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Abstract

The prime purpose of this paper is to outline tests which yield information about the number of real roots, and the positivity, of a prescribed polynomial. The Sturm-Routh test is reviewed and use of a Routh array for executing a Sturm test is explained; a Markov test based on Markov parameters is reviewed, and a modified version of a test due to Fujiwara is given, this with its relation to the Markov parameter test. Finally, a test in the language of linear system theory is given, and a relation is noted between this and the Fujiwara test.

1. Introduction

The object of this paper is to set out tests, and to note their interdependence, which check that a prescribed real polynomial $f(\cdot)$ of a real variable x is nonnegative for all x in the range $(-\infty,\infty)$ (or which evaluate the number of real zeros of $f(x)$).

The requirement of carrying out such a test arises in problems of network theory (e.g., checking that a function is positive real or bounded real [1]), of linear optimal control theory (e.g., in studying the inverse optimal control problem [2,3]), and of stability theory (e.g., in using the Popov criterion [4] or circle criterion [5]).

The traditional technique espoused in texts on network theory (e.g., [6]) and in fact theory of equations (e.g. [7]) for checking the distribution of real roots of a real polynomial (essentially the same as checking nonnegativity) is the Sturm test, discussed more fully below. Recently, there appeared the suggestion that when $f(x)$ is an even polynomial, the Routh test could also be used [8], and unpublished work by Siljak, author of [8], explores connections between the Sturm and Routh test. Below we argue that essentially these tests are the same, for both the case when $f(x)$ is an even polynomial, and when it is unrestricted.

We also recall a test based on the use of Markov parameters, see [9], which appears to have originated in a paper by Hurwitz [10], better known for the test stated therein for checking the signs of the real parts of the roots of a polynomial. Fujiwara, [11], in an extension of Hermite's ideas [12], studies the problem of specifying the distribution of real roots of a polynomial, and a very simple extension of his ideas leads to a test of the sort desired here. Finally, there is a test presentable in the language of linear system theory which bears some tenuous relationship with Lyapunov's test for the eigenvalues of a prescribed matrix to all possess negative real part.

2. Cauchy Indices, Sturm Chains and Routh Arrays

Suppose $g(x)$ and $h(x)$ are real polynomials in a real variable x, and suppose that the degree of $g(x)$ does not exceed that of $h(x)$.

Definition [9] The Cauchy index of $g(x)/h(x)$ between the limits a and b, written $I_a^b[g(x)/h(x)]$, is the difference between the number of jumps of g h from $-\infty$ to $+\infty$ and from $+\infty$ to $-\infty$ as the argument changes from a to b. It is understood that a and b are real and $a < b$, with $a = -\infty$, $b = +\infty$ permitted.

Remark 1 It is easy to show by considering a partial fraction expansion of $f'(x)/f(x)$ that $I_a^b \frac{f'(x)}{f(x)}$ is equal to the number of distinct real roots of the equation $f(x) = 0$ in the interval (a,b). (Here $f'(x)$ is, of course, the derivative of $f(x)$).

Sturm's contributions were really three: first, to characterize $I_a^b[g(x)/h(x)]$ in terms of the signs of a number of polynomials evaluated at a and b; second, to provide an algorithm to evaluate $I_a^b[g(x)/h(x)]$ which does not require the factoring of any polynomial; third, to note that his algorithm was applicable to the case when $g(x) = h'(x)$, so that the number of distinct real zeros of $h(x)$ could be specified.

Definition [9] A sequence of real polynomials $f_1(x), f_2(x) \ldots f_m(x)$ is a Sturm chain in the interval $[a,b]$ if the following two properties hold:

(1) $m > k > 1$ and $f_k(x_0) = 0$ for $a < x_0 < b$ implies $f_{k-1}(x_0) f_{k+1}(x_0) < 0$

(2) $f_m(x)$ is identically nonzero in $[a,b]$.

We denote by $V(x)$ the number of variations in sign of the sequence $f_1(x), \ldots, f_m(x)$ for fixed x.

Theorem 1 (Sturm's Theorem) If $f_1(x), f_2(x), \ldots, f_m(x)$ is a Sturm chain in $[a,b]$ and $V(x)$ is the number of variations in sign at an arbitrary fixed point x,

$$I_a^b[f_2(x)/f_1(x)] = V(a) - V(b) \tag{2-1}$$

Remark 2 By taking $f_1(x) = f(x)$, $f_2(x) = f'(x)$, $a = -\infty$, $b = +\infty$, we obtain a technique for counting this distinct real roots of $f(x) = 0$, provided that a Sturm chain $f(x), f'(x), \ldots$ can be constructed. Then to check that a prescribed $f(x)$ is positive for all x, it is sufficient to show that it is positive for one x and has no real zero. To check that it is nonnegative requires a little more manipulation if a technique based on theorem 1 is used.

Remark 3 If the degree of $f_1(x)$ is not less than that of $f_2(x)$, one way of constructing a Sturm chain is by means of the Euclidean algorithm. Thus, see [7,9], we divide $f_2(x)$ into $f_1(x)$, and take $-f_3(x)$ as the remainder. (Note the negative sign!) Then we divide $f_3(x)$ into $f_2(x)$ and take $-f_4(x)$ as the remainder. Eventually the procedure stops, with $f_m(x)$ the greatest common divisor of $f_1(x)$ and $f_2(x)$. If these polynomials are relatively prime, $f_m(x)$ is constant.

The term Sturm test is applied to a certain procedure for checking the number of real roots of a polynomial $f(x)$ in a prescribed interval $[a,b]$. The procedure consists of constructing a Sturm chain whose first two members are $f(x)$ and $f'(x)$, and whose remaining members are found via the Euclidean algorithm. It then follows that

$$I_a^b \frac{f'(x)}{f(x)} = V(a) - V(b) \tag{2-2}$$

$$= \text{number of distinct real zeros of } f(x) \text{ in } [a,b]$$

Actually, even if f(x) has multiple roots in [a,b], which turns out to imply that the last entry of the Sturm chain fails to be identically nonzero in [a,b], this formula is still valid [9].

