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Information Architecture and Control Design for Rigid Formations*

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Abstract: Formations of robots, underwater vehicles and autonomous airborne vehicles are progressively being deployed to tackle problems of surveillance, bush fire control, and the like. Much formation behavior mimics the behavior of formations of living organisms, such as birds and fish. This paper reviews a number of concepts and results relevant to the design of control schemes and information architectures to maintain the shape of a formation of autonomous agents. The task of providing satisfactory sensing, communication and control architectures within a formation of autonomous agents is emphasized and elaborated in the paper. The paper provides a set of technical tools for characterizing and designing information architectures, which largely rest on graph theoretic considerations, as well as a control scheme exemplifying a class of decentralized controllers for maintaining the shape of a formation.

Key Words: Autonomous Multiagent System, Formation Control, Graph Theory

1 INTRODUCTION

Human beings have seen examples of collective behaviour by groups of insects, birds, fish, etc since time immemorial. Such behaviour is an evolutionary consequence of the ability of a group to secure types of behaviour promoting survival that would never be achievable by individual members of the group[1,2]. The behaviour may serve the needs of defence against predators, or aggression against prey, of foraging for food, of mating, etc. Some creatures, fish and birds particularly, as part of their group behaviour, often display formation type behaviour; in this sort of behaviour while the whole formation moves as a cohesive whole, the relative positions of the fish or birds are preserved, at least more or less. Of course, from time to time, a formation may rearrange itself in a minor way, perhaps to remove a burden on one or members of the formation; it may rearrange itself in a major way, perhaps for obstacle avoidance or predator avoidance, even to the point of splitting; and it may also merge with another formation. To some degree nature has probably been a conscious or unconscious motivator for human kind, and so today formations of robots, underwater vehicles and autonomous airborne vehicles are slowly being deployed to tackle problems in both civilian and defence spheres-for example, bush fire control, surveillance, underwater exploration and the like.

A formation of vehicles may constitute a much more effective sensor than a single vehicle, for various reasons. First, with many vehicles one can effectively synthesise a large dimension antenna, for receiving electromagnetic or acoustic signals, and large antennas have higher sensitivity and so may allow better source localization; accurate knowledge and even control of the relative positions of the agents of the formation is of course essential for this application. A second reason is that some tasks inherently require multiple sensors with known relative positions; for example, in three dimensions, if one can measure distances to an object of interest at an unknown position, and if one wants to determine the position of that object, one needs four separate distance measurements from sensors with known positions to determine that object's position (otherwise termed localizing the object). A third reason is that multiple sensors may have individually differing functionalities; the collection of functionalities in aggregate gives a new functionality for the formation of sensors. There are cost drivers in this direction too: small mobile sensors are likely to be much cheaper to deploy, but then weight and other considerations will set upper bounds on the on-board functionality of any one sensor.

Sometimes all these factors are simultaneously operative. For example, in one application with which the authors are familiar, the agents of the formation fulfill their localizing objective by determining angle information associated with the object at unknown position, but issues of sensor noise, limited cone of visibility and the like, mean that more agents need to be used in the formation that might be at first thought. Formations in naturally or artificially hazardous environments also may require larger numbers of agents, in order to have redundancy that will cope with outages of some of them.

From a control point of view, it is clear that there are tasks at both the level of the whole formation, determining for example waypoints for a path which the centre of gravity of the formation should follow, as well as control tasks for the individual agents of the formation, such as maintaining their relative positions, or shifting from one formation shape to another formation shape. Now certainly in formations occurring in nature, and commonly in man-made formations, there is no single all-powerful master agent exercising control over every other agent. Control tasks in some way have to be handled on a decentralized basis, and it is this fact which gives rise to many of the scientific and technical challenges associated with man-made formations.

In fact, apart from the need to design decentralized control laws, many systems problems arise, and standing at the apex of these problems is the task of defining practical architectures for control, communications and sensing. Of course, these three architectures are conceptually interrelated, and cannot be defined independently of one another. An overarching requirement for all three is that they be scalable. This scalability requirement inherently imposes a need for significant decentralization of information and control structures. Without being a biologist, one knows that in a formation of birds, no one bird can be expected to watch all other birds and compute its own trajectory using even partial knowledge of the trajectories of all other individual birds. The idea carries over to man-made formations: the amount of sensing, communication and control computation by any one agent has to be limited.

We have sketched, very crudely and with much omission, a vast

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problem domain above. This paper will traverse only one corner of the whole problem domain, albeit an important corner. We shall mainly focus on the question of what sort of sensing and control architectures are needed to maintain the shape of a formation, while the formation moves as a cohesive whole. We shall present some details on types of control law that can ensure such motion occurs. In addition though, we shall consider various change scenarios, including splitting, merging, and closing ranks (where the formation has to deal with the loss of one or more agents). The architectures need to be able to sustain these tasks.

Within this limited domain, we will impose yet a further limitation. Obviously, to maintain a formation shape, agents have to sense some aspect of the geometry of the formation, i.e. some geometrically relevant variable involving some, at least, of the other agents in the formation. This will be needed in order to apply a control to correct any error in formation shape. There are many things that can in principle be sensed, involving distances and angles, for example agent j might sense: (a) The distance from agent j to agent k; (b) The difference in the distance from agent j to agent k and the distance from agent j to agent m ("time difference of arrival"); (c) The bearing relative to north of agent k; (d) The declination/inclination relative to the horizon of agent k; (e) The angle subtended at agent j by the lines joining agent j to agent k and agent k to agent m etc.

