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A QUADRATIC PERFORMANCE INDEX MAXIMIZATION PROBLEM

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1. STATEMENT OF THE PROBLEM

The usual quadratic loss optimization problem requires the minimization of a performance index of the form

$$V(x_0,u(\cdot)) = \int_0^T [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt + x'(T)Ax(T) \tag{1}$$

for a linear system

$$\dot{x} = F(t)x + G(t)u \tag{2}$$

The matrices Q, R, F and G are assumed to have continuous entries, and u may be a vector. With an avowed aim of achieving a regulator design, $Q(t)$ and A are taken as nonnegative definite or even positive definite. In order to limit the magnitude of $u(\cdot)$, the matrix $R(t)$ is taken to be positive definite Here, we consider instead a maximization problem; more precisely we seek to maximize

$$V(u(\cdot)) = \int_0^T x'(t)Q(t)x(t)dt + x'(T)Ax(T) \tag{3}$$

where again $Q(t)$ and A are nonnegative or positive definite and $Q(t)$ has continuous entries. As before, equation (2) describes the system under consideration. We impose a limit on $u(\cdot)$ by demanding that

$$\int_0^T u'(t)u(t)dt \le k \tag{4}$$

for some constant k. Finally, we restrict consideration to the case $x(0) = 0$.

An example of such an optimization problem might arise in the following situation. Consider a motor driving an electromechanical system, including a flywheel; with an upper constraint on the amount of energy supplied to the system over a certain interval, it is desired to maximize the speed of the flywheel at the end of the interval, assuming the system to be initially at rest. In this case, we would have $Q = 0$, and A would consist of a diagonal matrix with but the one nonzero entry in that row and column corresponding to that entry of the state vector representing flywheel speed.

2. STATEMENT OF SOLUTION IN TERMS OF EIGENVALUES AND EIGENFUNCTIONS

It turns out [1] that the performance index (3) can be written in the form

$$V(u(\cdot)) = \int_0^T \int_0^T u'(t)[K(t,\tau) + L(t,\tau)]u(\tau)dtd\tau \tag{5}$$

where $K(\cdot,\cdot)$ and $L(\cdot,\cdot)$ are integral kernels, both nonnegative definite, and given by

$$K(t,\tau) = G'(t)\int_t^T \phi'(\lambda,t)Q(\lambda)\phi(\lambda,\tau)d\lambda G(\tau)1(t-\tau)$$

$$+ G'(t) \int_\tau^T \phi'(\lambda,t)Q(\lambda)\phi(\lambda,\tau)d\lambda G(\tau)1(\tau-t) \tag{6a}$$

and

$$L(t,\tau) = G'(t)\phi'(T,t)A\phi(T,\tau)G(\tau) \tag{6b}$$

The maximization of (5) subject to an integral squared constraint on $u(\cdot)$ is a standard problem considered in functional analysis and integral equation theory. The kernel $K(\cdot,\cdot) + L(\cdot,\cdot)$ possesses a denumerable set of nonnegative real eigenvalues λ_i, with $\infty > \lambda_1 \geq \lambda_2 \geq .. \geq 0$. The maximum value of (5) is determined by λ_1 and (4) as $k\lambda_1$. Moreover, the maximizing $u(\cdot)$, call it $u_1(\cdot)$, satisfies (4) with equality as well as the eigenvalue equation

$$\lambda_1 u_1(t) = \int_0^T [K(t,\tau) + L(t,\tau)]u_1(\tau)d\tau \tag{7}$$

Formally, this completes the solution of the optimization problem. But the fact that $K(\cdot,\cdot)$ and $L(\cdot,\cdot)$ have the special structures of equation (6) leads to a number of interesting properties of the eigenvalue-eigenfunction system associated with $K(\cdot,\cdot)$ and $L(\cdot,\cdot)$. These properties can be used to assist in the numerical computation of λ_1 and $u_1(\cdot)$.