As remarked above, the Sturm test can be used as a basis for determining whether a polynomial is positive for all real x (take $a = -\infty$, $b = +\infty$, check that $V(-\infty) = V(\infty)$, and that $f(x) > 0$ for some x). Nonnegativity can be checked, with some additional complexity in the calculations [6,9].

Routh [13] was primarily concerned with setting up conditions ensuring that all roots of a polynomial have negative real parts. Routh's contribution lies in (a) the application of the Sturm test with ingeniously chosen $f_1(x)$ and $f_2(x)$; (b) the construction of numerical devices to speed up the calculation, the principal such device being the Routh array. We take no cognizance here of the choice of $f_1(x)$ and $f_2(x)$; rather we wish to note properties of the Routh array, which we now define.

The Routh array starts its construction from two prescribed rows of numbers, and builds from these a third row according to a certain rule; from the second and new third row, a fourth row is constructed using the same rule as employed in the construction of the third row from the first and second. A fifth row is constructed from the third and fourth, and so on. The process terminates in a finite number of steps.

The rule for obtaining one row of the Routh array from the two preceding rows can be noted by studying the following sequence of three rows.

$$
\begin{array}{llll}
a_0 & a_1 & a_2 & \cdots \\
b_0 & b_1 & b_2 & \cdots \\
c_0 = \dfrac{b_0a_1 - a_0b_1}{b_0} & c_1 = \dfrac{b_0a_2 - a_0b_2}{b_0} & c_2 = \dfrac{b_0a_3 - a_0b_3}{b_0} & \cdots
\end{array}
$$

The entries d_i of the next row are constructed in the same way as the c_i, thus $d_0 = \dfrac{c_0b_1 - b_0c_1}{c_0}$, etc.

Various special adjustments are made in case division by zero might be required in the array construction [9,13].

3. Sturm-Routh Test

In this section, we consider two separate situations, the first where a prescribed real polynomial $f(x)$ is even in x, and the second where it is not. We wish to note why the writing down of a Routh array is equivalent to the construction of a Sturm chain with first two members $f(x)$ and $f'(x)$ via the Euclidean algorithm (or a minor modification of it). We state a theorem for each case which yields necessary and sufficient conditions for $f(x)$ to be positive in terms of the coefficients of the Routh array.

The reason why we single out even polynomials for special consideration is twofold. First, those polynomials which have to be tested for nonnegativity in the applications mentioned earlier [1-4] always have the evenness property. Thus on the grounds of practical application, it is worthwhile considering even $f(x)$. Second, the adjustments which can be made to the Routh array associated with a general $f(x)$ when it is in fact even are so great as to warrant special consideration from the beginning of even $f(x)$. We remark that Routh himself [13] saw the Routh array primarily as a representation of a Sturm chain with first entry an even polynomial, second entry an odd polynomial.

Even polynomials Suppose

$$\begin{aligned} f(x) &= a_0x^n - a_1x^{n-2} + a_2x^{n-4} - \cdots \\ f'(x) &= b_0x^{n-1} - b_1x^{n-3} + b_2x^{n-5} - \cdots \end{aligned} \tag{3-1}$$

(Note the sign pattern!) Of course, $b_0 = na_0$, $b_1 = (n-2)a_1$, etc. The connection between Sturm chains and the Routh array is as follows. The terms $f_3(x)$, $f_4(x)$,... in the Sturm chain with $f_1(x) = f(x)$, $f_2(x) = f'(x)$, and later terms determined from the Euclidean algorithm are

$$\begin{aligned} f_3(x) &= c_0x^{n-2} - c_1x^{n-4} + c_2x^{n-6} - \cdots \\ f_4(x) &= d_0x^{n-3} - d_1x^{n-5} + d_2x^{n-7} - \cdots \end{aligned} \tag{3-2}$$

where the coefficients c_i, d_i and so on are precisely the entries of the Routh array

$$\begin{matrix} a_0 & a_1 & a_2 & \cdots \\ b_0 & b_1 & b_2 & \cdots \\ c_0 & c_1 & c_2 & \cdots \\ d_0 & d_1 & d_2 & \cdots \end{matrix} \tag{3-3}$$

This property is immediately verifiable by direct calculation. We stress that it was precisely the observation above which Routh used in setting up his array, which he correctly saw as nothing but a systematic way of carrying the numerical data inherent in the polynomials of a Sturm chain without having to carry powers of x.

It follows that the checking of the signs of the polynomials $f_i(x)$ for any fixed x can be done with the aid of the Routh array, while the checking of the signs of $f_i(\infty)$ and $f_i(-\infty)$ requires merely examination of the first term of the ith row of the Routh array. In fact, with $V(x)$ denoting the sign variations in $f_1(x)$, $f_2(x)$,... for some fixed x, $V(-\infty) = V(a_0,-b_0,c_0,-d_0)$ and $V(+\infty) = V(a_0,b_0,c_0,\ldots)$.

In the normal or regular case, a_0,b_0, etc. are nonzero, and there are $(n+1)$ rows in the array. (Note: the array could in theory be continued to $(n+2)$ rows, as will be seen upon checking that the nth row will have two nonzero elements in general, and the (n+1)th row one nonzero element. The (n+2)th row is not, however, constructed since its coefficient would not correspond to any polynomial in a Sturm chain).

With $(n+1)$ rows then, we see by inspection of the expressions for $V(-\infty)$ and $V(\infty)$ in terms of a_0,b_0, etc. that $V(-\infty) = n-V(\infty)$; so $V(-\infty) - V(+\infty) = n - 2V(\infty)$. Consequently:

Theorem 2 (Sturm-Routh test for even polynomials). Let $f(x)$ be an even polynomial, and suppose the Routh array (3-3) is formed, the first two rows being defined from equation (3-1). Assuming a regular case, where there are $(n+1)$ rows in the array with the first entry of each row nonzero, the number of real roots of $f(x) = 0$ in the interval $(-\infty,\infty)$ is given by $n-2V(a_0,b_0,c_0,\ldots)$ and $f(x)$ is strictly positive if $a_0 > 0$ and $V(a_0,b_0,c_0,\ldots) = \frac{1}{2}n$.

Remark 4 In the "irregular" case, techniques as described in [9,13] will yield the appropriate modification. Our main purpose here is to indicate broad connections of results, not the details of special case situations.