On occasions, one agent may sense more than one class of measurement. For example, a recent study on coordination and control of mobile vehicles using both distance and bearing measurements can be seen in ref. [3].

We note that biological organisms may sense more complicated things again than the above, and matters are far from clear anyway. When we asked a biologist colleague what birds sense when they are in a formation, he suggested that they effectively project angles within a cone onto a hemisphere, and at the same time sense perhaps one distance (so that the formation is not scale free, which would be the case were angles alone to be sensed).

In this paper, and restrictively, we shall confine our attention to the use of distance measurements. Obviously, a typical agent then in two dimensions is likely to require at least two distance measurements to neighboring agents to maintain its position, assuming it has no degrees of freedom through being a leader or co-leader of the formation. Many of the tools we use for this work (and they will be introduced here) are graph theoretic. Graph theory is useful for focussing on the higher-level question of defining the architecture behind the control laws that maintain the shape of the formation; the control laws themselves are a separate matter, but we shall say something about these too. There has been considerable earlier work on control laws, see e.g. [3-10]. Should make sure Wang reference is in this list. Though not discussed here, it turns out that the graph theoretic tools we describe can often be applied to formations with other sensed variables than just distance, including those with mixtures of angles and distances, see e.g. [11]. However, we have not explored that concept in relation to [3].

The paper is organized as follows. In Sections 2 and 3, we present and elaborate two different control structures that can be used in maintaining shapes of autonomous formations, the symmetric and asymmetric control structures, and characteristics of formations controlled by such control structures. Two particular graph theoretical notions are introduced and used in analysis in these two sections: graph (formation) rigidity and persistence. These two sections describe conditions, in 2 and 3 dimensions, that must be fulfilled by architectures that allow maintenance of formation shape during formation movement.

In Section 4, we summarize some recent results on the construction of decentralized control laws to ensure that the shape of a formation is preserved. In this section, we particularly focus on autonomous formations with asymmetric control structure.

Section 5 presents certain classes of topology modification scenarios or formation operations, including merging, splitting and closing ranks. Closing ranks is the task of repairing a formation when one or more agents are lost; it is a task that obviously nature has solved in a number of cases. Generally speaking, a minor rearrangement of the architecture is needed.

In Section 6, the merging operation is elaborated further, and a relevant framework for analyzing large formation, the *meta-formation framework* is introduced.

Section 7 provides concluding remarks.

2 FORMATIONS WITH SYMMETRIC CONTROL STRUCTURE

There are two types of control structures that can be used to maintain the shape of a formation: symmetric control and asymmetric control. Formations with symmetric control structure can be represented with undirected graphs, while directed graphs are used to represent those with asymmetric control structure.

In the symmetric control structure, to keep the distance between each pair (A_1, A_2) of neighbor agents, there is a joint effort of both A_1 and A_2 to simultaneously and actively maintain their relative positions. The underlying graph of the formation will have an undirected edge between vertices 1 and 2 (representing agents A_1 and A_2). If enough agent pairs explicitly maintain distances, all inter-agent distances and hence the formation shape will be maintained, i.e. the formation will be rigid.

In contrast, in the asymmetric control structure, only one of the agents in each neighbor agent pair, e.g., A_1 in the agent pair (A_1, A_2), actively maintains its position relative to its neighbor. This means for the neighbor agent pair (A_1, A_2) that only A_1 has to receive the position information broadcast by A_2, or sense the position of A_2 and make decisions on its own. Therefore, in the asymmetric control structure, both the overall control complexity and the communication complexity (in terms of message sent or information sensed) for the formation are reduced by half, compared to symmetric control. This is modelled in the associated directed graph by a directed edge from vertex 1 to vertex 2 (representing a directed constraint link from A_1 to A_2).

In this section and the next we elaborate more on characteristics of formations with symmetric and asymmetric control structures, mainly using *rigid graph theory* [12−14]. To fit in the (rigid) graph theory framework, a multi-agent system model is used where agents are assumed to be points, and agent pairs for which the inter-agent distance is actively constrained to be constant are thought of as being joined by bars with lengths enforcing the inter-agent distance constraints[9,11,15,16]. The multi-agent system can be therefore modelled by a graph where vertices represent point-like agents and inter-agent distance constraints are abstracted as edges. (Naturally, one can contemplate other constraints than distance, e.g. those involving angle, or angle and distance, although omitted in this paper for the sake of simplicity.) Rigid graph theory is used to state properties of graphs which ensure that the formation being modelled

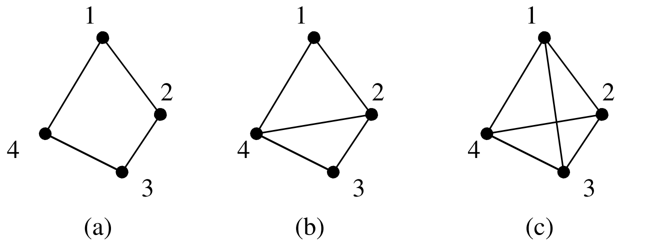

Fig. 1 Illustration of (a) non-rigid formation, (b) (minimally) rigid formation, and (c) non-minimally but rigid formation

by the graph will be rigid; formal definitions are available of course, but roughly speaking, a *rigid formation* is one in which the only smooth motions are those corresponding to translation or rotation of the whole formation.

