THE TRACKING PROBLEM: A COMPARISON OF THE MODERN AND CLASSICAL APPROACHES

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Abstract

Two distinct formulations of the optimum tracking problem have been presented in the literature. The classical formulation is in frequency domain terms, assumes stationary and stability of the underlying systems, and assumes zero initial conditions; the optimal control is given in open-loop form via a frequency domain formula, and its evaluation requires the extraction of spectral factors with certain analyticity properties, and expansion in partial fractions with subsequent deletion of some of these partial fractions. The modern formulation is in time domain terms, does not assume zero initial conditions, but normally assumes finite-dimensionality of the underlying systems. The optimal control is given in time domain terms and partly in feedback form. It is computed by solving a matrix Riccati differential equation or a quadratic matrix equation. By posing a tracking problem with constraints permitting solution via both techniques, the two techniques are shown to yield the same control law.

The study of the tracking problem via two techniques suggests technical reasons why the classical procedures are not as effective as they might be.

1. INTRODUCTION

The first serious attempts at formulating and solving quadratic optimization problems are well summed up in the books of Newton, Gould and Kaiser (1) and Chang(2). These books took a frequency domain point of view, and solutions to the various optimization problems formulated were normally achieved using the idea of polynomial spectral factorization. Subsequently, the work of Kalman (3), (4) appeared which generalized the earlier results. More specifically, the restriction on the time-invariance of the underlying systems was removed, and the restriction that the systems considered be initially in the zero state was removed. As it turns out though in retrospect, the fact that the original design procedures only apparently worked for zero initial states is not so important, since implementation of the resulting optimal control in feedback form serves to handle the non-zero initial state case equally well. Further, although the modern approach would appear superior, it should also be noted in fairness to the authors of (1) and (2), that the approaches of (3) and (4) are more restrictive in the sense that they assume the underlying systems to be finite-dimensional, whereas (1) and (2) do not make this assumption.

The purpose of this paper is to interrelate the approaches of (1) and (2) with that of (3) for the tracking problem, where with $u(\cdot)$ the input to the system and $e(\cdot)$ amount by which its output leviates from a certain desired output, the aim is to choose $u(\cdot)$ to minimize an index of the type

$$V = \int_0^\infty (\alpha e^2(t) + u^2(t))dt \qquad (\alpha > 0) \qquad (1)$$

The precise nature of the underlying systems for which the interrelation is established will be discussed subsequently.

In (5), regulator designs via both techniques were related, the regulator problem being the special case of the tracking problem with desired output zero. In making the connection between designs via both techniques, it was shown that for a special initial state vector of the underlying system, the two designs yielded the same controller. Then, by an argument depending on the existence of a special coordinate representation of the state-space equations of the underlying system, the restriction on the initial state-vector was removed. This technique is inherently very difficult to extend to multiple input systems, and in any case could with conceptual advantage be replaced by a more direct argument. This is done implicitly in this paper where the regulator problem appears as a special case in our approach to the general tracking problem.

2. REVIEW OF THE TWO TRACKING PROBLEM FORMULATIONS AND SOLUTIONS

The classical version of the optimum tracking problem proceeds as follows.

We are given $G(s)$, the transfer function of an asymptotically stable plant, whose input and output we denote in the time domain by $u(t)$ and $y(t)$, and in the Laplace-transform domain by $U(s)$ and $Y(s)$. We are also given a reference signal $r(t)$, with Laplace transform $R(s)$. The aim is to choose a transfer function $G_c(s)$ of a compensator, such that with the compensator driven by $R(s)$ and driving the plant, i.e. with

$$U(s) = G_c(s)R(s) \tag{2}$$

the following integral is minimized:

$$\int_0^\infty \{\alpha[y(t) - r(t)]^2 + [u(t)]^2\}dt \tag{3}$$

where α is a positive constant. It is tacitly assumed that the states of plant and compensator are zero at time $t = 0$.

Figure 1 shows the control scheme.