<u>Arbitrary polynomials</u> Suppose now that

$$f(x) = a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots$$
$$f'(x) = b_0x^{n-1} - b_1x^{n-2} + b_2x^{n-3} - \cdots \qquad (3\text{-}4)$$

Direct calculation shows easily that

$$f(x) = \frac{a_0}{b_0} xf'(x) - (c_0x^{n-1} - r\, x^{n-2} + c_2x^{n-3} - \cdots)$$
$$\triangleq \frac{a_0}{b_0} xf'(x) - f_3(x) \qquad \text{say}$$

Then

$$f'(x) = \frac{b_0}{c_0} f_3(x) - (d_0x^{n-2} - d_1x^{n-3} + \cdots)$$
$$\triangleq \frac{b_0}{c_0} f_3(x) - f_4(x) \qquad \text{say}$$

And

$$f_3(x) = \frac{c_0}{d_0} xf_4(x) - (e_0x^{n-2} - e_1x^{n-3} + \cdots)$$
$$\triangleq \frac{c_0}{d_0} xf_4(x) - f_5(x) \qquad \text{say}$$

In the above situations, it is easily seen by direct calculation that the coefficients c_i, d_i, etc. are the entries of the Routh array (3-3). The sequence of polynomials $f_1(x) = f(x)$, $f_2(x) = f'(x)$, $f_3(x) \cdots$ also constitutes a Sturm chain, albeit not one of minimum length, and not identical with that computed by the Euclidean algorithm, though almost so. Nonetheless, it is a Sturm chain* as may easily be checked on applying the definition, and accordingly Theorem 1 may be applied.

There will in this case be 2n members of the Sturm chain, asssuming a regular case: one member of degree n, 2 of degree (n-1), (n-2),...,2, and one of degree 1.

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*At least this is true when f(x) and f'(x) have no common zeros, which corresponds to the regular case. The irregular case can be dealt [illegible]

If n is even, and $V(x)$ denotes the sign variations in $f_1(x)$, $f_2(x), \ldots$ for some fixed x, then we have

$$V(\infty) = V(a_0, b_0, c_0, \ldots, r_0, s_0, t_0)$$

and

$$V(-\infty) = V(a_0, -b_0, -c_0, d_0, e_0, \ldots, -r_0, -s_0, t_0)$$

Hence

$$\begin{aligned} V(-\infty)-V(\infty) &= V(a_0,-b_0)+V(-b_0,-c_0)+V(-c_0,d_0)+\cdots+V(-s_0,t_0) \\ &\quad - V(a_0,b_0)-V(b_0,c_0)-V(c_0,d_0)-\cdots-V(s_0,t_0) \\ &= [V(a_0,-b_0)-V(a_0,b_0)]+[V(-c_0,d_0)-V(c_0,d_0)]\cdots \\ &\quad + [V(-s_0,t_0)-V(s_0,t_0)] \\ &= [1-2V(a_0,b_0)]+[1-2V(c_0,d_0)]+\cdots+[1-2V(s_0,t_0)] \\ &= n-2[V(a_0,b_0)+V(c_0,d_0)+\cdots+V(s_0,t_0)] \qquad (3.5) \end{aligned}$$

It is easy to check that this formula holds also in the case when n is odd.

Now combining Theorem 1 with the above formula, we deduce

Theorem 3 (Sturm-Routh Test for Arbitrary Polynomials) Let $f(x)$ be a polynomial as in equation (3-4), and suppose the Routh array (3-3) is formed, the first two rows being derived from equation (3-4). Assuming a regular case, where there are $2n$ rows in the array with the first entry of each row nonzero, the number of real roots of $f(x) = 0$ in the interval $(-\infty,\infty)$ is given by $n-2[V(a_0,b_0)+V(c_0,d_0)+\cdots+V(s_0,t_0)]$ and $f(x)$ is strictly positive for all x if $a_0 > 0$ and $V(a_0,b_0)+V(c_0,d_0)+\cdots+V(s_0,t_0) = n/2$.

Note that a polynomial of odd degree can never be strictly positive for all x, so that the condition for positivity can reasonably involve $\frac{n}{2}$.

The first entries of each Routh array row also carry information about the number of distinct real zeros of $f(x)$ in the interval $[0,\infty)$. Suppose that $f(x)$ has no zero at $x = 0$. (If it has, it will be obvious, and removable by division.) Then the number of real zeros of $f(x)$ in $[0,\infty)$ is the same as the number of real zeros of $g(x)$ in $(-\infty,\infty)$ where $g(x) = f(x^2)$. The polynomial $g(x)$ is even, and the first two rows of its Routh array, formed according to the rule associated with even polynomials, are identical with the first two rows of the Routh array of $f(x)$ formed according to the rule associated with arbitrary polynomials, save for an inessential multiplier of +2 for the second row. Theorem 2 then yields the number of positive real zeros of $f(x)$ as $n-2V(a_0,b_0,c_0,\ldots,t_0)$.

This result was also essentially given in [14], and represents a considerable improvement from the computational point of view over that noted in [15]. It can also be derived by evaluating the variations in the signs of the last entry of each row of the Routh array, and relating their variation to the variation in the signs of the first entry of each row.

The leading coefficients of each row of the Routh array are related to the <u>Hurwitz determinants</u> [9]. The <u>Hurwitz matrix</u> is square, and defined by the coefficients of $f(x)$ and $f'(x)$ in (3-1) or (3-4) as

$$\begin{bmatrix} b_0 & b_1 & b_2 & \cdot & \cdot & \cdot & \cdot \\ a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot \\ 0 & b_0 & b_1 & b_2 & \cdot & \cdot & \cdot \\ 0 & a_0 & a_1 & a_2 & \cdot & \cdot & \cdot \\ 0 & 0 & b_0 & b_1 & \cdot & \cdot & \cdot \\ 0 & 0 & a_0 & a_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

It is of size nxn in case $f(x)$ is even and of degree n, and of size 2nx2n for a general $f(x)$ of degree n. An entry a_i or b_i is replaced by zero when there is no corresponding power of x in $f(x)$ or $f'(x)$.