Fig. 1 shows several examples of 2-dimensional graphs, two of which are rigid and one of which is not rigid. In a nonrigid graph part of the graph can flex or move, while the rest of the graph stays still. The notion of rigidity conforms to one's normal intuition.

There exist two key tool sets for rigidity analysis. The first is from linear algebra and matroid theory: Given knowledge of the positions of the agents at any one time, one can construct a matrix, the so-called *rigidity matrix*[12,13], and the dimensions and rank of this matrix allow one to conclude that the formation is or is not rigid. The dimensions and rank are the same for almost all positions of the agents. This means that rigidity matrices formed from two formations differing from each other only in terms of the values for the constrained distances will have the same rank, except for very special sets of the agent positions. The concept is valid in both 2 and 3-dimensions.

The second tool set is a combinatorial one composed of a number of purely counting-type characterization conditions. The following theorem, which is well known in the rigid graph literature, implies that it is also possible in 2-dimensions to characterize rigidity of a generic formation corresponding to a given graph in purely combinatorial conditions related to the graph (discarding the agent coordinates):

Theorem 1 *(Laman's Theorem, ref.* [17]*) A graph $G = (V, E)$ in $\mathbf{R}^2$ of $|V|$ vertices and $|E|$ edges is rigid if and only if there exists a subgraph $G' = (V, E')$ with $2|V| - 3$ edges such that for any subset $V"$ of V, the induced subgraph $G" = (V", E")$ of G' obeys $|E"| \leqslant 2|V"| - 3$.*[1]

There does not yet exist any 3-dimensional equivalent of Theorem 1. In $\mathbf{R}^3$, differing necessity and sufficiency conditions are known for a graph to correspond to a formation which will be rigid for generic values of the constrained inter-agent distances[12]. A set of such conditions is given in the following theorem:

Theorem 2 *A graph $G = (V, E)$ in $\mathbf{R}^3$ of $|V|$ vertices and $|E|$ edges is rigid only if (a) there exists a subgraph $G' = (V, E')$ with $3|V| - 6$ edges such that for any subset $V"$ of V, the induced subgraph $G" = (V", E")$ of G' obeys $|E"| \leqslant 3|V"| - 6$, and (b) if $G"$ obeys $|E"| = 3|V"| - 6$, then it is 3-connected, i.e. between any two vertices of $G"$, there are three paths which pairwise have no vertices in common.*

1 The subgraph of G induced by a vertex set $V' \subset V$ is the graph $G' = (V', E')$ where E' includes all the edges of E that are incident on a vertex pair in V'.

A rigid formation is further called *minimally rigid* if no single inter-agent distance constraint can be removed without losing rigidity. A graph is minimally rigid if almost all formations to which the graph corresponds are minimally rigid. Minimal rigidity is easily described in two and 3-dimensions with the rigidity matrix, characterizable in 2-dimensions with Theorem 1, and the subject of some necessary conditions in 3-dimensions on the graph determined by a formation. Necessary conditions in 2 and 3-dimensions are that $|E| = 2|V| - 3$ and $|E| = 3|V| - 6$ respectively, where $|E|$ and $|V|$ are the numbers of edges and vertices of the graph. See Fig. 1 for an illustration.

In a minimally rigid formation (with $N > 0$ agents), the formation shape is maintained using the minimum possible number of information links (for the given number of agents, N), which makes use of minimally rigid information architectures an optimal choice. However, in many cases, there are good reasons to use non-minimally rigid graphs rather than minimally rigid ones to underpin the shape of a formation. In a minimally rigid formation, there is no protection against loss of a sensor, a communication link or a control actuator, and in practice, it will often be necessary to obtain robustness through the use of some measure of redundancy. Measures of robustness are needed to reflect ability to sustain agent and link losses (whether from a sensing, communication or control failure). Certain agent and link loss cases are further discussed in Section 5.

A particular class of non-minimally rigid formations is the class of *globally rigid formations*. A 2-dimensional or 3-dimensional formation, in which agents are labelled and certain inter-agent distances are prescribed, is called *globally rigid* if it differs from any formation with the same agent labelling and the same inter-agent distance prescription at most by translation, rotation and reflection. The notion of global rigidity is particularly useful in various agent/sensor network localization tasks to establish uniqueness of location estimates. Further details on this matter can be found in [18, 19].

3 FORMATIONS WITH ASYMMETRIC CONTROL STRUCTURE

In this section, we consider extensions of the notions and results presented in Section 2 to formations with asymmetric control structure. The basic task we address is again maintenance of formation shape during motion, i.e. ensuring that the motions of the formation are restricted to translation or rotation. This task is examined in [20] for 2-dimension, and in [16] for 3-dimensional formations.

We call a formation with asymmetric control structure *persistent* if it is rigid (where rigidity is as defined in Section 2) and satisfies another condition named *constraint consistence*[16,20], which is equivalent to the requirement that *it is possible* to maintain the nominated inter-agent distances. Note that the rigidity property says that *if* certain inter-agent distances are maintained, then all inter-agent distances are maintained when the formation moves smoothly. The notions of persistence and constraint consistence are illustrated further in Fig. 2 Similarly, a *minimally persistent formation* is that which is minimally rigid and constraint consistent.

