Proceedings of the European Control Conference 2007
Kos, Greece, July 2-5, 2007

A "Mixed" Small Gain and Passivity Theorem for an Interconnection of Linear Time-Invariant Systems

Wynita M. Griggs, Brian D. O. Anderson and Alexander Lanzon

***Abstract*— We show that the negative feedback interconnection of two causal, stable, linear time-invariant systems with a "mixed" small gain and passivity frequency domain property is guaranteed to be finite-gain stable. This "mixed" small gain and passivity property refers to the characteristic that the frequency range $-\infty < \omega < \infty$ can be divided into intervals for which the two systems in the interconnection are both: a) "input and output strictly passive" (and one or both of the systems may or may not have "gain less than one"); or b) "input and output strictly passive and with gain less than one"; or c) "with gain less than one" (and one or both of the systems may or may not be "input and output strictly passive"). The "mixed" small gain and passivity property is described mathematically using the notion of dissipativity of systems, and finite-gain stability of the interconnection is proven via a stability result for dissipative interconnected systems.**

***Index Terms*— Dissipative systems, linear systems, negative feedback stability, finite gain, passivity.**

I. INTRODUCTION

The small gain and passivity theorems are two of the most important results in the theory of stability of input-output systems. The small gain theorem states that if the product of the gains of two stable systems is less than one then the feedback interconnection of the two systems is stable [1]–[4]. The passivity theorem guarantees stability of a feedback interconnection of two stable systems if, for instance, both of the systems are passive, and one of them is input strictly passive with finite gain [1]–[3], [5]. Of course, there exist many situations where stability cannot be guaranteed by use of the small gain or passivity theorems because the classes of systems under consideration are not compatible.

For instance, it has been observed that high frequency dynamics can frequently destroy the passivity property of an otherwise passive system. A celebrated controversy in adaptive control [6] depended on the observation that passivity conditions normally forming part of the hypotheses of the proofs of convergence of certain adaptive control algorithms should not be assumed to be valid in practice (because high frequency dynamics often neglected for modelling purposes will always be present in a real system). Failure of the passivity condition invalidated the applicability of the associated theorem on the algorithm convergence to most real-life applications, and left a cloud hanging over the real-life use of the algorithm. Simulations of [6] confirmed that adverse behavior could occur when high frequency dynamics were explicitly taken into account.

The book [7] (see also [8] and [9]) described tools for establishing stability of adaptive systems of the type examined in [6]; that is, where passivity properties hold only for low frequency signals (in a sense made precise later in this paper). Stability is established if additionally, and in a rough manner of speaking, gains are small at high frequencies, ie: a small gain property in the sense of the small gain theorem holds in the frequency band where the passivity condition fails (again, more precise discussion of what is meant by small gain just at high frequencies appears later). The fact that there is an important class of applications in which passivity and small gain ideas have to be blended is a motivation for this paper.

Indeed, the idea of "merging" the passivity and small gain theorems to provide stability results for feedback interconnections containing a class of systems, broader than those dealt with by the small gain and passivity theorems alone, would be extremely useful. For example, consider two open-loop causal, stable, single-input single-output (SISO) systems with linear time-invariant (LTI) transfer functions, say

$$m_1(s) = \frac{3}{(s+1)(s+2)}$$

and

$$m_2(s) = \frac{13}{(s+3)(s+4)}$$

with Nyquist diagrams shown in Fig's. 1 and 2. It is clear that, if in some frequency range $[0, \Omega]$ the systems are passive (ie: the real part of each of the transfer functions is positive), and if in the frequency range $[\Omega, \infty)$ the product of the amplitudes of the transfer functions is less than one, then there is no way that the Nyquist diagram of the cascade would encircle the point $-1 + j0$. Accordingly, the closed-loop would be stable. For example, Fig. 3 shows that the Nyquist diagram of $m_1(s)m_2(s)$ does not encircle $-1 + j0$.

Continuing with the example, note that one could not simply determine stability by scaling one of the systems with transfer functions $m_1(s)$ or $m_2(s)$ to have gain less than one, as this would result in an increase in the other system's gain.

This work was supported by ARC Discovery-Projects Grants (DP0342683, DP0664427) and National ICT Australia. National ICT Australia is funded through the Australian Government's *Backing Australia's Ability* initiative, in part through the Australian Research Council.

W. Griggs and B. Anderson are with Department of Information Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia; and National ICT Australia Limited, Locked Bag 8001, Canberra ACT 2601, Australia.

A. Lanzon is with Control Systems Group, School of Electrical and Electronic Engineering, The University of Manchester, Sackville Street Building, Manchester M60 1QD, UK.

Corresponding author: Wynita M. Griggs, Department of Information Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia wynita.griggs@anu.edu.au

ISBN: 978-960-89028-5-5

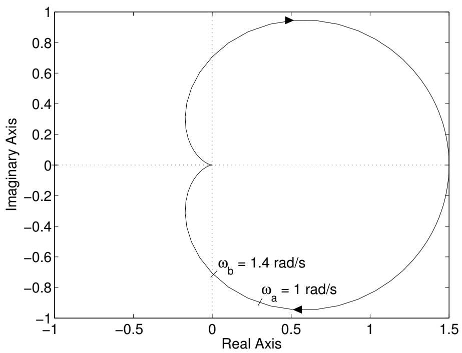

Fig. 1. Nyquist diagram of $m_1(s)$.