The Hurwitz determinants Δ_i are the minors of the first i rows and columns of the Hurwitz matrix. That these minors are related to the leading entries of the Routh array is reasonably well-known, the calculation being set out in, for example, [9].

In fact,

$$\Delta_1 = b_0,\ \Delta_2 = b_0c_0,\ \Delta_3 = b_0c_0d_0,\ \text{etc.} \tag{3-6}$$

Thus we get readily analogies of Theorems 2 and 3, based on the obvious formulas

$$V(a_0,b_0,c_0,d_0,\ldots) = V(a_0,\Delta_1,\frac{\Delta_2}{\Delta_1},\frac{\Delta_3}{\Delta_2},\ldots) \tag{3-7}$$

and

$$\begin{aligned} &V(a_0,b_0)+V(c_0,d_0)+V(e_0,f_0)+\cdots\cdot \\ &= V(a_0,\Delta_1)+V(\Delta_1,\Delta_3)+V(\Delta_3,\Delta_5)+\cdots\cdot \\ &= V(a_0,\Delta_1,\Delta_3,\Delta_5,\ldots.) \end{aligned} \tag{3-8}$$

The simplest way from the computational point of view to check positivity of $f(x)$ is not to use a Hurwitz test, but the Sturm test, in modified Routh array format if desired.

The next group of tests we examine involve the characterization of root properties in terms of rank and signature properties of symmetric matrices. The first such test was linked by Hurwitz to the test bearing his own name in the original paper outlining the Hurwitz test, [10].

4. Markov-Parameter Test

Suppose given a real rational function, $g(x)/h(x)$, with $g(\cdot)$ and $h(\cdot)$ real polynomials. The Markov parameters are obtained in the following fashion. The function $g(x)/h(x)$ is written as a series in descending powers of x, e.g.,

$$\frac{g(x)}{h(x)} = s_{-(k+1)} x^k + \cdots + s_{-2}x + s_{-1} + \frac{s_0}{x} + \frac{s_1}{x^2} + \cdots \qquad (4\text{-}1)$$

The numbers s_i are known as the Markov parameters of the rational function $g(x)/h(x)$. Associated with the Markov parameters s_i, $i \geq 0$, is an infinite Hankel matrix whose i-j element is s_{i+j-2}. Thus,

$$\mathscr{H} = \begin{bmatrix} s_0 & s_1 & s_2 & \cdot & \cdot & \cdot \\ s_1 & s_2 & s_3 & \cdot & \cdot & \cdot \\ s_2 & s_3 & s_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad (4\text{-}2)$$

In [9], the following result is established:

Lemma A Let n be the degree of $h(x)$ when $g(x)$ and $h(x)$ are relatively prime real polynomials. Let $\mathscr{H}$ be the associated Hankel matrix formed from the Markov parameters of $g(x)/h(x)$. Then $\mathscr{H}$ has rank n. Conversely, if $\mathscr{H}$ has rank n, the series $\frac{s_0}{x} + \frac{s_1}{x^2} + \cdots$ is a real rational function of x, representable in the form $g(x)/h(x)$, $g(\cdot)$ and $h(\cdot)$ being relatively prime polynomials and $h(\cdot)$ of degree n.

Since the rationality of $\frac{g(x)}{h(x)}$ guarantees finite rank for $\mathscr{H}$, it makes sense to talk of the signature (the number of positive eigenvalues less the number of negative eigenvalues) of $\mathscr{H}$. This will be a finite number. In fact, there is a strong connection between the Cauchy index of $\frac{g(x)}{h(x)}$ and the eigenvalue properties of $\mathscr{H}$, [9]:

<u>Theorem 4</u> Let $g(x)/h(x)$ be a real rational function, $g(\cdot)$ and $h(\cdot)$ being relatively prime polynomials with $h(\cdot)$ of degree n. Let $\mathscr{H}$ be the associated Hankel matrix formed from the Markov parameters of $g(x)/h(x)$. Then $I_{-\infty}^{+\infty}\, g(x)/h(x)$ is equal to the signature of $\mathscr{H}$, and in fact the signature of $\mathscr{H}_{mm}$ for all $m \geq n$, where $\mathscr{H}_{mm}$ is the matrix consisting of the first m rows and columns of $\mathscr{H}$.

<u>Remark 5</u> The signature of $\mathscr{H}_{nn}$ is readily determined from the sign pattern of the leading principal minors, call them $D_1, D_2, \ldots, D_n$, of $\mathscr{H}_{nn}$. (Thus $D_1 = s_0$, $D_2 = \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}$, etc.). The rule is as follows, [9]. If all D_i are nonzero, then

$$I_{-\infty}^{+\infty} \frac{g(x)}{h(x)} = n - 2V(1, D_1, D_2, \ldots, D_n) \tag{4-3}$$

If among $D_1, \ldots, D_n$ there is a group of vanishing determinants $(D_h \neq 0)$ $D_{h+1} = \cdots = D_{h+p} = 0$, $(D_{h+p+1} \neq 0)$, then we can set

$$\begin{aligned} V(D_h, D_{h+1}, \ldots, D_{h+p+1}) &= \frac{p+1}{2} \text{ for p odd} \\ &= \frac{p+1-\varepsilon}{2} \text{ for even p} \end{aligned} \tag{4-4}$$

where

$$\varepsilon = (-1)^{1/2} \operatorname{sign} \frac{D_{h+p+1}}{D_h}$$

And now the way is clear towards using the Markov parameters for checking the number of distinct real zeros, and checking positivity of a polynomial $f(x)$. We identify $g(x)$ with $f'(x)$, $h(x)$ with $f(x)$.

Theorem 5 Let $f(x)$ be a polynomial of degree n. Let $\mathcal{H}$ be the Hankel matrix constructed from the Markov parameters of the rational function $f'(x)/f(x)$. Then $\mathcal{H}$ has rank n if and only if $f(x)$ has no repeated zeros and rank $n' < n$ otherwise, n' being the number of distinct zeros of $f(x)$. In both cases $\mathcal{H}_{nn}$ has signature equal to the number of (distinct) real roots of $f(x) = 0$, and the conditions for $f(x)$ to be positive for all x are that it be positive for one x and that the signature of $\mathcal{H}_{nn}$ be zero.