Formal definitions and detailed characteristics of constraint consistence and persistence can be found in [16, 20], where directed graph notions are used. Here we note two key facts:

- Any 2-dimensional graph (representing a 2-dimensional formation with asymmetric structure) which has no more

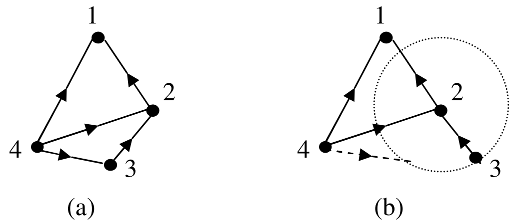

Fig. 2 Illustration of a 4-agent directed formation that is not constraint consistent: Suppose agents 1 and 2 are fixed, with agent 2 at its correct distance from 1. Suppose also that agent 3 is at its correct distance from agent 2 and agent 4 at its correct distance from 1, 2 and 3. Now observe that agent 3 has only one distance constraint, thus it can move, while maintaining its distance from 2, on a circle centered at agent 2. It is unconscious of the constraint which 4 is supposed to maintain on the distance between agents 3 and 4. When agent 3 moves, agent 4 then has an impossible task. There are only two possible positions where agent 4 can be in order to maintain its correct distances from agents 1 and 2; for generic allowable positions of agent 3, agent 4 will not be able to maintain the correct distance from agent 3 from either of these two positions. We describe such an arrangement as being *not constraint consistent*. Evidently, too much is being asked of one agent.

than two outgoing edges (constraint links) from any vertex (agent) is constraint consistent (though there are constraint consistent graphs where some vertices have outdegree greater than two).

- A graph (representing a formation with asymmetric structure) can be checked for persistence by testing a certain collection of its subgraphs (in which the edge directions are neglected) for rigidity.

These two facts form a basis for a new task related to construction of asymmetric formation control structures, acquisition of persistence[21]: Suppose that a 2-dimensional undirected graph (representing a 2-dimensional formation with asymmetric control structure, ignoring the directions of the constraint links) is rigid. The task is assignment of directions to the edges so that the resultant directed graph is constraint consistent and thus persistent. Although the persistence acquisition problem in general is still open, affirmative solutions have been established for minimally rigid graphs[20], and graphs with certain structures, including wheel graphs, trilateration graphs, complete graphs and power graphs of circle graphs[21].
Most of the persistence discussion above is made considering only 2-dimensional formations. In $\mathbf{R}^3$, much of the discussions will carry through. In particular, one can certainly define constraint consistence and persistence. However, there is a subtle twist. In effect, one needs the equivalent of constraint consistence for all subsets of vertices, as opposed to just each individual vertex considered one at a time. For three and indeed higher dimensions, a concept termed *structural persistence* is required[16]. The *structural persistence* condition, whose formal definition is given in [16], in effect is equivalent to the requirement of being able to provide enough interagent distance controls to ensure the formation behaves as a cohesive whole. A generalized check for structural persistence can be executed based on the following theorem.

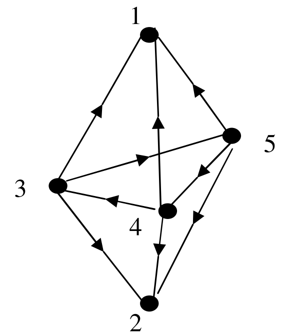

Fig. 3 A persistent but not structurally persistent formation: The underlying directed graph, and the associated graph is persistent, this being a consequence of every agent having no more than three out-going edges. However, it is evident that agents 1 and 2 are unconstrained, having no outgoing edges, and so in principle can move apart, thus destroying the shape of the formation. Hence despite the persistence property, this formation does not have a sensing and control architecture allowing retention of its shape. The reason is that it is not structurally persistent.

Theorem 3 [16] *A persistent graph in* $\mathbf{R}^d$ *(*$d \geqslant 1$*) is structurally persistent if and only if every one of its closed subgraphs* 2 *with less than* d *vertices is persistent.*

It is provable that all persistent graphs in R^2 and R^1 are also structural persistent [16]. Structural persistence of a 3-dimensional formation can be verified by checking its persistence *and* verifying there is at most one vertex of the graph with no outwardly directed edges. The concept of structural persistence is further illustrated in Fig. 3 For formal definitions and further analysis, the reader may refer to [16].

4 CONTROL LAWS FOR MINIMALLY PERSISTENT FORMATIONS

In this section, we summarize some recent results on the construction of decentralized control laws to ensure that the shape of a formation is preserved. The key references are[22,24], see also [25, 26]. However, we must not discount other work in this area. Among the earliest papers linking graph theory to problems of stabilization of the associated formations were those of Olfati-Saber and Murray[27]. The first of these papers pointed out the possibility of separating the navigation task of the formation (getting from A to B) from the shape stabilization task, and in this section we will focus *solely* on the latter task, given the now fairly wide understanding of the principle that the two tasks can be effectively separated. The second of the cited papers dealt extensively with formations for which the underlying graph was acyclic. It turns out that formation stabilization in this case is very straightforward: decentralization of the control laws is easy because there is one-way-only or triangular coupling between the agents. References [22-24] are concerned with formations where the underlying graph includes one or more cycles.
There are other references dealing with agents which have more complicated models than the point models we use here, such as unicycles, see e.g. [28, 29] and other problems where securing a formation of a specialized shape, e.g. equilateral polygon [30], is important. The literature also reveals that the formation stabilization problem can be simplified if all agents are

2 $G' = (V', E')$ is a closed subgraph of G if there is no directed path in G starting from V' and containing either a vertex or an edge that does not belong to G'.