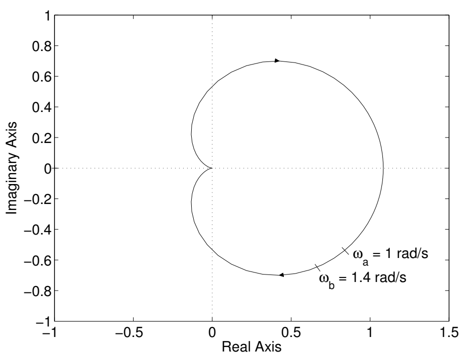

Fig. 2. Nyquist diagram of $m_2(s)$.

That is, absolute feedback loop gain is constant. Similarly, multipliers cannot always be used to transform the feedback loop such that both systems are passive and one is strictly passive with finite gain. There exist transformations in the literature that transform a passive system to a system with gain less than one; and vice versa [1], [2]. As an extension to this, one could consider doing a stability-preserving, frequency-dependent loop transformation on feedback loops consisting of systems with "mixed" small gain and passivity properties like those discussed in the example above, such that the product of the gains of the transformed systems is less than one say, and hence the small gain theorem could be applied to determine stability. Initial investigations hint that such a successful loop transformation in general may be difficult to find.

In this paper, we develop the idea of merging the passivity and small gain theorems. We consider multi-input multi-output (MIMO), causal, stable, LTI systems connected in a negative feedback loop, as illustrated in Fig. 4, which have a "mixed" small gain and passivity frequency domain property, demonstrated by the systems described by transfer functions $m_1(s)$ and $m_2(s)$ above. We exploit the notion of dissipativity, initiated by [10] and used by [11]–[14] to produce stability results for interconnected systems, to mathematically describe this "mixed" small gain and passivity frequency domain property. This description is given in Section II. The main result of the paper shows that finite-gain stability of the feedback interconnection is guaranteed.

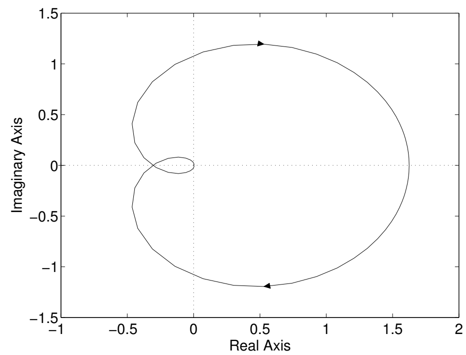

Fig. 3. Nyquist diagram of $m_1(s)m_2(s)$.

The feedback interconnection is described in Section III and the main result is given in Section IV. Section V contains the conclusions and outlines intended future development of the ideas presented in this paper.

Notation

The results of this paper are concerned with the frequency domain. We consider frequency domain signals $f \in \mathcal{L}_2(j\mathbb{R})$, where $\mathcal{L}_2(j\mathbb{R})$ denotes the real frequency domain Lebesgue space in which

$$\|f\| = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega) f(j\omega) d\omega \right\}^{\frac{1}{2}}$$

and the superscript $(\cdot)^*$ denotes the complex conjugate transpose. $\mathcal{L}_2(j\mathbb{R})$ is a Hilbert space under the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} g^*(j\omega) f(j\omega) d\omega.$$

$\mathcal{R}$ denotes the set of proper real rational transfer function matrices. For a transfer function matrix $G \in \mathcal{R}$, $G^*(s)$ is defined to mean $G(-s)^T$. $\mathcal{L}_\infty$ is a Banach space of matrix- (or scalar-) valued functions that are essentially bounded on $j\mathbb{R}$. The Hardy space, $\mathcal{H}_\infty$, is the closed subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane (RHP), with norm denoted $\|\cdot\|_\infty$. In other words, $\mathcal{H}_\infty$ is the space of transfer functions of stable, LTI, continuous-time systems. $\mathcal{RH}_\infty$ denotes the subspace of $\mathcal{H}_\infty$ whose transfer function matrices are proper and real rational.

II. MATHEMATICAL DESCRIPTION OF SYSTEMS

In this section, a mathematical description for a causal LTI system with transfer function matrix $M \in \mathcal{RH}_\infty$ and

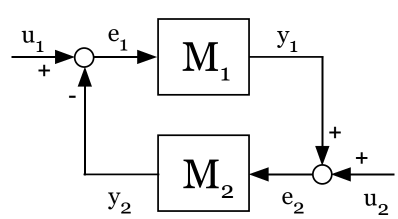

Fig. 4. Feedback interconnection of M_1 and M_2.

with the following frequency domain property is formulated. Consider the frequency range $-\infty < \omega < \infty$ and divide this range into intervals for which system M is: a) "input and output strictly passive"; b) "input and output strictly passive and with gain less than one"; or c) "with gain less than one". This property will be referred to throughout the paper as the "mixed" small gain and passivity frequency domain property of a system M. What is meant by a system being input and output strictly passive on a frequency interval, or having finite gain on a frequency interval, is defined below. The standard notions of input and output strict passivity and finite gain which refer to the full $j\omega$-axis are also provided.

Definition 1: [1], [5] Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$. This system is input and output strictly passive if $\exists \delta > 0$, $\epsilon > 0$ such that

$$\langle Mx, x\rangle \geq \delta\|x\|^2 + \epsilon\|Mx\|^2$$

$\forall x \in \mathcal{L}_2(j\mathbb{R})$.

In [13]–[15], input and output strict passivity is referred to as very strong passivity (VSP).