There is no inherent conceptual difficulty in extending Theorem 4 to cover the situation where $f(x)$ has repeated real roots, [9].

5. Hermite-Fujiwara Test

The Hermite-Fujiwara test is again a device for counting the real roots of a polynomial, but looks quite different from the earlier mentioned tests. The theorem of Fujiwara is as follows [11]:

Theorem 6 Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ be a real polynomial. Define $f^*(x) = (x-\xi)f'(x)$, for arbitrary but fixed real ξ with $f(\xi) \neq 0$, and form

$$K(f) = \frac{f(x)f^*(y) - f(y)f^*(x)}{x - y} = \sum_{i=1}^{n} c_{ij}x^{i-1}y^{j-1} \tag{5-1}$$

Then if $f(x) = 0$ does not possess multiple roots, the signature of the matrix $C = (c_{ij})$, which is symmetric, is equal to the number of real roots of $f(x) = 0$ greater than ξ, less the number less than ξ.

We now wish to improve on this theorem, to the extent of permitting multiple roots of $f(x)$, and counting all real roots, rather than those greater than ξ.

Accordingly, we define

$$K(f) = \frac{f(x)f'(y) - f(y)f'(x)}{x - y} \tag{5-2}$$

It is clear that $K(f)$ may be expressed as $\sum_{i,j=1}^{n} c_{ij}x^{i-1}y^{j-1}$ for some collection of coefficients c_{ij}. Moreover, since $K(f)$ is evidently symmetric in x and y, $C = (c_{ij})$ is also symmetric.

Lemma B If $f(x) = f_1(x)f_2(x)$, $f_1(x)$ and $f_2(x)$ being real polynomials, then

$$K(f) = f_1(x)f_1(y)K(f_2) + f_2(x)f_2(y)K(f_1) \tag{5-3}$$

This lemma is immediately verifiable by direct calculation. From it, we deduce the following important consequence.

Lemma C Let $f(x) = f_1(x)f_2(x)$ be a factoring of all real polynomials $f(x)$ of degree n into real polynomials $f_1(x)$ and $f_2(x)$ of degrees p and $q = n-p$. Define matrices $C^{(1)} = (c_{ij}^{(1)})$ and $C^{(2)} = (c_{ij}^{(2)})$ by

$$K(f_1) = \sum_{i,j=1}^{p} c_{ij}^{(1)} x^{i-1}y^{j-1} \qquad K(f_2) = \sum_{i,j=1}^{q} c_{ij}^{(2)} x^{i-1}y^{j-1} \tag{5-4}$$

Then if $f_1(x)$ and $f_2(x)$ are relatively prime, and $r(\cdot)$ and $\sigma(\cdot)$ are abbreviations for "rank of" and "signature of",

$$\begin{aligned} r(C) &= r(C^{(1)}) + r(C^{(2)}) \\ \sigma(C) &= \sigma(C^{(1)}) + \sigma(C^{(2)}) \end{aligned} \tag{5-5}$$

Proof Suppose $f_1(x) = a_0^{(1)}x^p + a_1^{(1)}x^{p-1} + \cdots + a_p^{(1)}$ and $f_2(x) = a_0^{(2)}x^q + \cdots + a_q^{(2)}$. Then the result of lemma B yields

$$\begin{aligned} \sum_{i,j=1}^{n} c_{ij}x^{i-1}y^{j-1} = &\sum_{i,j=1}^{q} c_{ij}^{(2)}(a_0^{(1)}x^{p+i-1}+\cdots+a_p^{(1)}x^{i-1}) \\ &\cdot (a_0^{(1)}y^{p+j-1}+\cdots+a_p^{(1)}y^{j-1}) \\ &+ \sum_{i,j=1}^{p} c_{ij}^{(1)}(a_0^{(2)}x^{q+i-1}+\cdots+a_q^{(2)}x^{i-1}) \\ &\cdot (a_0^{(2)}y^{q+j-1}+\cdots+a_q^{(2)}y^{j-1}) \end{aligned} \tag{5-6}$$

Now let $u_1,\ldots,u_n$ be a set of independent real variables, and set

$$w_i = a_0^{(1)}u_{i+p}+a_1^{(1)}u_{i+p-1}+\cdots+a_p^{(1)}u_i \qquad i = 1,2,\ldots,q \tag{5-7a}$$

and

$$w_{i+q} = a_0^{(2)}u_{i+q}+a_1^{(2)}u_{i+q-1}+\cdots+a_q^{(2)}u_i \qquad i = 1,2,\ldots,p \tag{5-7b}$$

With a superscript T denoting vector transposition, we define the vectors $\hat{u}$ and $\hat{w}$ by $\hat{u}^T=[u_1,u_2,\ldots,u_n]$ and $\hat{w}$ similarly. We see then from (5-6) and the definition of the w_i, eq. (5-7), that

$$\hat{u}^T C \hat{u} = \hat{w}^T \begin{bmatrix} C^{(1)} & 0 \\ 0 & C^{(2)} \end{bmatrix} \hat{w} \tag{5-8}$$

The lemma conclusion follows if the defining relations for $\hat{w}$ are invertible. That they are follows from the fact that the determinant of the coefficients of the transformation yielding the w_i from the u_i is precisely the resultant of the polynomials f_1 and f_2 [11], and this is nonzero because these two polynomials have no zero in common. Δ

Corollary Suppose $f_1(x)$ and $f_2(x)$ are arbitrary real polynomials of degree p and q. Suppose that $K(f_1) = \sum_{i,j=1}^{p} c_{ij}^{(1)} x^{i-1} y^{j-1}$ and that $f_2(x) f_2(y) K(f_1) = \sum_{i,j=1}^{p+q} d_{ij} x^{i-1} y^{j-1}$. Then with $C^{(1)} = (c_{ij}^{(1)})$ and $D = (d_{ij})$, $r(C^{(1)}) = r(D)$ and $\sigma(C^{(1)}) = \sigma(D)$.

The proof of this corollary is essentially embedded within the above lemma proof.

Now we can state and prove the following theorem.