The optimum compensator is given by the formula[1], [2]

$$G_c(s) = \frac{\alpha}{R(1+\alpha G\bar{G})^+}\left[\frac{R\bar{G}}{(1+\alpha G\bar{G})^-}\right]_+ \tag{4}$$

In this as in later expressions, the superscript bar denotes paraconjugation, i.e. $\bar{G}(s) = G(-s)$; the superscript + around parentheses denotes the operation of factoring the enclosed function, assumed rational in s, and eliminating (i.e. replacing by unity) those numerator and denominator factors with right half plane zeros; finally, the subscript + around square brackets denotes the operation of performing a partial fraction expansion and discarding (replacing by zero) those partial fractions with right half plane poles, it being tacitly assumed that this operation is applied to a rational function of s whose denominator polynomial has higher degree than the numerator polynomial.

From the point of view of achieving optimality, many other control schemes are possible. For example, if in Figure 2 $G_1(s)$ and $G_2(s)$ are chosen so that

$$G_c(s) = \frac{G_1(s)}{1+G(s)G_2(s)} \tag{5}$$

optimality will be achieved, because both schemes yield the same relation between $R(s)$ and $U(s)$, and $R(s)$ and $Y(s)$.

The common feature of all possible schemes is that the same $U(s)$ must be used in every case; we see from (2) and (4) that the optimal $U(s)$ is given by

$$U(s) = \frac{\alpha}{(1+\alpha G\bar{G})^+}\left[\frac{R\bar{G}}{(1+\alpha G\bar{G})^-}\right]_+ \tag{6}$$

By contrast, the modern version of the optimum tracking problem proceeds as follows, [3,4].

We are given a set of state-space equations describing a plant

$$\dot{x} = Fx + gu \qquad y = h'x \tag{7}$$

with an initial state x_0. Here x is an n-vector, F an $n \times n$ matrix and g and h are n-vectors. Without loss of generality, the pair $[F,g]$ is assumed completely controllable and $[F,h]$ completely observable. We do <u>not</u> in general assume that F is asymptotically stable. We are also given a reference signal $r(\cdot)$, and the aim is to choose a control $u(\cdot)$ such that the following integral is minimized:

$$\int_0^\infty \{\alpha[h'x - r(t)]^2 + [u(t)]^2\}dt \tag{8}$$

The solution to this problem is obtained in the following way, [3,6]. Let Π be the unique, positive definite matrix (known to exist) satisfying

$$\Pi F + F'\Pi - \Pi gg'\Pi + \alpha hh' = 0 \tag{9}$$

Let

$$k = \Pi g \tag{10}$$

Then the optimal $u(\cdot)$ is given by

$$u(t) = -k'x(t) + g'\lim_{T\to\infty}\int_t^T e^{(F-gk')'(\tau-t)}hr(\tau)d\tau \tag{11}$$

assuming the limit exists. The control arrangement is shown in Figure 3, where $v(t)$ is shorthand for the second term on the right side of (11).

3. FORMAL STATEMENT OF THE PROBLEM

Before meaningful comparison can be made between the two tracking system designs, we must ensure that the two design procedures are being applied to the same situation. This means that the plants under consideration must be the same, the reference signals must be the same, and the same value of α must be used in both performance indices, (3) and (8). The latter two requirements are satisfied by fiat, and the first requirement by insisting that

$$G(s) = h'(sI-F)^{-1}g \tag{12}$$

The classical solution procedure requires that $G(s)$ have all its poles in $\mathrm{Re}[s] < 0$ (although there is no corresponding requirement in the modern procedure), and therefore we make the assumption that the eigenvalues of F all have negative real part.

Furthermore, we must also find a technique for dealing with nonzero initial conditions on the plant using the frequency domain design approach. This is in fact quite easy,[5]. Figure 4 illustrates that a plant of transfer function $G(s)$ with a nonzero initial condition is equivalent to a plant of transfer function $G(s)$ with a zero initial condition and with an additive signal at the output (depicted in the Figure by its Laplace transform $Y_0(s)$), which is the zero-input response of the plant.