Definition 2: Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$ and consider frequencies in the interval $[a, b]$. Call the system input and output strictly passive on the frequency interval $[a, b]$ if $\exists \delta > 0$, $\epsilon > 0$ such that

$$\langle Mx, x\rangle_{[a,b]} \geq \delta\|x\|^2_{[a,b]} + \epsilon\|Mx\|^2_{[a,b]}$$

$\forall x \in \mathcal{L}_2(j\mathbb{R})$, where, given $x, y \in \mathcal{L}_2(j\mathbb{R})$,

$$\langle y, x\rangle_{[a,b]} := \frac{1}{2\pi}\int_a^b x^*(j\omega)y(j\omega)d\omega \tag{1}$$

and

$$\|(\cdot)\|^2_{[a,b]} := \langle(\cdot), (\cdot)\rangle_{[a,b]}. \tag{2}$$

If $x \in \mathcal{L}_2(j\mathbb{R})$ is the Fourier transform of a real-valued signal, it follows that $\langle Mx, x\rangle_{[-b,-a]} \geq \delta\|x\|^2_{[-b,-a]} + \epsilon\|Mx\|^2_{[-b,-a]}$ $\forall x \in \mathcal{L}_2(j\mathbb{R})$.

Definition 3: [1], [2] Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$. This system is said to have finite gain if $\exists k < \infty$ such that

$$\|Mx\| \leq k\|x\|$$

$\forall x \in \mathcal{L}_2(j\mathbb{R})$.

It is worth noting that input and output strict passivity is equivalent to input strict passivity with finite gain [5], [13], [15].

Definition 4: Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$ and consider frequencies in the finite interval $[a, b]$. Say that the system has finite gain on the frequency interval $[a, b]$ if $\exists k < \infty$ such that

$$\|Mx\|_{[a,b]} \leq k\|x\|_{[a,b]}$$

$\forall x \in \mathcal{L}_2(j\mathbb{R})$, where $\|(\cdot)\|_{[a,b]}$ is defined by (2). If $x \in \mathcal{L}_2(j\mathbb{R})$ is the Fourier transform of a real-valued signal, it follows that $\|Mx\|_{[-b,-a]} \leq k\|x\|_{[-b,-a]}$ $\forall x \in \mathcal{L}_2(j\mathbb{R})$.

Finite frequency intervals $[a, b]$ are considered in the above definitions of input and output strict passivity on a frequency interval and finite gain on a frequency interval. However, infinite frequency intervals $[a, b)$, $(a, b]$ or (a, b), where a or b may be equal to $\pm\infty$, may be considered by taking improper integrals in (1) and (2) where appropriate.

The "mixed" small gain and passivity frequency domain property of a system M can be described mathematically using the notion of dissipativity of systems as follows. First we give a definition of a dissipative system.

Definition 5: Consider a causal system with transfer function matrix $M \in \mathcal{RH}_\infty$. Denote the system's input and output signals, $e \in \mathcal{L}_2(j\mathbb{R})$ and $y \in \mathcal{L}_2(j\mathbb{R})$, respectively. The system is said to be dissipative with respect to the triple $(Q(\omega), S(\omega), R(\omega))$ if

$$\langle y, Q(\omega)y\rangle + 2\langle y, S(\omega)e\rangle + \langle e, R(\omega)e\rangle \geq 0$$

$\forall e \in \mathcal{L}_2(j\mathbb{R})$, where $Q(\omega)$ and $R(\omega)$ are self-adjoint at every ω (ie: $Q(\omega)^T = Q(\omega)$ and $R(\omega)^T = R(\omega)$) and $Q(\omega)$ is also negative semi-definite at every ω.

Define a real continuous (even) function of frequency that is: i) equal to one on frequency intervals for which M is considered "input and output strictly passive"; ii) equal to zero on frequency intervals for which M is considered to have "gain less than one"; and iii) is strictly greater than zero and strictly less than one on frequency intervals for which M is considered "input and output strictly passive with gain less than one". Denote this function $\alpha(\omega)$. Then the "mixed" small gain and passivity frequency domain property of system M can be described by letting

$$\begin{aligned} Q_m(\omega) &:= Q(\omega) = -(\epsilon\alpha(\omega) + 1 - \alpha(\omega))I \\ S_m(\omega) &:= S(\omega) = \alpha(\omega)I \\ R_m(\omega) &:= R(\omega) = (k^2(1 - \alpha(\omega)) - \delta\alpha(\omega))I \end{aligned}$$

in Definition 5, where $k < 1$, $\delta > 0$ and $\epsilon > 0$. That is, the statement that system M is dissipative with respect to the triple $(Q_m(\omega), S_m(\omega), R_m(\omega))$ means that

$$\langle y, Q_m y\rangle + 2\langle y, S_m e\rangle + \langle e, R_m e\rangle \geq 0 \tag{3}$$

$\forall e \in \mathcal{L}_2(j\mathbb{R})$.

To see that the desired "mixed" property of a system M is accurately described using the notion of dissipativity as above, note that the left hand side (LHS) of (3) is equal to

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} Q_m(\omega)e^*(j\omega)M^*(j\omega)M(j\omega)e(j\omega)d\omega \\ &+\frac{1}{2\pi}\int_{-\infty}^{\infty} S_m(\omega)e^*(j\omega)[M^*(j\omega) + M(j\omega)]e(j\omega)d\omega \\ &+\frac{1}{2\pi}\int_{-\infty}^{\infty} R_m(\omega)e^*(j\omega)e(j\omega)d\omega. \end{aligned} \tag{4}$$

We continue by illustrating with a simple example. Suppose that system M has finite gain less than one on the frequency intervals $(-\infty, -\omega_b]$ and $[\omega_b, \infty)$; is input and output strictly passive and has finite gain less than one on the frequency intervals $(-\omega_b, -\omega_a)$ and (ω_a, ω_b); and is input and output strictly passive on the frequency interval $[-\omega_a, \omega_a]$. For the example system described by the transfer function $m_1(s)$ in Section I, we could take $\omega_a = 0.924$ and $\omega_b = 1.414$.