Theorem 7 Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ be a real polynomial, and define $K(f) = (x-y)^{-1}[f(x)f'(y) - f(y)f'(x)] = \sum_{i,j=1}^{n} c_{ij} x^{i-1} y^{j-1}$. With $C = (c_{ij})$, $r(C)$ is the number of distinct zeros, real and complex, of $f(x)$, and $\sigma(C)$ is the number of distinct real zeros of $f(x)$

Proof The proof is analogous to the proof of Theorem 6 published in [11]. Write

$$f(x) = \prod_k (x-\alpha_k)^{\nu_k} \prod_\ell [(x-\alpha_\ell)^2 + \beta_\ell^2]^{\nu_\ell} \qquad (\beta_\ell \neq 0) \tag{5-9}$$

By Lemma C, it is sufficient to prove that with $f_1 = (x-\alpha_k)^{\nu_k}$, rank $C^{(1)} = 1$ and signature $C^{(1)} = 1$, and that with $f_2 = [(x-\alpha_\ell)^2 + \beta_\ell^2]^{\nu_\ell}$, rank $C^{(2)} = 2$, and signature $C^{(2)} = 0$. This we now do.

It is straightforward to calculate that

$$K(f_1) = \nu_k(x-\alpha_k)^{\nu_k-1}(y-\alpha_k)^{\nu_k-1} \tag{5-10}$$

Now apply the corollary above, taking f_1 of the corollary to be $(x-\alpha_k)^{\nu_k-1}$. Evidently $r(C^{(1)}) = \sigma(C^{(1)}) = 1$.

Second, with $f_2 = [(x-\alpha_\ell)^2 + \beta_\ell^2]^{\nu_\ell}$, we find easily that

$$K(f_2) = 2\nu_\ell[(x-\alpha_\ell)^2+\beta_\ell^2]^{\nu_\ell-1}[(y-\alpha_\ell)^2+\beta_\ell^2]^{\nu_\ell-1}[(x-\alpha_\ell)(y-\alpha_\ell)-\beta_\ell^2] \tag{5-11}$$

By the corollary again, it is sufficient to consider simply $[(x-\alpha_\ell)(y-\alpha_\ell)-\beta^2_\ell] = xy-\alpha_\ell x-\alpha_\ell y+(\alpha_\ell^2-\beta_\ell^2)$. Direct calculation yields that the matrix

$$\begin{bmatrix} \alpha_\ell^2-\beta_\ell^2 & -\alpha_\ell \\ -\alpha_\ell & 1 \end{bmatrix}$$

has rank 2 and signature zero. ▲

With $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, the matrix $C = (c_{ij})$ such that $K(f) = \sum_{i,j=1}^{n} c_{ij}x^{i-1}y^{i-1}$ can be expressed in terms of the a_i. The easiest way to do this is to note that the coefficient of x^iy^j in $f(x)f'(y)-f(y)f'(x)$ is $(j+1)a_{n-1}a_{n-j-1}-(i+1)a_{n-i-1}a_{n-j}$. In $(x-y)\sum_{i,j=1}^{n} c_{ij}x^{i-1}y^{j-1}$, the coefficient of x^iy^j is $c_{ij+1}-c_{i+1\,j}$. Hence,

$$(j+1)a_{n-1}a_{n-j-1}-(i+1)a_{n-i-1}a_{n-j} = c_{ij+1}-c_{i+1\,j}$$

from which

$$c_{ij} = \sum_{k\geq 0}[(j-k)a_{n-i-k}a_{n-j+k}-(i+k+1)a_{n-i-k-1}a_{n-j+k+1}] \tag{5-12}$$

where $a_m = 0$ for $m < 0$, $m > n$.

Since the polynomial $f(x)$ will be positive for all x if and only if it is positive for one x and has no real zero, we have the following special version of Theorem 7:

Theorem 8 Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ be a real polynomial, and define $C = (c_{ij})$ by equation (5-12). Then $f(x)$ is positive for all x if and only if $f(x)$ is positive for one x, and the symmetric matix C has signature zero.

Remark 6 In contrast to the Sturm-Routh test and the Markov test, only integral, as distinct from rational, functions of the coefficients of the polynomial $f(x)$ need be constructed to apply the test.

In the remainder of this section, we shall indicate a simple connection between the Hermite-Fujiwara and Markov parameter test. Though each of these tests can be established separately, this connection does enable either test to be derived from the other. A little more than this is true however. The checking of the signature of the matrix C is straightforwardly done if all principal leading minors are nonzero, but if some are zero this is not the case. In the case of the Markov parameter test, precisely because the matrix whose signature is being examined is a submatrix of a Hankel matrix, the signature can still, with some minors zero, be calculated by a special rule, [9]. But this rule for dealing with the case when some minors are zero cannot a priori be applied to C, leaving us for the moment without a test based on leading minors for computing signature. The connection exhibited below, however, yields a test for C which permits leading minors of zero.

Into the formula

$$\sum_{i,j=1}^{n} c_{ij}x^{i-1}y^{j-1} = \frac{f(x)f'(y)-f(y)f'(x)}{x-y} \tag{5-13}$$

we shall substitute

$$f'(x) = f(x)\left[\frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \cdots\right] \qquad (5\text{-}14)$$

where, of course, the s_i are Markov parameters associated with $f'(x)/f(x)$. We have then

$$\Sigma\, c_{ij}x^{i-1}y^{j-1} = \frac{f(x)f(y)}{xy}\,\frac{1}{\frac{1}{y}-\frac{1}{x}}\left[s_0\left(\frac{1}{y}-\frac{1}{x}\right)+s_1\left(\frac{1}{y^2}-\frac{1}{x^2}\right)+s_2\left(\frac{1}{y^3}-\frac{1}{x^3}\right)+\cdots\right]$$

$$= \frac{f(x)f(y)}{xy}\left[s_0 + s_1\left(\frac{1}{y}+\frac{1}{x}\right) + s_2\left(\frac{1}{y^2}+\frac{1}{yx}+\frac{1}{x^2}\right)+\cdots\right]$$

$$= f(x)\left[\frac{1}{x}\;\frac{1}{x^2}\;\frac{1}{x^3}\cdots\right]\mathscr{H}\left[\frac{1}{y}\;\frac{1}{y^2}\;\frac{1}{y^3}\cdots\right]^T f(y) \qquad (5\text{-}15)$$

$$= [f_1(x)+r_1(x)\;\; f_2(x)+r_2(x)\cdots]\;\mathscr{H}[f_1(y)+r_1(y)\;\; f_2(y)+r_2(y)\cdots]^T$$

where the $f_i(x)$ are polynomials in x and the $r_i(x)$ polynomials in $\frac{1}{x}$, the latter with no constant term, and all formed in the obvious way. Thus with