There is a restriction on $Y_0(s)$ because it is a zero-input response of the plant. Thus suppose that

$$G(s) = \frac{q(s)}{p(s)} = h'(sI-F)^{-1}g \tag{13}$$

where p and q are polynomials with no common factor and with the degree of p, $\delta(p)$ for short, greater than $\delta(q)$. Note that

$$p(s) = \det|sI-F| \tag{14}$$

Then any zero-input response must be of the form

$$y_0(s) = \frac{n(s)}{p(s)} = h'(sI-F)^{-1}x_0 \tag{15}$$

for a certain polynomial $n(s)$ with $\delta(n) < \delta(p)$ and for a certain n-vector x_0. There is no constraint requiring nonexistence of common factors of $u(s)$ and $p(s)$.

Now successful tracking requires that $y(t)$ approach $r(t)$, or, see Figure 4, that the inverse transform of $G(s)U(s)$ should approach $r(t) - y_0(t)$, which is a standard tracking problem for the classical approach, with reference signal $r(t) - y_0(t)$.

We can now give a formal statement setting out what has to be done to demonstrate equivalence of the two approaches: given $G(s) = h'(sI-F)^{-1}g$, with $[F,g]$ completely controllable, $[F,h]$ completely observable and all eigenvalues of F possessing negative real part; given $n(s)$, an arbitrary polynomial with degree less than that of $p(s) = \det(sI-F)$, and given the associated initial state x_0 defined via $n(s)/p(s) = h'(sI-F)^{-1}x_0$; given a desired output $r(t)$; show that the control $U(s)$ defined by

$$U(s) = \frac{\alpha}{(1+\alpha G\bar{G})^{+}}\left[\frac{(R-\frac{n}{p})\bar{G}}{(1+\alpha G\bar{G})^{-}}\right]_{+} \tag{16}$$

is the Laplace transform of

$$u(t) = -k'x(t) + \alpha g'\int_t^\infty e^{(F-gk')'(\tau-t)}hr(\tau)d\tau \tag{11}$$

with k defined according to equations (9) and (10), and x obeying the differential equation (7) with initial condition x_0.

4. DETAILS OF THE CONNECTION PROCEDURE

We first note the following two lemmas:

LEMMA 1 With quantities as defined previously, the following relation holds:

$$(1 + \alpha G\bar{G})^{+} = 1 - k'(sI-F)^{-1}g \tag{17}$$

LEMMA 2 With quantities as defined previously, and the definition of the polynomial $\gamma(s)$ by

$$\frac{\gamma(s)}{p(s)} = 1 - k'(sI-F)^{-1}g \tag{18}$$

the following relation holds:

$$\alpha\frac{n(s)}{p(s)}\frac{\bar{q}(s)}{\bar{\gamma}(s)} = k'(sI-F)^{-1}x_0 + x_0'\Pi[-sI-(F-gk')]^{-1}g \tag{19}$$

We can now prove the desired connection. Starting with (16), write

$$U(s) = U_1(s) + U_2(s)$$

where

$$U_1(s) = \frac{\alpha}{(1+\alpha G\bar{G})^{+}}\left[\frac{R\bar{G}}{(1+\alpha G\bar{G})^{-}}\right]_{+} \tag{20}$$

and

$$U_2(s) = \frac{\alpha}{(1+\alpha G\bar{G})^{+}}\left[\frac{-\frac{n\bar{G}}{p}}{(1+\alpha G\bar{G})^{-}}\right]_{+} \tag{21}$$

The transforms $U_1(s)$ and $U_2(s)$ prove convenient to compute separately.

We shall evaluate $U_2(s)$ first. Substituting for $\bar{G}$ and $(1+\alpha G\bar{G})^{-}$ by using the relations $G = q/p$, and the relation $(1+\alpha G\bar{G})^{+} = \gamma/p$ derived in lemma 1 – see also equation (18) – we obtain

$$U_2 = \frac{\alpha p}{\gamma}\left[\frac{-\frac{n\bar{q}}{p}}{\frac{\bar{\gamma}}{\bar{p}}}\right]_{+} = \frac{\alpha}{\gamma}\left[-\frac{n\bar{q}}{\bar{\gamma}}\right]_{+}$$

Now lemma 2 may be applied. Observe that F has all eigenvalues in $\mathrm{Re}[s] < 0$, and $-(F-gk')$ has all its eigenvalues in $\mathrm{Re}[s] > 0$. Hence all poles of $k'(sI-F)^{-1}x_0$ are in $\mathrm{Re}[s] < 0$, and all poles of $x_0'\Pi[-sI-(F-gk')]^{-1}g$ are in $\mathrm{Re}[s] > 0$. Accordingly,

$$U_2 = \frac{p}{\gamma}[-k'(sI-F)^{-1}x_0] = -k'[sI-(F-gk')]^{-1}x_0 \tag{22}$$

on using (18).

