"THE RELATION BETWEEN CONTINUOUS AND DISCRETE TIME QUADRATIC MINIMIZATION PROBLEMS" *

by

Konrad L. Hitz
Department of Mechanical Engineering
and
Brian D. O. Anderson
Department of Electrical Engineering
University of Newcastle
New South Wales, 2308, Australia

1. A variety of optimal control problems involving linear systems with quadratic performance indices, and several network theory problems, require the computation of the limiting, or steady state, solution of a matrix Riccati differential equation. It is the aim of this paper to show that this solution can also be obtained by computing the limiting solution of an associated matrix difference equation. In many cases, this approach appears to offer significant computational advantages.

2. Consider the classic optimal control problem: given the system

$$\dot{x} = Fx + Gu \quad ; \qquad x(0) = x_0 \qquad (1)$$

find the control which minimizes

$$J(x_0, u) = \int_0^{\infty} [u'Ru + x'LL'x]\,dt \qquad (2)$$

where $R = R' > 0$ and where $[F,G]$ is completely controllable and $[F,L]$ completely observable. The solution is [1]

$$u^\circ(t) = -R^{-1}G'\Pi x(t) \qquad (3)$$

where

$$\Pi = \lim_{t \to -\infty} \{\bar{P}(t,t_1)\} \qquad (4)$$

and where

$$-\frac{d\bar{P}}{dt} = \bar{P}F + F'\bar{P} - \bar{P}GR^{-1}G'\bar{P} + LL' \ ; \ \bar{P}(t_1,t_1) = 0 \qquad (5)$$

The minimum value of (2) is $x_0'\Pi x_0$; the closed-loop system (1) and (3) is asymptotically stable; and Π is the symmetric positive definite solution of the equation

$$\Pi F + F'\Pi - \Pi GR^{-1}G'\Pi + LL' = 0 \qquad (6)$$

with the property that if ϕ is any other solution of (6), then $\Pi - \phi \geq 0$.

* Work supported by the Australian Research Grants Committee

3. The minimization problem stated in 2 can be reformulated so as to eliminate from the performance index the quadratic term in the state. Define a control by

$$u_e = u + K'x \tag{7}$$

so that equation (1) becomes

$$\dot{x} = F_1 x + G u_e \qquad \text{where} \quad F_1 = F - GK' \tag{8}$$

The matrix K in (8) is chosen so as to make F_1 asymptotically stable. Procedures for finding such K are set out, for example, in [2]. The performance index is now, with $Q = LL' + KRK'$,

$$J(x_0,u) = \int_0^\infty [u_e'Ru_e - 2u_e'RK'x + x'Qx]\,dt \tag{9}$$

Define P as the symmetric nonnegative definite solution of

$$PF_1 + F_1'P = -Q \tag{10}$$

Hence $x'Qx = -x'PF_1x - x'F_1'Px = -\frac{d}{dt}(x'Px) + 2x'PGu_e$ by (8).
Substitute into (9):

$$J(x_0,u) = \int_0^\infty [u_e'Ru_e + 2x'(PG - KR)u_e]\,dt + x_0'Px_0 - \lim_{T\to\infty}\{x'(T)Px(T)\} \tag{11}$$

Since the optimal closed-loop system is known to be asymptotically stable, the last term in (11) is zero when J achieves its minimum value. Setting $H = PG - KR$ and

$$J_1(x_0,u\,) = \int_0^\infty [u_e'Ru_e + 2x'Hu_e]\,dt \tag{12}$$

it is clear that the minimization of (12) subject to (8) is equivalent to the minimization of (2) subject to (1), with the minimizing controls related by $u^\circ(t) = u_e^\circ(t) - K'x(t)$. The minimization of a performance index of the form (12) subject to (8) is discussed in [3]. The results reported there rely on the assumption that the pair $[F_1,H]$ is completely observable, which in our case is not true in general. However, since F_1 is asymptoticall stable, it may be shown that $[F_1,H]$ need not be completely observable for the results of [3] to be valid. It follows then from [3] that a necessary and sufficient condition for the existence of an optimal control is that there exist a number $\alpha > 0$ such that the matrix $Z(s) = \frac{1}{2}(R-\alpha I)+H'(sI-F_1)^{-1}G$ is positive real. The optimal control is then given by $u_e^\circ(t) = -R^{-1}(G'\hat{\Pi} + H')x(t)$, where

$$\hat{\Pi} = \lim_{t\to-\infty} \{\hat{P}(t,t_1)\} \qquad (14)$$

and where $$-\frac{d\hat{P}}{dt} = \hat{P}F_1 + F_1'\hat{P} - (\hat{P}G+H)R^{-1}(G'\hat{P}+H'); \; \hat{P}(t_1,t_1) = 0 \qquad (15)$$