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

$$f_1(x) = a_0x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1} \;;\; r_1(x) = \frac{a_n}{x}.$$

$$f_2(x) = a_0x^{n-2} + a_1x^{n-3} + \cdots + a_{n-2} \;;\; r_2(x) = \frac{a_{n-1}}{x} + \frac{a_n}{x^2}$$

and so on. Observe that $f_{n+1}(x) = 0$ for all $i > 0$.

Now since the left hand side of (5-15) has no terms in $\frac{1}{x}$ or $\frac{1}{y}$, it must be true that

$$\Sigma c_{ij}x^{i-1}y^{j-1} = [f_1(x)\cdots f_n(x)\quad 0\cdots 0\cdots]\;\mathscr{H}[f_1(y)\cdots f_n(y)\quad 0\cdots 0\cdots]^T$$

$$= [f_1(x)\cdots f_n(x)]\;\mathscr{H}_{nn}[f_1(y)\cdots f_n(y)]^T$$

It is readily verified that

$$[f_1(x)\dots f_n(x)] = [1 \quad x\dots x^{n-1}] \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ a_{n-2} & a_{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_0 & \cdots & 0 & 0 \\ a_0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and so with

$$A = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ a_{n-2} & a_{n-3} & \cdots & a_0 & \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & a_0 & \cdots & 0 & 0 \\ a_0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

we have

$$C = A' \mathcal{H}_{nn} A \tag{5-16}$$

Thus the rank and signature of C are the same as the rank and signature of $\mathcal{H}_{nn}$. Theorems 5 and 7 then imply each other. Whilst it is not immediately possible to relate leading principal minors of C and $\mathcal{H}_{nn}$ especially simply, we may proceed as follows: Define the matrix $D = (d_{ij})$ through

$$\sum_{i,j=1}^{n} c_{ij} x^{i-1} y^{j-1} = \sum_{i,j=1}^{n} d_{ij} x^{n-i} y^{n-j} \tag{5-17}$$

Thus D is C with its row and column ordering reversed, and certainly has the same rank and signature C. Then

$$D = \hat{A}' \mathcal{H}_{nn} \hat{A} \tag{5-18}$$

where

$$\hat{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ 0 & 0 & \cdots & a_{n-3} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & a_0 \end{bmatrix}$$

6. The Linear System Theory Viewpoint

Again we are interested in determining the number of real roots of $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, and determining conditions for the positivity of this polynomial. We shall make use of the language of linear system theory, see for example [16], to discuss the problem.

We form the transfer function

$$w(s) = \frac{f'(s)}{f(s)} \tag{6-1}$$

and construct for $w(s)$ a realization [16] completely controllable $\{F,g,h\}$ with F of size n, that is, an $n\times n$ matrix F and n-vectors g and h such that

$$w(s) = h^T(sI - F)^{-1}g \tag{6-2}$$

and

$$\text{rank } [g \;\; Fg \;\cdots\; F^{n-1}g] = n \tag{6-3}$$

One such triple is

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ & & & & \vdots \\ & & & & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-2}}{a_0} & & -\frac{a_1}{a_0} \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad h = \begin{bmatrix} \frac{a_{n-1}}{a_0} \\ \frac{2a_{n-2}}{a_0} \\ \vdots \\ \frac{(n-1)a_1}{a_0} \\ n \end{bmatrix} \tag{6-4}$$

The theorem we shall prove is the following:

Theorem 9 Let $f(s) = a_0s^n + a_1s^{n-1} + \cdots + a_n$ and let $w(s) = f'(s)/f(s)$ have a completely controllable n-dimensional realization $\{F,g,h\}$. Then there exists a unique symmetric matrix P such that and now the $\underline{\nu}$th leading principal minor of D is equal to $a_0^{2\nu}$ times the $\underline{\nu}$th leading minor of $\mathcal{H}_{nn}$. Accordingly, we have the same rule for computing the signature of C as we have for $\mathcal{H}_{nn}$, provided that we work with principal minors which are the opposite of leading, i.e.,

$$c_{nn}, \quad \begin{vmatrix} c_{n-1,n-1} & c_{n-1,n} \\ c_{n-1,n} & c_{nn} \end{vmatrix}, \text{ etc.}$$

$$PF = F^T P \qquad Pg = h \tag{6-5}$$

an the number of distinct zeros of $f(s)$ is rank P and the number of distinct real zeros of $f(s)$ is the signature of P. Further, P is computable from F, g and h via algebraic operations which exclude polynomial factoring or its equivalent.

Proof We shall first define a matrix P, prove its symmetry, next the fact that it satisfies (6-5), and then its uniqueness. Then we shall establish the properties claimed for its rank and signature. Computability of P will be evident from the proof.

Define $V = [g \; Fg \; \cdots \; F^{n-1}g]$, $W = [h \; F^T h \; \cdots \; (F^T)^{n-1}h]$ and $P = WV^{-1}$. Then $PV = W$ and $V^T PV = V^T W$. The i-j term of $V^T W$ is $g^T (F^{i-1})^T (F^T)^{j-1} h = h^T F^{i+j-2} g$, and thus symmetry holds. That $Pg = h$ follows by equating the first column of PV and of W. From $PV = W$ and the Cayley-Hamilton theorem it follows that $Pe^{Ft}g = e^{F^T t}h$. Therefore $F^T P e^{Ft} g = F^T e^{F^T t} h = \frac{d}{dt}(e^{F^T t}h)$. On the other hand, $\frac{d}{dt}(e^{F^T t}h) = \frac{d}{dt}(Pe^{Ft}g) = PFe^{Ft}g$. Hence, $(F^T P - PF)e^{Ft}g = 0$, and $F^T P = PF$ follows by complete controllability.