Now turn to $U_1(s)$. We have from (20) and the expression for G and $(1+\alpha G\bar{G})^{+}$ that

$$U_1(s) = \frac{\alpha p}{\gamma}\left[R\frac{\bar{q}}{\bar{\gamma}}\right]_{+} \tag{23}$$

Now consider Figure 5. Instead of driving the system $\dot{x} = Fx + gu$, $y = h'x$ with an input $u_1(t)$, we can equally well drive the system $\dot{x} = (F-gk')x + gu_3$ with an appropriately chosen $u_3(t)$ and achieve the same state trajectory. Using Figure 5, it is easy to check that

$$\frac{U_1(s)}{U_3(s)} = \frac{p}{\gamma}$$

and accordingly use of the control

$$U_3(s) = \left[R\frac{\bar{q}}{\bar{\gamma}}\right]_{+} \tag{24}$$

in Figure 5 will guarantee that $U_1(s)$ in Figure 5 is given by (23).

The overall optimal control required according to the classical theory is $U_1(s) + U_2(s)$. Observe now from (22) and Figure 5 that such an input will be achieved if the scheme of Figure 5 is altered to replace the zero initial state by the initial state x_0. No alteration of $u_3(t)$ is required. Thus the scheme of Figure 6 generates the correct optimal control according to classical notions. This figure also shows that $u(t) = -k'x(t) + u_3(t)$.

Therefore, bearing in mind the formula for the optimal control derived by the modern approach, repeated, for convenience in a slightly different format as

$$u(t) = -k'x(t) + \alpha\int_t^\infty [h'e^{(F-gk')'(\tau-t)}g]r(\tau)d\tau \qquad (11)$$

we see that it remains to show that $U_3(s)$, as given by (24), is the Laplace transform of the quantity

$$v(t) = \alpha\int_t^\infty [h'e^{(F-gk')'(\tau-t)}g]r(\tau)d\tau$$

This may be shown as follows. Introduce the shorthand notation $w(t) = h'e^{(F-gk')t}g$. Then

$$\int_0^\infty e^{-st}v(t)dt = \alpha\int_0^\infty dt \int_t^\infty d\tau[e^{s(\tau-t)}w(\tau-t)e^{-s\tau}r(\tau)]$$

Let $\sigma = \tau-t$, and integrate with respect to τ and σ. Then

$$\begin{aligned}\int_0^\infty e^{-st}v(t)dt &= \alpha\int_0^\infty d\sigma\int_0^\infty d\tau[e^{s\sigma}w(\sigma)e^{-s\tau}r(\tau)] \\ &= \alpha[\bar{W}(s)R(s)]_+ \\ &= \alpha \left[R\frac{\bar{q}}{\bar{\gamma}}\right]_+\end{aligned}$$

Comparing this with (24), we see that the desired result is established.

CONCLUSIONS

The significance of the result established lies not so much in the fact that classical and modern results may be related (*although, to be sure, there is significance in this*) but rather in a fact that has been alluded to, but not emphasised up till this point: relation of the two sorts of results for the case of an unstable plant is a *difficult* problem. In this paper, we took the fairly *easy* way out, by requiring the plant to be asymptotically stable. The effect of this was to allow us to prove lemma 1, and to use the result of lemma 2 with the operator $[\cdot]_+$ applied to both sides of equation 19.

If F is unstable, this is not possible. The difficulty seems to lie in the fact that classical solutions of quadratic minimization problems demand application of the operator $(\cdot)^+$, requiring extraction of the left half plane poles and zeros *of the function to which it is applied.* By contrast, the result (effectively derived in lemma 1) $1 + \alpha g'(-sI-F')^{-1}hh'(sI-F)^{-1}g = [1-k'(-sI-F)^{-1}g][1-k'(sI-F)^{-1}g]$ amounts to a factorization of the function on the left side of the equation in which left half plane zeros only are factored out, and if F is unstable, left half plane poles are certainly not factored out. This sort of factorization appears to be behind any frequency domain description of the modern approach to quadratic optimization, but is not the sort of factorization used in classical frequency domain approaches.