Breaking the integrals from $-\infty$ to ∞ of (4) into integrals from $-\infty$ to $-\omega_b$, $-\omega_b$ to $-\omega_a$, $-\omega_a$ to ω_a, ω_a to ω_b and ω_b to ∞; grouping the integrals from each respective frequency range together and adding the integrands; and substituting into the integrands values of $\alpha(\omega) = 1$ for the integrals from $-\omega_a$ to ω_a, and $\alpha(\omega) = 0$ for the integrals from $-\infty$ to $-\omega_b$ and ω_b to ∞, gives

$$\frac{1}{2\pi}\int_{\omega_b}^{\infty} e^*(k^2 I - M^*M)e\,d\omega \quad (5)$$
$$+\frac{1}{2\pi}\int_{\omega_a}^{\omega_b} e^*(Q_m M^*M + S_m(M^* + M) + R_m)e\,d\omega \quad (6)$$
$$+\frac{1}{2\pi}\int_{-\omega_a}^{\omega_a} e^*(M^* + M - \epsilon M^*M - \delta I)e\,d\omega \quad (7)$$
$$+\frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} e^*(Q_m M^*M + S_m(M^* + M) + R_m)e\,d\omega \quad (8)$$
$$+\frac{1}{2\pi}\int_{-\infty}^{-\omega_b} e^*(k^2 I - M^*M)e\,d\omega. \quad (9)$$

Now integrals (5) and (9) are greater than or equal to zero since M has finite gain less than one on the frequency intervals $(-\infty, -\omega_b]$ and $[\omega_b, \infty)$. Integral (7) is greater than or equal to zero since M is input and output strictly passive on the frequency interval $[-\omega_a, \omega_a]$. It remains to show that integrals (6) and (8) are greater than or equal to zero. Note that integral (6) is equal to

$$\frac{1}{2\pi}\int_{\omega_a}^{\omega_b} \alpha e^*(M^* + M - \epsilon M^*M - \delta I)e\,d\omega$$
$$+\frac{1}{2\pi}\int_{\omega_a}^{\omega_b} (1-\alpha)e^*(k^2 I - M^*M)e\,d\omega,$$

which is greater than or equal to zero because $0 \le \alpha(\omega) \le 1$ and M is both input and output strictly passive and has finite gain less than one on the frequency interval (ω_a, ω_b). Similarly, integral (8) is greater than or equal to zero.

We conclude the section with the following comment on the division of the frequency range.

Remark 1: The division of the frequency range $-\infty < \omega < \infty$ into intervals for which a system M is: a) "input and output strictly passive"; b) "input and output strictly passive and with gain less than one"; or c) "with gain less than one" should be interpreted to mean to divide the frequency range $-\infty < \omega < \infty$ into intervals for which a system M is a) "input and output strictly passive" (*and may or may not have "gain less than one"*); b) "input and output strictly passive and with gain less than one"; or c) "with gain less than one" (*and may or may not be "input and output strictly passive"*).

For example, consider Nyquist diagrams of transfer functions of SISO systems with the "mixed" small gain and passivity frequency domain property, such as $m_1(s)$ and $m_2(s)$ given in Section I. Remark 1 indicates that it is not required that the divisions of the frequency range occur precisely at those frequencies for which the Nyquist diagrams cross the unit circle and the $j\omega$-axis. For example, the Nyquist diagram of $m_1(s)$ indeed crosses the unit circle at frequencies ± 0.924 and crosses the $j\omega$-axis at frequencies ± 1.414, and so one could take $\omega_a = 0.924$ and $\omega_b = 1.414$. However, the notion of using dissipativity to describe the "mixed" small gain and passivity frequency domain property of a system still holds if we take $1.414 > \omega_b > \omega_a > 0.924$.

The above comment is important in the following manner. In the forthcoming sections, we will be interested in determining stability of negative feedback interconnections of two systems, where each system has a "mixed" small gain and passivity frequency domain property. To determine stability, we require that common frequency intervals can be found on which both systems in the feedback interconnection are "input and output strictly passive with gain less than one". That is, we require that the frequency range $-\infty < \omega < \infty$ can be divided into intervals for which the two systems in the interconnection are both: a) "input and output strictly passive" (and one or both of the systems may or may not have "gain less than one"); or b) "input and output strictly passive and with gain less than one"; or c) "with gain less than one" (and one or both of the systems may or may not be "input and output strictly passive"). For instance, consider the interconnection of the two systems described by the transfer functions $m_1(s)$ and $m_2(s)$ given in Section I. These systems are "input and output strictly passive with gain less than one" on, say, the common frequency intervals $(-1.4, -1)$ and $(1, 1.4)$ (as shown in Fig's. 1 and 2). It would therefore be satisfactory to set $\omega_a = 1$ and $\omega_b = 1.4$.

Discussion on the relaxation of the requirement of "input and output strict passivity on a frequency interval" for one of the systems in the feedback loop occurs later in the paper. This discussion is important because this situation is analogous to the passivity theorem's supposition that one system is input strictly passive with finite gain, while the other system is passive. We also discuss, by giving an example, how multipliers (or weights) can be used to scale the interconnection when one or both of the original systems in the feedback loop do not exhibit the "mixed" small gain and passivity frequency domain property, hence increasing the system class size for which the results of this paper are applicable.