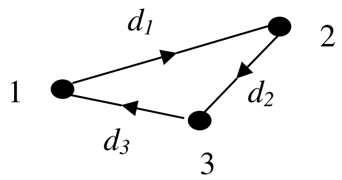

Fig. 4 Illustration of a triangular formation with three coleaders

equipped with a compass [9]; that assumption will not be made here.

The assumptions we do make are as follows: the formation exists in the plane, the agents are point agents, their velocities are controllable, they can measure the distance to their neighbors, and they know the desired distance to each of their neighbor(s). We restrict attention to *minimally* persistent formations, and we consider first the simplest formation containing a cycle–a triangular formation of three agents.

The potential for instability when a cycle is present should be reasonably clear. If agents i, j, k form a cycle, i tries to fix its position relative to j, j tries to fix its position relative to k, and k tries to fix its position relative to i, then there is a clear feedback mechanism around the cycle, and so a worry that the feedback under some circumstances could be destabilizing.

4.1 Control of a Triangular Formation with Three Coleaders

In this subsection, we summarise results of [23-25]. Denote the three agents by 1,2 and 3, and suppose their positions at any instant of time are denoted by $x_i, i = 1, 2, 3$. Suppose that the nominal distances from 1 to 2, 2 to 3 and 3 to 1 are d_1, d_2, d_3, with these distances satisfying the triangle inequality, see Fig. 4 Let z_1, z_2, z_3 denote the relative positions for 1 with respect to 2, etc, i.e. $z_1 = x_1 - x_2$ with two similar equations. The formation stabilization task is to ensure that $\lim_{t\to\infty} ||z_i(t)|| - d_i = 0, i = 1, 2, 3$. The question arises as to whether this behavior can reasonably be expected for all initial conditions, or only those for which $|||z_i(t)|| - d_i|$ is small; as it turns out, it is possible to secure this behavior for almost all initial conditions.

The control laws of references [23-25] are different, but have a common form. The law is

$$\dot{x}_i = -k_i z_i \tag{1}$$

In [23] and [24], the gains k_i are respectively:

$$k_i = [||z_i|| - d_i]/||z_i|| \tag{2}$$
$$k_i = [||z_i||^2 - d_i^2] \tag{3}$$

Evidently, movement of x_i is always directly towards or directly away from its neighbor, and the direction of movement is such as to reduce the error between the actual and desired distance. This is intuitively reasonable. The gain is different between the two algorithms. Actually, the two algorithms are special cases of the algorithm

$$\dot{x}_i = -\phi_i(e_i, d_i) z_i \tag{4}$$

where ϕ_i for fixed d_i is a smooth first-third quadrant nonlinearity in $e_i = ||z_i||^2 - d_i^2$.

Evidently, the whole system is nonlinear in a nontrivial way, and its analysis is by no means straightforward. Nevertheless, much can be shown. The following properties are probably true for the general family of laws just identified; however, at the moment, the properties have just been established for one at least of the two particular laws.

- **Property 1**. Suppose that the initial positions of all agents are such that they are not collinear. If no two of the d_i are the same, and the second control law is used, the errors $e_i = ||z_i||^2 - d_i^2, i = 1, 2, 3$ will all converge exponentially fast to zero, or equivalently the correct triangle shape is assumed exponentially fast. (The authors of [24] expect to remove the constraint that no two of the d_i can be the same. Also, under the first control law, a similar result holds. There is no restriction requiring no two of the d_i to be the same; but the initial positions from which convergence to a correct shape occurs have not been identified with the set of points corresponding to noncollinear agents)

- **Property 2**. If the initial positions of the agents are collinear, the subsequent trajectory will retain the collinearity property. However, the associated manifold is not stable, so that a random perturbation will result in the property 1 situation applying. (This has effectively been proved for the two laws)

- **Property 3**. If the initial positions result in the correct triangle shape being assumed exponentially fast, then the total translation of the triangle's centre of mass, and the total rotation of the triangle about the centre of mass over the interval $[0, \infty)$ are bounded, and all agents come to rest. (This has been formally proved for the first law, but is easy to extend to the second law)

- **Property 4**. If the initial positions of the agents are collinear but are otherwise generic, under the second law the agents will exponentially fast assume a common nonzero velocity.

4.2 Control of a Minimally Persistent Formation with Leader and First Follower

In this subsection, the graphs under consideration may have any number of vertices N greater than 2. Given the minimally persistent property, the number of edges is precisely $2N - 3$. A further consequence of minimal persistence[16] is that vertices can have an out-degree of at most 2, and apart from such vertices either there are exactly three vertices with an out-degree of 1, or one vertex with an out-degree of zero and one with an out-degree of 1. In the latter case, the vertex with zero out-degree is termed a *leader*. Having no neighbor from which its distance must be maintained, it is not constrained at all in its motion. When the vertex with out-degree 1 has the leader as its neighbor, it is termed a *first follower*. The graphs we consider here are leader-first-follower graphs. It turns out that some results for minimally persistent graphs without a leader-first-follower structure are, or appear to be, easily derivable from the result for a leader-first-follower structure. So leader-first-follower graphs are a suitable subclass to investigate initially. Also, in this subsection we restrict attention to graphs which contain one or more cycles, for reasons explained earlier.

It is also clear that before one could contemplate formation control for a non-minimally persistent graph, one would have to be able to treat minimally persistent graphs, and it might be straightforward to use that as a basis for non-minimally persistent graphs, though this has however yet to be done. It is this thinking anyway which has driven the work on the ideas of this subsection.