To see that P is unique, we start with (6-5), and observe that $PFg = F^T Pg = F^T h$, $PF^2 g = (F^T)^2 Pg = (F^T)^2 h$ and so on, so that $PV = W$. Since V is nonsingular, P is uniquely determined.

To establish that the rank and signature of P have the properties claimed, we shall prove the claimed properties for the matrix

$$V^T PV = V^T W \tag{6-6}$$

whose i-j term is $h^T F^{i+j-2} g$. Now it is straightforward to establish from (6-2) that

$$w(s) = \frac{h^T g}{s} + \frac{h^T F g}{s^2} + \frac{h^T F^2 g}{s^3} + \cdots$$

so that the Markov parameter s_i is $h^T F^i g$ and

$$V^T W = \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ \vdots & & & \\ s_{n-1} & \cdots & & s_{2n-2} \end{bmatrix} \tag{6-7}$$

This is none other than the matrix $\mathscr{H}_{nn}$, and Theorem 5 yields the desired result. ▲

An alternative proof of this theorem, not making use of earlier results, can be found by writing w(s) in a partial fraction expansion, constructing a completely controllable realization for each partial fraction set with the same poles, establishing a P matrix of readily computable rank and signature for each such realization, and then taking a direct sum of the P matrices and the realizations. We leave the details to the reader.

We can make an interesting connection with the Hermite-Fujiwara test by showing that for a particular completely controllable realization of w(s)--in fact that quoted in (6-4)--the matrix P is to within a positive constant, the matrix C of the Hermite--Fujiwara test. Let us note

Lemma D Let F, g, and h be as in (6-4), and let $\Pi(s) = [1 \; s \; s^2 \cdots s^{n-1}]^T$. Then $F\,\Pi(s) = s\,\Pi(s) \bmod f(s)$ and $h^T\Pi(s) = \dfrac{f'(s)}{a_0}$.

Proof is immediate by direct calculation.

Theorem 10 Let F, g, h be as given in (6-4), and P as given in the statement of Theorem 9. Let C be as defined in Theorem 7. Then $C = a_0^2 P$.

Proof We have

$$\frac{f(x)f'(y)-f(y)f'(x)}{x-y} = \sum_{i,j=1}^{n-1} c_{ij}x^{i-1}y^{j-1}$$

$$= \Pi^T(x)C\Pi(y)$$

Hence,

$$f(x)f'(y)-f(y)f'(x) = x\Pi^T(x)C\Pi(y)-\Pi^T(x)C\Pi(y)y \qquad (6\text{-}8)$$

Now reduce both sides mod $f(x)$ and then mod $f(y)$. The left hand side becomes zero, while the right hand side becomes such that Lemma D applies. We obtain

$$\Pi^T(x)(F^TC-CF)\Pi(y) = 0$$

Since this holds for all x and y,

$$F^TC = CF \qquad (6\text{-}9)$$

Next, observe that in equation (6-8), the coefficient of y^n on the left hand side is $-a_0f'(x) = -a_0^2h^T\Pi(x)$ by Lemma D. The coefficient on the right side is evidently $-\Pi^T(x)Cg$. Consequently

$$\Pi^T(x)[Cg - a_0^2h] = 0$$

whence

$$Cg = a_0^2h \qquad (6\text{-}10)$$

It is now immediate from (6-5), (6-9) and (6-10), and the uniqueness property of P that $C = a_0^2P$.

This result is analogous to some recently established connections between tests for checking stability, that is, for checking whether all zeros of a prescribed polynomial $f(x)$ have negative real parts. Suppose as before that $f_1(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$; let us take $a_0 = 1$ for convenience. Define the polynomial $f^*(x)$ by $f^*(x) = (-1)^nx^n + (-1)^{n-1}a_1x^{n-1} + \cdots + a_n$, and set

$$K(f) = \frac{f(x)f^*(y) - f(y)f^*(x)}{x - y} = \sum_{i,j=1}^{n} c_{ij}x^{i-1}y^{j-1}$$

Then $C = (c_{ij})$ is termed the Hermite matrix, and is positive definite if and only if $f(x)$ has all zeros with negative real part [11].

A second approach to checking stability is to find an nxn matrix F with $\det(Ix - F) = f(x)$, together with an n-vector h such that $[F^T,h]$ is completely controllable. Then [16], $f(x)$ has all roots with negative real parts *if and only if there is a* positive definite P such that

$$PF + F^TP = -hh'$$

In [17], it was shown that a choice of F as in (6-4) and a special h led to P being the matrix C, at least to within a reordering of the rows and columns.

This is still not really a parallel. But then in [18] it was shown that there was a triple $[\hat{F},\hat{g},\hat{h}]$, readily constructible from $f(x)$, such that the P just referred to satisfied

$$P\hat{F} = -\hat{F}^TP$$

$$P\hat{g} = \hat{h}$$

The connection between this $\hat{F}$, $\hat{g}$, $\hat{h}$ and the F and h of [17] was simply $h = \hat{h}$, $\hat{g} = [0 \quad 0\cdots0 \quad 1]^T$ and $F = \hat{F}-\hat{g}\hat{h}^T$. There is a good deal of significance to the triple $\hat{F}$, $\hat{g}$, $\hat{h}$ other than that mentioned here which is discussed in [18].

7. Conclusions

The preceding tests really fall into two groups, one consisting of the Sturm test, the Sturm test in Routh array format, and the Hurwitz test. The other group of tests distinguish themselves in that symmetric matrices are involved, and the rank and signature of these matrices carry the same sort of information that the relevant coefficients in the earlier tests carry.

From the computational point of view, the Sturm test in Routh array format is probably the most straightforward, save for the possible complication of dealing with nonnormal arrays, i.e., ones which have fewer rows than normal, or do not have nonzero elements at the start of each row of the array. Of the three symmetric matrix based tests, perhaps the Hermite-Fujiwara has a slight edge, since only integral functions of the polynomial coefficients are involved, and the procedure for handling nonnormal cases is straightforward.

From the theoretical point of view, the Sturm test, and the tests depending on symmetric matrices must be viewed as the most important.

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