Again, $x_0'\hat{\Pi}x_0$ is the minimum value of (12); the optimal closed-loop system is asymptotically stable; and $\hat{\Pi}$ is the symmetric nonpositive definite solution of

$$\hat{\Pi}F_1 + F_1'\hat{\Pi} - (\hat{\Pi}G + H)R^{-1}(G'\hat{\Pi} + H') = 0 \qquad (16)$$

with the property that if $\hat{\phi}$ is any other solution of (16), then

$$\hat{\Pi} - \hat{\phi} \geq 0 \qquad (17)$$

Now since the optimal control $u_e^\circ(t)$ is known to exist, being given by $u_e^\circ(t) = u^\circ(t) + K'x(t)$, it follows that the condition necessary for the existence of $\hat{\Pi}$ as defined by (14) must be satisfied. Moreover, by considering (11) and the fact that $x_0'\Pi x_0$ and $x_0'\hat{\Pi}x_0$ minimize (2) and (12), respectively, it follows that

$$\Pi = \hat{\Pi} + P \qquad (18)$$

Guided by the bi-linear transformation $s = (z-1)(z+1)$, which can be used to define the z-transfer function of a discrete system from the transfer function of a continuous system, we now define matrices A, B, C, U and Ψ by the relations

$$F_1 = (A+I)^{-1}(A-I); \quad G = 2(A+I)^{-2}B; \quad H = C;$$
$$U = R + C'(A+I)^{-1}B + B'(A'+I)^{-1}C; \; \hat{\Psi} = 2(A'+I)^{-1}\hat{\Pi}(A+I)^{-1} \qquad (19)$$

Substituting these into eqn. (16) results, after a lengthy manipulation, in the equation

$$A'\hat{\Psi}A - \hat{\Psi} - (A'\hat{\Psi}B+C)[U+B'\hat{\Psi}B]^{-1}(B'\hat{\Psi}A+C') = 0 \qquad (20)$$

where the required inverse exists since it can be shown that $U + B'\hat{\Psi}B = R + B'(A'+I)^{-1}[2\Pi+Q](A+I)^{-1}B$, which is positive definite. Eqn. (20) may not have a unique solution. However, in view of (17) and the last of eqns. (19), it is clear that $\hat{\Psi}$ must be the nonpositive definite solution of (20) with the property that

$$\hat{\Psi} - \Phi \geq 0 \qquad (21)$$

where Φ is any other solution of (20). It now remains to show that $\hat{\Psi}$ is the limiting solution of a matrix difference equation.

4. Consider the discrete-time minimization problem: given a system $x(n+1) = Ax(n) + Bu(n)$; $x(0) = x_0$, find the control sequence $\{u^\circ(k)\}$ which minimizes $J_d(x_0,u) = \sum_{k=0}^{\infty} [u'(k)Uu(k) + 2x'(k)Cu(k)]$ where $U = U' > 0$, A is asymptotically stable, and $[A,B]$ is completely controllable. The theory of this problem is entirely analogous to that of the continuous-time problem of minimizing (12) subject to (8). Moreover, whenever the matrices A, B, C and U are related to the corresponding matrices of the continuous-time problem by (19), as they are in our case, it may be shown that the existence of the solution of the continuous-time problem implies the existence of the solution of the discrete-time problem, and conversely. Therefore, the discrete-time problem has a solution in our case; it can be shown to be given by $u^\circ(k) = -[U+B'\bar{\Psi}B]^{-1}(B'\bar{\Psi}A+C')x(k)$ where

$$\bar{\Psi} = \lim_{N\to-\infty}\{\Psi(n,N)\} \tag{22}$$

and

$$\Psi(n,N) = A'\Psi(n+1,N)A - [A'\Psi(n+1,N)B+C][U+B'\Psi(n+1,N)B]^{-1}[B'\Psi(n+1,N)A+C'] \tag{23}$$

with $\Psi(N,N) = 0$.

The optimal closed-loop system turns out to be asymptotically stable, which implies that $\bar{\Psi}$ is the symmetric nonpositive definite solution of (20) which also satisfies (21). Hence $\hat{\Psi} = \bar{\Psi}$. Thus to compute Π in (4), we compute $\hat{\Psi}$ above, and make use of (18) and the last equation of (19).

References

[1] R.E. Kalman, "Contributions to the Theory of Optimal Control," Boletin de la Sociedad Matematica Mexicana, 1960, pp.102-119.

[2] B.D.O. Anderson and D.G. Luenberger, "Design of Multivariable Feedback Systems," Proceedings IEEE, Vol.114, No.3, March 1967, pp. 395-399.

[3] B.D.O. Anderson, "Quadratic Minimization, Positive Real Matrices and Spectral Factorization," Technical Report EE-6812, Department of Electrical Engineering, University of Newcastle, Australia, June 1968.