By way of a final qualifying remark, in this subsection, we are only able to present control laws for restoring a formation to its correct shape when it has undergone a *small* perturbation. The entire analysis is a linear one, and assumes that a linearized model is a valid approximation. Thus though in comparison to the previous section, the vertex number is general the convergence result is weaker.

We will now explain the system model and indicate the structure of the decentralized control law. We focus first on an agent j which has two neighbors, agents k and m, from which it must maintain its distance. Suppose that the three agents concerned are all displaced from their nominal positions, call them x_{0j}, x_{0k}, x_{0m}, by small amounts $\delta x_j, \delta x_k, \delta x_m$. Agent j determines to where it would need to move, in order to be at the correct distance from agents k and m, assuming the latter do not move while it is moving. Identify this target position as $x_{0j} + \delta^* x_j$. Note that $\delta^* x_j$ is a function of δx_k and δx_m. Agent j actually moves to reduce the distance between where it currently is and this target position. To do this it uses the following law:

$$\dot{\delta x_j} = A_j(\delta^* x_j(\delta x_k, \delta x_m) - \delta x_j) \tag{5}$$

with the following condition guaranteeing the distance reduction property

$$A_j + A_j^T > 0 \tag{6}$$

By simple geometric arguments, it is possible to express δx_j^* in terms of $\delta x_k, \delta x_m$ and the nominal agent positions x_{0j}, x_{0k}, x_{0m}. There results

$$\dot{\delta x_j} = A_j B_j \begin{bmatrix} \delta x_j \\ \delta x_m \\ \delta x_k \end{bmatrix} \tag{7}$$

where B_j is a matrix depending on differences of the position coordinates x_{0j}, x_{0k}, x_{0m}, and is actually a submatrix of the rigidity matrix of the formation. Similar equations can be written for every agent, and for the leader and first follower. For details, the reader is referred to [26]. When the equations are put together with the first follower and leader labelled as vertices $N-1$ and N, there results:

$$\frac{d}{dt} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_{N-1} \\ \delta x_N \end{bmatrix} = \Gamma \begin{bmatrix} R \\ 0_{3\times 2n} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_{N-1} \\ \delta x_N \end{bmatrix} \tag{8}$$

where Γ is obtained by stacking together into a block diagonal matrix the 'gain' matrices A_j each multiplied by a nonsingular 2×2 matrix , and R is actually the rigidity matrix of the formation with agents at their nominal locations.

It is convenient to drop the last three rows of this equation (and the last three columns of R), to form a smaller size equation. The associated modes are attributable to the *degrees of freedom (DOF)*[3] of the leader and the first follower, which if exercised would allow the whole formation to translate or rotate. From the point of view of shape stabilization or control of the formation however, they are not used.

3 In $\mathbf{R}^2$, a vertex has two, one or zero DOF(s) if it has no, one, or at least two outgoing edges; each outgoing edge uses up one DOF. A minor variation applies in 3 dimensions.

The key question now is: how should the entries of Γ –a block diagonal matrix whose block entries are the A_j multiplied by known 2×2 nonsingular matrices–be chosen, in order that the linear equation set be stable. This will then ensure that when the agents of the formation are displaced a small distance from their nominal positions, the corrective action taken by each agent will drive the displacements to zero.

It is incidentally easy to find examples where the choice of $A_j = I$ is *destabilizing*. Thus the problem is nontrivial.

The key is the following result, [22, 26], which actually comes with a constructive proof:

Theorem 4 *Consider the linear differential equation $\dot{x} = \Lambda A x$ in which a square real A is prescribed and Λ is diagonal, real and otherwise adjustable. Then a sufficient condition that there exists a choice of Λ such that the equation is asymptotically stable, i.e. the eigenvalues of ΛA all have negative real parts, is that the leading principal minors of A are nonzero.*

It is incidentally easy to show that actual eigenvalue positionability is impossible via choice of Λ in a number of cases where securing eigenvalues in the left half plane is possible.

The matrix Λ of the theorem corresponds to our Γ, minus its last three rows and columns. Notice that Γ, being *block* diagonal has more adjustable entries than does Λ, which is strictly diagonal, and it is an open question as to how that fact might be exploited. The matrix A of the theorem corresponds to the rigidity matrix R less its last three columns. Does it necessarily have nonzero leading principal minors (perhaps after vertex reordering)? The answer is yes. The argument is far from trivial and depends on various properties of minimally persistent graphs, see [22, 26].

Evidently, the preceding results should only be the first of many. Ultimately, one would like to know which formations are easy to control, and to understand how to exploit the freedom in the choice of control laws for each agent to achieve some kind of tradeoffs. One would like results which were not just local or linearized, and one would like results which could deal with agents with their own dynamics, and one could envisage control laws other than memoryless ones.

5 FORMATION OPERATIONS

In many autonomous multi-agent formation applications, one needs to analyze certain scenarios (with significant probability of occurrence), as a matter of guaranteeing robustness in the presence of such scenarios. There exist three particular classes of such scenarios investigated in the recent literature (see, e.g. [31]) under the names *splitting*, *merging* and *closing ranks*.

Splitting refers to the scenario or the formation operation where a single rigid formation splits into two separate formations, i.e. its agents are divided into two subsets, and the distance constraints between agents in the different subsets are suppressed. Splitting may occur because of a change of objective, or to avoid an obstacle etc. An illustration of the problem is given in Fig. 5 Note that any of the two sub-formations after the split may be non-rigid. The main task in his scenario is introduction of additional distance constraints in the separate subformations to ensure rigidity of them both.