It is therefore possible that hitherto well-established classical procedures might benefit from some revision. And when we realize that certain problems, e.g. some involving stochastic variables, have not really been tackled via modern approaches, we are led to wonder whether all the possible classical results have been ful y exposed In particular, we could expect the existence of many results for unstable, as well as asymptotically stable, plants.

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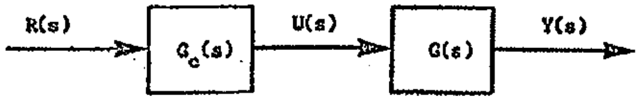

Fig. 1. First Classical Control Arrangement

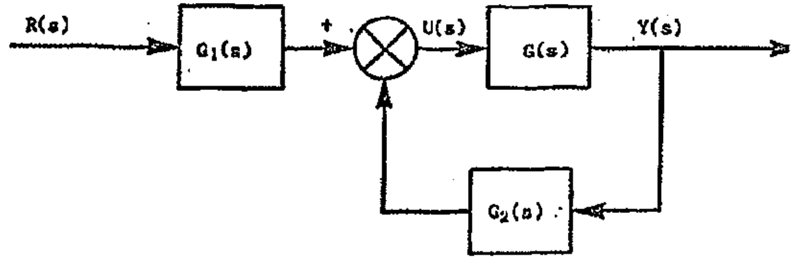

Fig. 2. Alternative Classical Control Arrangement

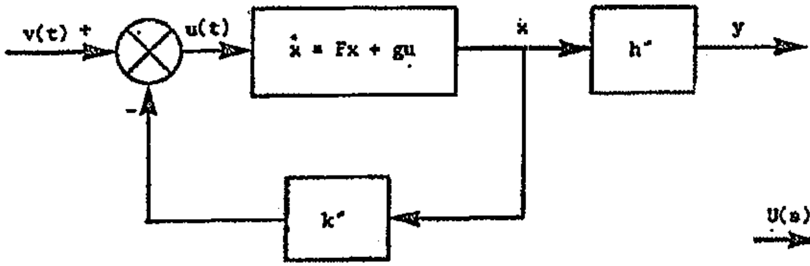

Fig. 3. Modern Control Arrangement

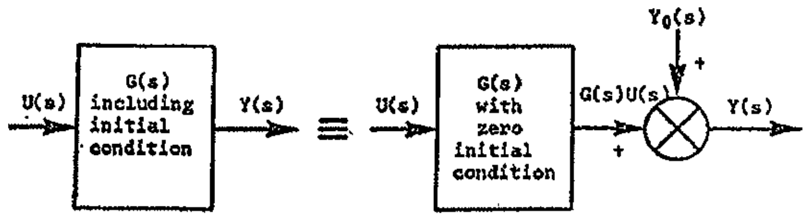

Fig. 4. Inclusion of Initial Conditions in the Classical Approach

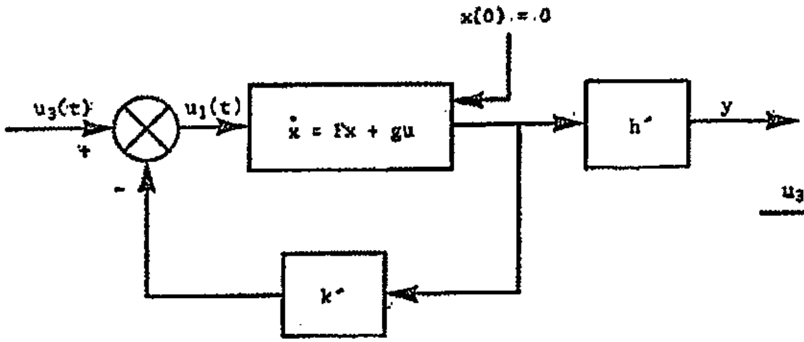

Fig. 5. Definition of u_1 and u_3