III. INTERCONNECTION OF SYSTEMS

Giving reference to Remark 1 and the second last paragraph of the previous section, we now consider the feedback interconnection of two systems M_1 and M_2 as shown in Fig. 4, which are dissipative in the sense of Definition 5. Let the $(Q(\omega), S(\omega), R(\omega))$ triple associated with system M_i, $i = 1, 2$, be given by

$$Q_i(\omega) = -(\epsilon_i \alpha(\omega) + 1 - \alpha(\omega))I \quad (10)$$
$$S_i(\omega) = \alpha(\omega) I \quad (11)$$
$$R_i(\omega) = (k_i^2(1 - \alpha(\omega)) - \delta_i \alpha(\omega))I \quad (12)$$

where $\alpha(\omega)$ is as described previously. In the spirit of [11]–[14], where constant (Q_i, S_i, R_i) triples are considered as opposed to frequency-dependent triples, we show that the interconnected system is also dissipative in a sense to be described. This description of dissipativity of the closed-loop provides us with a tool to prove finite-gain stability of the interconnection of systems M_1 and M_2, which is done in the next section.

Denote the interconnection of systems M_1 and M_2 by M_{sys}. So M_1 and M_2 are interconnected via

$$e_1 = u_1 - y_2 \tag{13}$$

$$e_2 = u_2 + y_1 \tag{14}$$

as indicated in Fig. 4. The input and output signal space for M_{sys} is the product space $\mathcal{L}_2(j\mathbb{R}) \times \mathcal{L}_2(j\mathbb{R})$, and the elements of the input and output signal space are $u := \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$ and $y := \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$, respectively. Note that inner products in these spaces are derived by summing inner products in the component spaces.

Assume that the system M_{sys} is well-posed in the sense of [4]. Write (13) and (14) in the compact form

$$e = u - Hy \tag{15}$$

where $H := \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Define

$$\tilde{Q} := \begin{pmatrix} Q_1(\omega) & 0 \\ 0 & Q_2(\omega) \end{pmatrix}$$
$$\tilde{S} := \begin{pmatrix} S_1(\omega) & 0 \\ 0 & S_2(\omega) \end{pmatrix}$$
$$\tilde{R} := \begin{pmatrix} R_1(\omega) & 0 \\ 0 & R_2(\omega) \end{pmatrix}.$$

Then similarly to [11]–[14], it can be shown that M_{sys} is $(\bar{Q}, \bar{S}, \tilde{R})$ dissipative, where

$$\begin{aligned} \bar{Q} : &= \tilde{Q} + H^T\tilde{R}H - \tilde{S}H - H^T\tilde{S}^T \\ &= \begin{pmatrix} -\bar{q}_1 I & 0 \\ 0 & -\bar{q}_2 I \end{pmatrix} \end{aligned}$$

with $\bar{q}_1 := (1-k_2^2)(1-\alpha(\omega)) + (\epsilon_1+\delta_2)\alpha(\omega) > 0$, $\bar{q}_2 := (1-k_1^2)(1-\alpha(\omega)) + (\epsilon_2+\delta_1)\alpha(\omega) > 0$ and

$$\begin{aligned} \bar{S} : &= \tilde{S} - H^T\tilde{R} \\ &= \begin{pmatrix} \alpha(\omega)I & \bar{s}_1 I \\ -\bar{s}_2 I & \alpha(\omega)I \end{pmatrix} \end{aligned}$$

with $\bar{s}_1 := k_2^2(1-\alpha(\omega)) - \delta_2\alpha(\omega)$, $\bar{s}_2 := k_1^2(1-\alpha(\omega)) - \delta_1\alpha(\omega)$, by adding inequalities

$$\langle y_i, Q_i y_i\rangle + 2\langle y_i, S_i e_i\rangle + \langle e_i, R_i e_i\rangle \geq 0$$

with $i = 1, 2$ and substituting (15) in as follows:

$$\begin{aligned} &\langle y_1, Q_1 y_1\rangle + \langle y_2, Q_2 y_2\rangle + 2\langle y_1, S_1 e_1\rangle + 2\langle y_2, S_2 e_2\rangle + \\ &\langle e_1, R_1 e_1\rangle + \langle e_2, R_2 e_2\rangle \geq 0 \\ &\Leftrightarrow \langle y, \tilde{Q}y\rangle + 2\langle y, \tilde{S}e\rangle + \langle e, \tilde{R}e\rangle \geq 0 \\ &\Leftrightarrow \langle y, \tilde{Q}y\rangle + 2\langle y, \tilde{S}u - \tilde{S}Hy\rangle + \langle u - Hy, \tilde{R}u - \tilde{R}Hy\rangle \\ &\quad \geq 0 \\ &\Leftrightarrow \langle y, \tilde{Q}y\rangle + \langle y, H^T\tilde{R}Hy\rangle + \langle y, -\tilde{S}Hy\rangle + \langle y, -H^T\tilde{S}^T y\rangle \\ &\quad + 2\langle y, \tilde{S}u\rangle + 2\langle y, -H^T\tilde{R}u\rangle + \langle u, \tilde{R}u\rangle \geq 0 \\ &\Leftrightarrow \langle y, \bar{Q}y\rangle + 2\langle y, \bar{S}u\rangle + \langle u, \tilde{R}u\rangle \geq 0. \end{aligned}$$

IV. STABILITY THEOREM

We now show that input-output stability of the interconnected system M_{sys}, as described in the previous section, is always guaranteed. This is the main contribution of the paper.