In *merging*, two given rigid formations are required to be merged in a single formation via adding some extra links such

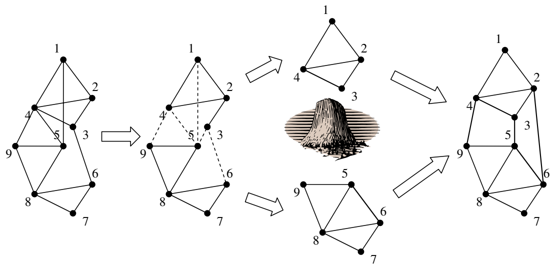

Fig. 5 Illustration of a formation splitting and merging process to avoid obstacles

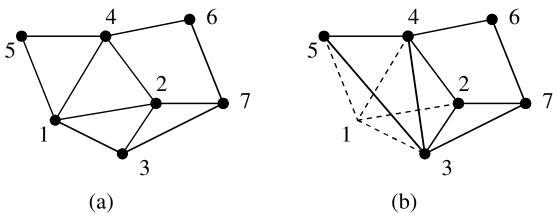

Fig. 6 Illustration of a closing rank process where agent 1 and all associated links are lost, and new links (5,3) and (4,3) are inserted to re-establish the rigidity. Agents 6 and 7 are not affected during the process.

that the resultant single formation is also rigid. More specifically, the task is to determine the additional distance constraints, with one agent in each formation, such that the union of the agents of the two formations, and the union of the distance constraints in the original formations and the new distance constraints, will describe a single rigid formation. The problem is also illustrated in Fig. 5.

In the *closing ranks* scenario (or operation), in a given single rigid formation, one agent is removed, and, consequentially, any distance constraints that applied between this agent and the remaining agents of the formation; and the task is to insert new distance constraints appropriately in order that the formation can be re-rendered rigid. The closing ranks problem is illustrated in Fig. 6.

The splitting, merging and closing ranks problems can be defined in different ways for different settings such as the case where the starting formation(s) is (are) minimally rigid, the case where the formations of interest are 2- or 3-dimensional, or a directed case where persistence is required to be maintained after the formation operation. In addition, the closing ranks problem can be generalized to contemplate formations in which two or more agents are simultaneously removed, with the associated distance constraints.

For all the three formation operations or scenarios above, one can also consider questions of algorithm complexity, and the possibility of posing computational constraints on individual agents if there is a wish to perform calculations on a decentralized basis.

Perhaps surprisingly, it turns out that the splitting problem is actually a particular case of the closing ranks problem. One sub-formation can regard the agents of the other sub-formation as the lost agents. Furthermore, and the result is not trivial, the closing ranks problem can always be solved by introducing new links (edges in the representative graph) between former neighbors of the lost agents (vertices) of the formation. In connection with the splitting problem, this means that any new links can be restricted to connecting pairs of those agents in one sub-formation that were previously neighbors of agents which ended up in the other sub-formation.

Having this unified framework for the formation operations, a so-called *minimal cover* problem was introduced and solved in [31]: in the minimal cover problem, a formation is presented which is not minimally rigid. One is required to determine a minimal set of links (minimality being in the sense of the actual number) which when added to the formation will render it rigid. The solution of the minimal cover problem can be applied to solve each of the problems of formation merging, splitting and closing ranks.

In the next section, a more recent and different approach to formation operation problems, particularly merging, is described. These formation operation problems can also be contemplated for directed graphs although little work has so far been done. A recent study on merging of persistent formations is presented in [32].

6 FORMATION MERGING AND META-FORMATIONS

In formation merging, it is of interest to establish what are the general rules concerning the connection of formations to form larger formations, particularly ensuring preservation of rigidity. Observing in general that the internal connections of the individual formations are unimportant or largely irrelevant in the merging operation, we refer to the larger post-merge formation as a *meta-formation*[22].

As the nomenclature implies, a (rigid) *meta-formation* is composed of a number of smaller (rigid) sub-formations (the pre-merge formations) and a number of inter-sub-formation links bringing these sub-formations together (and maintaining the rigidity). In the meta-formation framework, the sub-formations and the inter-sub-formation links composing the meta-formation are called *meta-vertices* (or *meta-agents*) and *meta-edges* (or *meta-links*), respectively[33].

The concept of *meta-formation* has been investigated in the literature under different names as well. The papers [34,35] use the *body-bar-joint* terminology. In this terminology, a *body* is like a generalization of a point agent. Any rigid formation of agents can be replaced by a body, a rigid object that in 2 dimensions has three DOFs, two displacements and one rotation. (In contrast, a point agent in 2 dimensions has two DOFs, both translational). Each body can be deemed to have a set of connection points on its surface, with the property that distances can be constrained between two connection points in different bodies. One can imagine a formation comprising a set of bodies, which might also be considered as meta-vertices or meta-agents, with certain distance constraints between them (usually more than one connection point on the surface of a body is used; for if only one connection point were used, the body or the meta-vertex could rotate about it). The term meta-vertex is however probably best restricted to applying to the graph equivalent of a body. One can also pose the question: when will such a formation be rigid? Of course, it is desired to answer this question taking no account of the internal structure of the bodies.

