha t}x(t)f(t),\ \hat{m}(t-\lambda) * \hat{w}(\lambda-\tau) * e^{\alpha\tau}u(\tau) >_{t_1}$$

$$= \underline{f} < e^{\alpha t}u(t),\ \hat{m}(t-\lambda) * \hat{w}(\lambda-\tau) * e^{\alpha\tau}u(\tau) >_{t_1}$$

$$+ < e^{\alpha t}u(t)[f(t)-f],\ \hat{m}(t-\lambda) * \hat{w}(\lambda-\tau) * e^{\alpha\tau}u(\tau) >_{t_1}$$

$$+ < k(t)e^{\alpha t}y(t)f(t),\ \hat{m}(t-\lambda) * \hat{w}(\lambda-\tau) * e^{\alpha\tau}u(\tau) >_{t_1}$$

These equations are now set up so that the same reasoning as in [1] given for the case $\alpha = 0$, may be used to yield the results that we require. Suffice it to say that by applying the lemmas, Parseval's Theorem, Schwarz inequality and the Mean Value Theorem the above equation may be manipulated to yield the inequality

$$\bar{f} \sup_{-\infty<w<\infty} \left| m(jw-\alpha)w(jw-\alpha) \right| \; ||e^{\alpha t}x(t)||_{t_1} ||e^{\alpha t}u(t)||_{t_1}$$

$$\geq \underline{f}\delta_1 ||e^{\alpha t}u(t)||^2_{t_1}$$

Where $||\cdot||_{t_1}$ denotes an L_2 norm on $[0,t_1]$. The value ot the supremium is finite as in [1], for some $\delta_1 > 0$ and all t_1.

We have therefore that

$$||e^{\alpha t}u(t)||_{t_1} \leq \delta_2 ||e^{\alpha t}x(t)||_{t_1}$$

for some $\delta_2 > 0$. But since $e^{\alpha t}x(t) \in L_2$, we have that $e^{\alpha t}u(t) \in L_2$

and thus condition 2 ensures that $e^{\alpha t}y(t) \in L_2$. Our claim is thereby established.

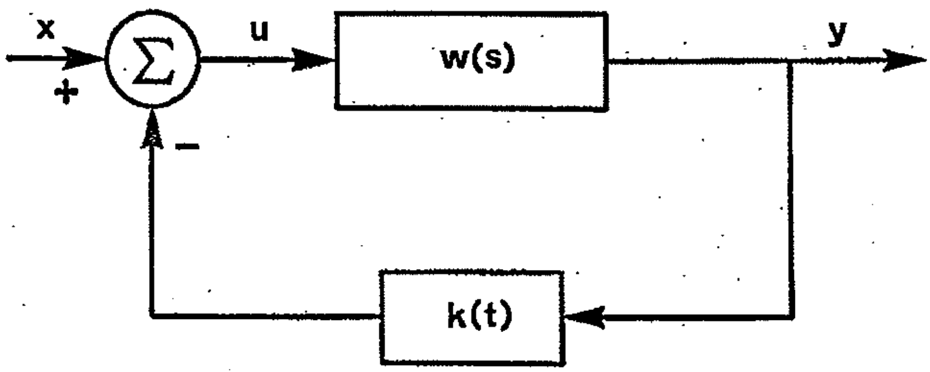

Figure 1. System Diagram

REFERENCE

[1] M. Freedman and G. Zames, "Logarithmic Variation Criteria for the stability of systems with Time-Varying gains", SIAM J. of control, Vol. 6. No. 3. August 1968, pp. 487-507.