3. PROPERTIES OF THE EIGENFUNCTIONS AND EIGENVALUES OF THE INTEGRAL KERNEL K+L

It turns out [1] that eigenfunction and eigenvalue properties of $K(\cdot,\cdot) + L(\cdot,\cdot)$ may be related to properties of the solution of the linear differential equation set

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} F & -\frac{GG'}{\lambda} \\ Q & -F' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{8}$$

wherein we suppose that λ is an arbitrary but fixed parameter. The transition matrix of (8) will be denoted by $\Psi(\cdot,\cdot;\lambda)$ and will be partitioned as

$$\Psi(t,\tau;\lambda) = \begin{bmatrix} \Psi_{11}(t,\tau;\lambda) & \Psi_{12}(t,\tau;\lambda) \\ \Psi_{21}(t,\tau;\lambda) & \Psi_{22}(t,\tau;\lambda) \end{bmatrix} \tag{9}$$

The first property is:

1. λ is an eigenvalue of $K(t,\tau) + L(t,\tau)$ if and only if $\Psi_{11}(0,T;\lambda) - \Psi_{12}(0,T;\lambda)A$ is singular.

This property gives a testing procedure, for checking whether a prescribed λ is an eigenvalue. However, of itself it says nothing about actually determining an eigenvalue λ, or the maximum eigenvalue λ_1.

The second property is:

2. If λ is an eigenvalue and if x_T is in the nullspace of $\Psi_{11}(0,T;\lambda) - \Psi_{12}(0,T;\lambda)A$, an associated eigenfunction of $K(\cdot,\cdot) + L(\cdot,\cdot)$ is defined by $-\frac{1}{\lambda}G'(t)p(t)$, where $p(t)$ is derived from (8) with boundary condition $x(T) = x_T$, $p(T) = -Ax_T$.

This property provides a simple way of computing an eigenfunction once the eigenvalue is known.

What is still required is some computational procedure for finding λ_1.

4. COMPUTATIONAL PROCEDURE FOR THE MAXIMUM EIGENVALUE

Let us temporarily replace the lower bound of 0 on the integral in the performance index (3) by t_0. Then λ_1 becomes a function of t_0. In reference [1] the following properties are established:

1. $\lambda_1(t_0)$ is a monotonically decreasing continuous function with $\lambda_1(T) = 0$

and

2. Let λ be arbitrary, and let $t_0 < T$ be the value of t closest to T for which $\Psi_{11}(t,T;\lambda) - \Psi_{12}(t,T;\lambda)A$ is singular. Then $\lambda_1(t_0) = \lambda$.

These properties suggest that λ_1 can be found as follows: guess a value for λ_1, call this guess λ, and find t_0 as the value of t closest to T for which $\Psi_{11}(t,T;\lambda) - \Psi_{12}(t,T;\lambda)A$ is singular. If $t_0 > 0$, guess a new λ larger than the earlier, while if $t_0 < 0$ guess a λ smaller than the first guess. (Of course, the step chosen in λ can be reasonably estimated at each stage of the iteration). Continue this procedure until $t_0 \approx 0$.

As an alternative to checking the singularity or otherwise of $\Psi_{11}(t,T;\lambda) - \Psi_{12}(t,T;\lambda)A$ for various t in order to determine t_0, it proves possible to use a different procedure, perhaps more convenient for computation. If the solution $P(\cdot)$ of

$$-\dot{P} = PF + F'P - PG\lambda^{-1}G'P - Q \qquad (10)$$

$$P(T) = -A$$

is computed backwards from T, t_0 is the escape time of this equation, [1]. Therefore, (10) is solved backwards in time for a fixed λ and its escape time t_0 determined. Then, as before, $\lambda = \lambda_1(t_0)$.

[1] B.D.O. Anderson, "A Quadratic Performance Index Maximization Problem," Technical Report EE-6907, University of Newcastle, August, 1969.

[2] A. N. Kolmogorov and S. V. Fomin, "Functional Analysis," Vols. 1 and 2, Graylock, Albany, New York, 1961.