Theorem 1: Consider two causal systems with transfer function matrices $M_1 \in \mathcal{RH}_\infty$ and $M_2 \in \mathcal{RH}_\infty$ which are interconnected as shown in Fig. 4. Furthermore, suppose that systems M_1 and M_2 are dissipative in the sense of Definition 5 with respect to the triples $(Q_i(\omega), S_i(\omega), R_i(\omega))$, $i = 1, 2$, given at the beginning of Section III. Then the interconnection of the systems, denoted M_{sys}, is finite-gain stable.

Proof: Note that $\bar{\bar{Q}} := -\bar{Q}$ is positive definite. As in [11]–[14], but considering frequency-dependent (as opposed to constant) $\bar{\bar{Q}}$, it is shown that, since $\bar{\bar{Q}}$ is positive definite, M_{sys} is finite-gain stable.

From Definition 5, the statement that M_{sys} is $(\bar{Q}, \bar{S}, \tilde{R})$ dissipative means that

$$\langle y, \bar{\bar{Q}}y\rangle - 2\langle y, \bar{\bar{Q}}^{\frac{1}{2}}\bar{\bar{S}}u\rangle \leq \langle u, \tilde{R}u\rangle$$

$\forall u \in \mathcal{L}_2(j\mathbb{R})$, where $\bar{\bar{S}} := \bar{\bar{Q}}^{-\frac{1}{2}}\bar{S}$. The matrix $\tilde{R} + \bar{\bar{S}}^T\bar{\bar{S}}$ is a symmetric matrix, equal to

$$\begin{pmatrix} (\bar{s}_2 + \frac{\alpha^2}{\bar{q}_1} + \frac{\bar{s}_2^2}{\bar{q}_2})I & \alpha(\omega)(\frac{\bar{s}_1}{\bar{q}_1} - \frac{\bar{s}_2}{\bar{q}_2})I \\ \alpha(\omega)(\frac{\bar{s}_1}{\bar{q}_1} - \frac{\bar{s}_2}{\bar{q}_2})I & (\bar{s}_1 + \frac{\bar{s}_1^2}{\bar{q}_1} + \frac{\alpha^2}{\bar{q}_2})I \end{pmatrix}.$$

Then $\tilde{R} + \bar{\bar{S}}^T\bar{\bar{S}}$ is orthogonally similar to a diagonal matrix, ie:

$$\tilde{R}(\omega) + \bar{\bar{S}}(\omega)^T\bar{\bar{S}}(\omega) = U(\omega)^T D(\omega) U(\omega),$$

and so there always exists a finite scalar $\kappa > 0$ such that $\tilde{R} + \bar{\bar{S}}^T\bar{\bar{S}} \leq \kappa^2 I$, ie: $U(\omega)^T D(\omega) U(\omega) \leq \kappa^2 I = \kappa^2 U(\omega)^T U(\omega)$ and

$$U(\omega)^T \begin{pmatrix} \lambda_1(\omega) - \kappa^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_p(\omega) - \kappa^2 \end{pmatrix} U(\omega) \leq 0.$$

So $\exists \kappa > 0$ such that

$$\langle y, \bar{\bar{Q}}y\rangle - 2\langle y, \bar{\bar{Q}}^{\frac{1}{2}}\bar{\bar{S}}u\rangle \leq \kappa^2\langle u, u\rangle - \langle u, \bar{\bar{S}}^T\bar{\bar{S}}u\rangle \tag{16}$$

$\forall u \in \mathcal{L}_2(j\mathbb{R})$.

Inequality (16) is equivalent to

$$\begin{aligned} &\langle y, \bar{\bar{Q}}^{\frac{1}{2}}\bar{\bar{Q}}^{\frac{1}{2}}y\rangle - 2\langle y, \bar{\bar{Q}}^{\frac{1}{2}}\bar{\bar{S}}u\rangle + \langle u, \bar{\bar{S}}^T\bar{\bar{S}}u\rangle \leq \kappa^2\langle u, u\rangle \\ &\Leftrightarrow \|\bar{\bar{Q}}^{\frac{1}{2}}y - \bar{\bar{S}}u\|^2 \leq \kappa^2\|u\|^2 \\ &\Leftrightarrow \|\bar{\bar{Q}}^{\frac{1}{2}}y - \bar{\bar{S}}u\| \leq \kappa\|u\|. \end{aligned}$$

It follows easily that

$$\|\bar{\bar{Q}}^{\frac{1}{2}}y\| \leq (\kappa + \|\bar{\bar{S}}\|_\infty)\|u\|. \tag{17}$$

Finally, note that $y = (\bar{\bar{Q}}^{\frac{1}{2}})^{-1}\bar{\bar{Q}}^{\frac{1}{2}}y$ implies that $\|y\| \leq \|\bar{\bar{Q}}^{-\frac{1}{2}}\|_\infty\|\bar{\bar{Q}}^{\frac{1}{2}}y\|$, or $\|\bar{\bar{Q}}^{-\frac{1}{2}}\|_\infty^{-1}\|y\| \leq \|\bar{\bar{Q}}^{\frac{1}{2}}y\|$. Then from (17),

$$\begin{aligned} &\|\bar{\bar{Q}}^{-\frac{1}{2}}\|_\infty^{-1}\|y\| \leq (\kappa + \|\bar{\bar{S}}\|_\infty)\|u\| \\ &\Leftrightarrow \|y\| \leq \bar{k}\|u\|, \end{aligned}$$

where $\bar{k} := \|\bar{\bar{Q}}^{-\frac{1}{2}}\|_\infty(\kappa + \|\bar{\bar{S}}\|_\infty)$. ■

