KARHUNEN-LOEVE EXPANSIONS FOR A CLASS OF COVARIANCES*

by

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Introduction

Frequently in communication theory, one may be given a covariance $K(t,s)$, $0 \leq s, t \leq T$ for which a Karhunen-Loeve expansion is required, that is a decomposition of $K(t,s)$ into a particular kind of infinite sum. A problem equivalent to the determination of the Karhunen-Loeve expansion is the determination of the eigenvalues and eigenfunctions associated with $K(t,s)$, that is the determination of all constant λ and functions $\phi(\cdot)$ for which

$$\int_0^T K(t,s)\phi(s)ds = \lambda\, \phi(t) \qquad 0 \leq t \leq T \tag{1}$$

There are at most a countable number of solutions to this equation for which $\lambda > 0$, and no solutions for $\lambda < 0$. If $K(t,s)$ is positive definite, then all solutions λ are positive, and the associated eigenfunctions form a complete orthonormal set. In this paper, we present a procedure for finding all solutions of the integral equation for a class of positive definite $K(t,s)$, namely those which have the form

$$K(t,s) = a'(t)b(s)1(t-s) + b'(t)a(s)1(s-t) \tag{2}$$

where $a(\cdot)$ and $b(\cdot)$ are n-vector functions defined on $[0,T]$, assumed continuous, and $1(\cdot)$ is the unit step function. The process with $K(\cdot,\cdot)$ as covariance is a projection of a Markov process; otherwise, $K(t,s)$ can be regarded as the output covariance $E[y(t)y(s)]$ of a linear, finite-dimensional system described by equations of the form

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$$\dot{x} = Fx + gu; \qquad \dot{y} = h'x \tag{3}$$

with the input u being zero mean gaussian white noise, i.e. $E[u(t)u(s)] = \delta(t-s)$, and the initial state $x(0)$ is a zero mean gaussian random variable with covariance $P(0)$. Of course, knowledge of $K(\cdot,\cdot)$ does not immediately yield knowledge of F, g, h and $P(0)$ - indeed the determination of these quantities from $K(\cdot,\cdot)$ is a difficult problem. Accordingly, it is of some interest to have a procedure for the determination of eigenfunctions and eigenvalues of $K(\cdot,\cdot)$ which does not require knowledge of F, g, h and $P(0)$. Such a procedure is presented here, which should be compared with the procedure of [1] requiring knowledge of F, etc.

As in [1], the key to determining the eigenfunctions and eigenvalues is to define when a submatrix of the transition matrix of a certain differential equation is singular.

Determination of Eigenfunctions and Eigenvalues

In this section, a number of variables and equations will be defined. Then the constructive procedure will be presented in the form of a theorem.

Consider the differential equation set

$$\dot{z} = \begin{bmatrix} \dfrac{ba'}{\lambda} & -\dfrac{bb'}{\lambda} \\ \dfrac{aa'}{\lambda} & -\dfrac{ab'}{\lambda} \end{bmatrix} z \tag{4}$$

Here z is a 2n-vector, $a(\cdot)$ and $b(\cdot)$ are defined as in (2), and λ is a *fixed but arbitrary* positive constant. We shall be interested in solution of (4) for different values of λ. The transition matrix associated with (4) is the solution $\Psi(\cdot,0;\lambda)$ of the matrix differential equation obtained by replacing the 2n-vector z in (4) by a $2n \times 2n$ matrix Z, with an initial condition $Z(0) = I$. Let us write, with $\Psi_{ij}(\cdot,\lambda)$ an $n \times n$ matrix,

$$\Psi(t,0;\lambda) = \begin{bmatrix} \Psi_{11}(t,\lambda) & \Psi_{12}(t,\lambda) \\ \Psi_{21}(t,\lambda) & \Psi_{22}(t,\lambda) \end{bmatrix} \tag{5}$$

Defining n-vectors $x(t)$ and $y(t)$ by $z'(t) = [x'(t) \quad y'(t)]$, it is well known that (4) and (5) imply

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \Psi_{11}(t,\lambda) & \Psi_{12}(t,\lambda) \\ \Psi_{21}(t,\lambda) & \Psi_{22}(t,\lambda) \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \tag{6}$$

The theorem characterizes eigenfunctions and eigenvalues of (1) in terms of the $n \times n$ matrix $\Psi_{22}(T,\lambda)$

<u>Theorem</u> Let λ_0 be an eigenvalue of (1). Then $\Psi_{22}(T,\lambda_0)$ is singular. Conversely, if $\Psi_{22}(T,\lambda_0)$ is singular, λ_0 is an eigenvalue of (1) and an associated eigenfunction is given by

$$\phi_0(t) = \frac{1}{\lambda_0}[a'(t) \quad -b'(t)]\ \Psi(t,0;\lambda_0)\ [0 \quad \beta']' \tag{7}$$

where β is any n-vector in the nullspace of $\Psi_{22}(T,\lambda_0)$. Finally, the number of linearly independent eigenfunctions with eigenvalue λ_0 is the dimension of the nullspace of $\Psi_{22}(T,\lambda_0)$.

<u>Outline Proof</u> Suppose λ_0 is an eigenvalue, and $\phi_0(\cdot)$ the corresponding eigenfunction. One may show that $x_0(t) = \int_0^t b(s)\phi_0(s)ds$ and $y_0(t) = -\int_t^T a(s)\phi_0(s)ds$ satisfy (6). Then it follows that $y_0(0)$ is in the nullspace of $\Psi_{22}(T,\lambda_0)$. The converse is equally simple. The final part of the theorem follows by showing that $\Psi_{22}(T,\lambda_0)\beta = 0$ with $\beta \neq 0$ implies $\phi_0(t)$ is not identically zero, and thus the mapping $\{\beta\} \to \{\phi_0(\cdot)\}$ is one to one and onto.

Several points should be noted. First, $\det[\Psi_{22}(T,\lambda)]$ will be closely related to the Fredholm determinant [2], which is a function of a complex variable vanishing for values of the complex variable equal to $-\frac{1}{\lambda_0}$, where λ_0 is an eigenvalue of (1). The multiplicity of a zero of the Fredholm determinant is an upper bound on the number of associated eigenfunctions, but in general will not be the same as the number of eigenfunctions defined by a multiple zero; therefore the method of the theorem serves better in the case of eigenvalues of multiplicity greater than one to isolate the differing eigenfunctions.

The evaluation of the matrix $\Psi_{22}(T,\lambda_0)$ for differing values of λ_0 is clearly a tedious task. A simplification is however possible in the case where $K(t,s)$ is stationary. In this case, $a'(t)b(s) = h'e^{Ft}e^{-Fs}g$ for some constant n-vectors g and h and a constant $n \times n$ matrix F. It is then not difficult to show that eigenvalues and eigenfunctions are determined by singularity of the $n \times n$ bottom right hand submatrix of the transition matrix of

$$\dot{z} = \begin{bmatrix} F + \frac{gh'}{\lambda} & -\frac{gg'}{\lambda} \\ \frac{hh'}{\lambda} & -F' - \frac{hg'}{\lambda} \end{bmatrix} z \tag{8}$$

Of course, the solution of (8) is much more straightforward because of the time-invariance property.

References

[1] A. B. Baggeroer, "A State-Variable Approach to the Solution of Fredholm Integral Equations," Technical Report 459, Research Laboratory of Electronics, Massachusetts Institute of Technology, November 1967.

[2] F. Smithies, "Integral Equations," Cambridge University Press, Cambridge, England, 1958.