The question was answered for meta-formations of bodies in [34, 35], using both a generalization of the rigidity matrix, and a generalization of Theorem 1 for the 2-dimensional case. Recall that Theorem 1 provides necessary and sufficient conditions for generic rigidity of a graph corresponding to a formation of point agents, and the conditions are of a "counting" form; a simple adjustment of certain numbers appearing in the statement of Theorem 1 converts it to a theorem concerning generic rigidity of a graph corresponding to a 2-dimensional body-bar-joint framework. As for checking rigidity of a normal graph in $\mathbf{R}^3$, the available counting conditions for 3-dimensional body-bar framework are necessary for it to be rigid but not sufficient. The rigidity matrix ideas work in 3-dimensions (where the bodies are 3-dimensional and thus have six DOFs, three translational ones and three rotational ones)[34].

Interconnection of two formations is a matter of interconnection of two bodies, and the Theorem 1 extension easily provides the result that three distance constraints between connection points on each of the two bodies, with at least two connection points involved for each body, serves to give rigidity of the overall formation. This idea could be extended to the merging of more than two formations (meta-vertices) and agents (vertices)[33].

The problem described above of connecting two formations in 2-dimensions is investigated in [36]. [36] also investigates the problems of connecting (via insertion of additional edges) two formations in 3-dimensions to secure minimal rigidity, connecting two formations in 2 or 3-dimensions to secure global rigidity and connecting two formations when they are not disjoint, i.e., they are permitted to have a limited number of common vertices and/or a limited number of common edges.

By appealing to various results on rigidity and global rigidity, a series of conditions are established in [36] to solve these problems. The conditions are generally of the form: make m connections, involve at least n vertices of one formation, and at least p vertices of the second formation. The details of these results can be found in [36].

Directed versions of these results have in part been obtained[32]. Of course, the conditions for securing persistence always include those applicable to securing rigidity, as discussed before. The findings of [32] are summarized in the following paragraphs.

In order to merge two minimally persistent graphs (in $\mathbf{R}^3$) into a larger minimally persistent graph, one needs to add six directed interconnection edges that leave vertices with some positive DOF in the initial (pre-merging) graphs (one DOF for each out-going edge) but that can arrive at any vertices of the other initial (pre-merging) graphs. (In $\mathbf{R}^3$, the DOFs of a vertex are three, two, one or zero, according as the vertex has respectively no, one, two, three or more outgoing edges.) Not every selection of interconnection edges leads to a persistent merged graph, but it is always possible to find a set of interconnection edges that makes the merged graph structurally persistent, even when the initial (persistent) graphs are not structurally persistent.

When the merged graph needs to be persistent and not necessarily minimally persistent, one still needs to add six directed edges leaving vertices with some (positive) DOFs. The number of new edges leaving a vertex with positive DOF must be no greater than its DOF. Other edges (possibly leaving vertices without DOF) can under some conditions also be added, but they can always be avoided. As a consequence, at least six DOFs must be available in the two initial graphs; otherwise the two graphs cannot be merged.

If one of the two initial graphs has no DOF and if simultaneously the other one is not structurally persistent, then they cannot be merged into a persistent graph.

Further details on persistent formation merging and persistent meta-formations can be found in [32] and more recent work [37].

7 CONCLUDING REMARKS

In this paper, we have reviewed a number of concepts and results relevant to the design of control schemes and information architectures to maintain the shape of a formation of autonomous agents. Many of these concepts and results rest on ideas of rigid graph theory, some but not all of which are old. The background theory however has a number of gaps in it, and their elimination would help in applications. A particular gap in the relevant graph theory is non-existence of an analog for 3-dimensional graphs of Theorem 1, which provides for 2-dimensional graphs a combinatorial criterion for rigidity. Actually, it may be the case that one will need to study graphs in dimensions higher than 3, because the physical agents they are modelling may have orientation as well as position (see for example [3]). Generally speaking, results for dimensions higher than 3 are even less well documented.

Turning more to control problems, we can identify many very pertinent problems. In the ideal world of point agents, be it in two dimensions or three dimensions, it would be helpful to identify the class of decentralized control laws that could maintain formation shape. Of course, one must agree first on what variables are sensed, and here we have emphasised distance, and in the case of controlling distances, one must agree on what agent has the responsibility (or possibly shares the responsibility) for controlling what distance. Distance is naturally not the only variable, and other types of sensed variables should be contemplated.

It would be very helpful also to understand what formations were easy to control and what formations were hard to control: one that was easy to control would presumably require just small control signals to restore the formation shape when perturbed away from its ideal, and would be comparatively insensitive to noise perturbing the measurements. These questions also need to be posed when significant agent dynamics are introduced into the model; when there are speed, turning rate and sensing radius constraints; and when there are constraints on sensing involving typically blind cones (agents may not be allowed to have "eyes in the back of their head"). At a higher (architectural) level, it would be useful to know how extra sensed data might assist the control problem (through allowing smaller signals, or by diminishing the adverse effects of measurement noise), and how redundancy to protect against loss of a single control or communications link, and/or redundancy to protect against loss of a single agent and all the incident links, might best be secured by building extra links into the graphical models underpinning the control strategy; in particular, if one were allowed to add two extra links to a formation, where should they be located?

Another high-level architectural problem is exemplified by the formation merging and splitting problems. When one of these operations occurs, can the agents in a decentralized manner determine the corrective action (insertion of links in the underlying graph) that needs to be taken? The emphasis here is on the word "decentralized"; the discussion in the paper has emphasised a centralized approach for addressing such problems.

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