So by setting the $(Q(\omega), S(\omega), R(\omega))$ triples associated with systems M_1 and M_2 to be equal to the triples given by (10), (11) and (12), mathematical descriptions in terms of dissipativity can be given to describe the "mixed" small gain and passivity frequency domain property of each of the systems. With respect to the interconnection of the two systems, these mathematical descriptions allow for the frequency range $-\infty < \omega < \infty$ to be divided into intervals for which both systems M_1 and M_2 are: a) simultaneously "input and output strictly passive" (and may or may not have "gain less than one"); b) simultaneously "input and output strictly passive and with gain less than one"; or c) simultaneously "with gain less than one" (and may or may not be "input and output strictly passive"). Given the dissipative property of systems M_1 and M_2, it was shown that the interconnected system M_{sys} is $(\bar{Q}, \bar{S}, \bar{R})$ dissipative; and since $\bar{Q}$ is negative definite, then M_{sys} is finite-gain stable.

Suppose we let ϵ_1 and ϵ_2 from (10) be equal to zero. (We could say that this corresponds to relaxing input and output strict passivity on a frequency interval to "input strict passivity on a frequency interval".) Note that $\bar{Q}$ remains negative definite and so finite-gain stability of M_{sys} is still guaranteed. Alternatively, let δ_1 and δ_2 from (12) be equal to zero (which we could say corresponds to relaxing input and output strict passivity on a frequency interval to "output strict passivity on a frequency interval"). In this case, $\bar{Q}$ also remains negative definite and so finite-gain stability of M_{sys} is still guaranteed. Alternatively still, let ϵ_1 and δ_1 (or ϵ_2 and δ_2) of (10) and (12) be equal to zero (which corresponds to relaxing input and output strict passivity on a frequency interval of system M_1 (or system M_2) to what we could call "passivity on a frequency interval"). The matrix $\bar{Q}$ remains negative definite, and so finite-gain stability of M_{sys} is guaranteed in this case also.

Now we discuss how multipliers (or weights) can be used to scale the interconnection when one or both of the original systems in the feedback loop do not exhibit the "mixed" small gain and passivity frequency domain property. Appropriate scaling of the interconnection so that both weighted systems are of the "mixed" small gain and passive type increases the system class size for which the results of this paper are applicable. For example, let us consider the systems with transfer functions

$$n_1(s) = \frac{1}{s^2 + 0.2s + 1}$$

and

$$n_2(s) = \frac{10}{\frac{1}{0.009}s + 1}$$

with Nyquist diagrams shown in Fig's. 5 and 6. Clearly, the system with transfer function $n_1(s)$ is neither "input and output strictly passive", nor has "gain less than one", on the frequency interval between 1 rad/s and 1.4 rad/s. Yet the feedback interconnection of the two systems is stable, as shown by the Nyquist diagram of $n_1(s)n_2(s)$, illustrated in Fig. 7, as it does not encircle the point $-1 + j0$.

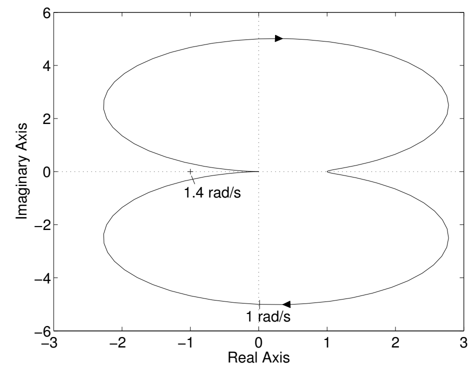

Fig. 5. Nyquist diagram of $n_1(s)$.

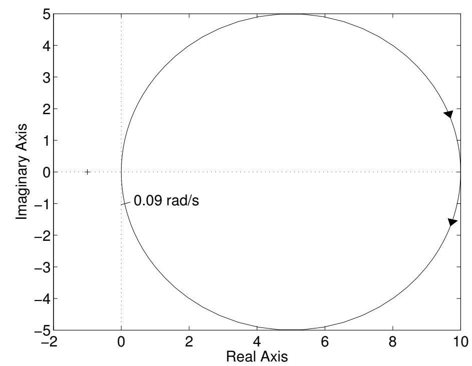

Fig. 6. Nyquist diagram of $n_2(s)$.

Suppose that we introduce the constant multiplier $\gamma = 0.1$ into the feedback interconnection. That is, let us scale the interconnection by (pre-)multiplying $n_1(s)$ by γ and (post-) multiplying $n_2(s)$ by γ^{-1} to give

$$\gamma n_1(s) = \frac{0.1}{s^2 + 0.2s + 1}$$

and

$$n_2(s)\gamma^{-1} = \frac{100}{\frac{1}{0.009}s + 1}$$

with Nyquist diagrams shown in Fig's. 8 and 9. It is clear that both of these scaled systems, ie: the systems with transfer functions $\gamma n_1(s)$ and $n_2(s)\gamma^{-1}$, do have the "mixed" small gain and passivity frequency domain property. Furthermore, it is possible to find a common frequency interval on which both scaled systems are "input and output strictly passive and with gain less than one" (we could choose the frequency interval $(0.91, 0.99)$ for example). In effect, we can apply the techniques of this paper to guarantee stability of the interconnection of the scaled systems, and since multiplier theory preserves stability, stability of the original feedback interconnection can be inferred, as expected.

Generally, one chooses weights which are units in $\mathcal{RH}_\infty$. Then the modified feedback interconnection consists of the scaled systems $W_1 M_1 W_2^{-1}$ and $W_2 M_2 W_1^{-1}$ replacing M_1

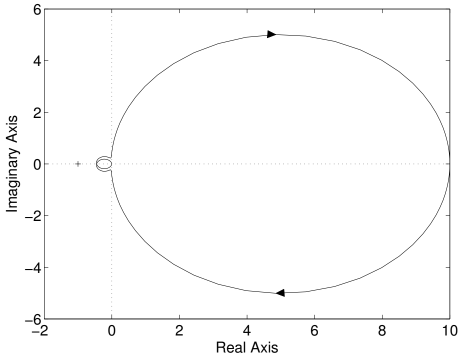

Fig. 7. Nyquist diagram of $n_1(s)n_2(s)$.

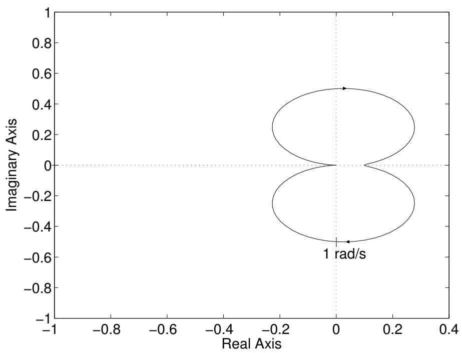

Fig. 8. Nyquist diagram of $\gamma n_1(s)$.

and M_2, respectively, where W_1 and W_2 represent the weights.

V. CONCLUSIONS AND FUTURE WORK

It was shown that finite-gain stability is guaranteed for a feedback loop, denoted M_{sys}, consisting of two causal, stable, LTI systems M_1 and M_2 which each have a "mixed" small gain and passivity frequency domain property. This property was described via the notion of dissipative systems. It is clear that, in the case of MIMO LTI systems, there already exist simple techniques to determine stability of a feedback interconnection. For example, one needs only to check that the transfer function matrix mapping signals u_1 and u_2 to e_1 and e_2 of Fig. 4 are in $\mathcal{RH}_\infty$. However, these simple techniques often fail in the time-varying and/or nonlinear case. It is expected that the technique of ensuring finite-gain stability presented in this paper will be able to be extended to the time-varying and/or nonlinear case by first reformulating the present results in the time domain. Preliminary investigations indicate that this is possible by considering frequency-dependent filters on the interconnection input and output signals. This work will be published elsewhere in due course.

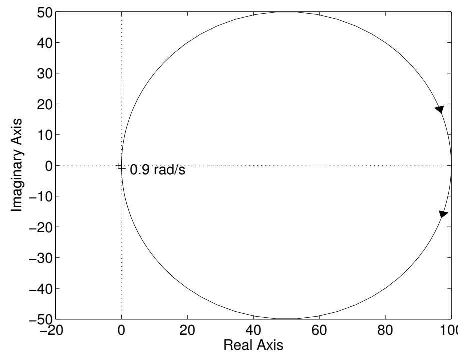

Fig. 9. Nyquist diagram of $n_2(s)\gamma^{-1}$.

VI. ACKNOWLEDGMENTS

The authors wish to thank Professor David Hill for his helpful comments.

REFERENCES

[1] C.A. Desoer and M.Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, NY; 1975.

[2] H.J. Marquez, *Nonlinear Control Systems: Analysis and Design*, John Wiley & Sons, Hoboken, NJ; 2003.

[3] M. Green and D.J.N. Limebeer, *Linear Robust Control*, Prentice Hall, Englewood Cliffs, NJ; 1995.

[4] K. Zhou, J.C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ; 1996.

[5] A.R. Teel, T.T. Georgiou, L. Praly and E. Sontag, "Input-output stability", in: W.S. Levine (Ed.), *The Control Handbook*, CRC Press, Boca Raton, FL; 1996, pp 895-908.

[6] C.E. Rohrs, L. Valavani, M. Athans and G. Stein, Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics, *IEEE Trans. on Automatic Control*, vol. 30, no. 9, 1985, pp 881-889.

[7] B.D.O. Anderson, R.R. Bitmead, C.R. Johnson, Jr., P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly and B.D. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis*, The MIT Press, Cambridge, MA; 1986.

[8] I.M.Y. Mareels, B.D.O. Anderson, R.R. Bitmead, M. Bodson and S.S. Sastry, "Revisiting the MIT rule for adaptive control", in *Second IFAC Workshop on Adaptive Systems in Control and Signal Processing*, Lund, Sweden, 1986, pp 161-166.

[9] B.D.O. Anderson, Failures of adaptive control thoery and their resolution, *Communications in Information and Systems*, vol. 5, no. 1, 2005, pp 1-20.

[10] J.C. Willems, Dissipative dynamical systems, Part I: General theory; Part II: Linear systems with quadratic supply rates, *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, 1972, pp 321-393.

[11] D.J. Hill and P.J. Moylan, General instability results for interconnected systems, *SIAM J. Control and Optimization*, vol. 21, no. 2, 1983, pp 256-279.

[12] P.J. Moylan and D.J. Hill, Tests for stability and instability of interconnected systems, *IEEE Trans. on Automatic Control*, vol. 24, no. 4, 1979, pp 574-575.

[13] P.J. Moylan and D.J. Hill, Stability criteria for large-scale systems, *IEEE Trans. on Automatic Control*, vol. 23, no. 2, 1978, pp 143-149.

[14] D.J. Hill and P.J. Moylan, Stability results for nonlinear feedback systems, *Automatica*, vol. 13, 1977, pp 377-382.

[15] D. Hill and P. Moylan, The stability of nonlinear dissipative systems, *IEEE Trans. on Automatic Control*, vol. 21, no. 5, 1976, pp 708-711.