RECIPROCAL PASSIVE IMPEDANCE SYNTHESIS VIA STATE-SPACE TECHNIQUES*

by

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It is not possible in general to synthesise a symmetric positive real impedance matrix $Z(s)$ by a passive network that simultaneously is reciprocal, i.e. uses no gyrators, has the minimal number of resistors and the minimal number of reactive elements.

The aim of the paper is to summarise three different procedures based on state-space ideas which result in gyratorless reciprocal passive synthesis of a symmetric rational p.r. $Z(s)$. The first two methods require twice the minimal number of resistors and twice the minimal number of reactive elements, but however have the feature of simplicity in respect of the computations involved. The third method achieves a synthesis using a minimal number of resistors, but in general is nonminimal in the number of reactive elements.

Review of State-Space Ideas

In a state-space representation of $Z(s)$, the port current i is the input u and the port voltage v the output y of a set of state-space equations, i.e.

$$\dot{x} = Fx + Gu \quad ; \quad y = H'x + Ju \tag{1}$$

where $u \equiv i$, $y \equiv v$. Thus

$$Z(s) = J + H'(sI-F)^{-1}G \tag{2}$$

Any quadruple $\{F, G, H, J\}$ for which (2) holds is termed a realization of $Z(s)$, and a minimum realization if F has the minimum dimension. Given one realization $\{F, G, H, J\}$, then other realizations of the same dimension are of the form $\{TFT^{-1}, TG, (T^{-1})'H, J\}$ for some nonsingular T.

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If $Z(s)$ is positive real, the para-Hermitian matrix $Z(s) + Z'(-s)$ is nonnegative definite for almost all s on the $j\omega$-axis. Alternatively, there exist real matrices $P = P' > 0$, L and W_0 such that [1]

$$PF + F'P = -LL' \; ; \quad PG = H-LW_0 \quad ; \quad J + J' = W_0'W_0. \tag{3}$$

Furthermore, there exists an infinity of matrices $W(s)$ such that

$$Z(s) + Z'(-s) = W'(-s)W(s) \tag{4}$$

By way of notation, $\{F_1, G_1, H_1, L_1\}$ will henceforth denote TFT^{-1}, TG, $(T^{-1})'H$, $(T^{-1})'L\}$ where T is any nonsingular matrix with $T'T = P$ of equation (3).

Passive Reciprocal Syntheses

The first two methods make use of the identity for symmetric $Z(s)$:

$$Z(s) = J + \frac{H'}{\sqrt{2}} (sI-F)^{-1} \frac{G}{\sqrt{2}} + \frac{G'}{\sqrt{2}} (sI-F')^{-1} \frac{H}{\sqrt{2}} \tag{5}$$

observed by Koga [2]. We shall state without proof the following theorem which gives a synthesis via reactance extraction.

Theorem 1 Let $Z(s)$ be $n \times n$, symmetric and p.r. with $Z(\infty) < \infty$, and let $\{F_1, G_1, H_1, J\}$ be a minimum realization with F a $p \times p$ matrix. The matrix equation

$$\begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & \frac{I_p}{\sqrt{2}} & \frac{I_p}{\sqrt{2}} \\ 0 & \frac{-I_p}{\sqrt{2}} & \frac{I_p}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} J & \frac{-H_1'}{\sqrt{2}} & \frac{G_1'}{\sqrt{2}} \\ \frac{G_1}{\sqrt{2}} & -F_1 & 0 \\ \frac{-H_1}{\sqrt{2}} & 0 & -F_1' \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & \frac{I_p}{\sqrt{2}} & \frac{-I_p}{\sqrt{2}} \\ 0 & \frac{I_p}{\sqrt{2}} & \frac{I_p}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ v_3 \end{bmatrix}$$

is a hybrid matrix description of an $(n+2p)$-port passive reciprocal network such that if the ports associated with v_2 and i_2 are terminated in unit inductors and the ports associated with v_3 and i_3 in unit capacitors, the impedance at the remaining ports is $Z(s)$.

Before stating the second theorem corresponding to a resistive extraction synthesis of $Z(s)$, we define first a set of state-space equations of a lossless network of $(n+2r)$ ports.

$$\dot{\hat{x}} = F_L\hat{x} + G_L\hat{u} \quad ; \quad \hat{y} = H_L'\hat{x} + J_L\hat{u} \tag{6}$$

where u and $\hat{y}$ are $(n+2r)$-vectors, r being the normal rank of $Z(s) + Z'(-s)$, and

$$F_L = \begin{bmatrix} 0 & -\frac{1}{2}(F_1'-F_1) \\ -\frac{1}{2}(F_1-F_1') & 0 \end{bmatrix} \quad ; \quad J_L = \begin{bmatrix} 0 & \frac{W_0'}{\sqrt{2}} \\ \frac{-W_0}{\sqrt{2}} & 0 \end{bmatrix} + 0_{r\times r}$$

$$G_L = H_L = \begin{bmatrix} \frac{1}{2}(G_1+H_1) & 0 & \frac{-L_1}{\sqrt{2}} \\ 0 & \frac{L_1}{\sqrt{2}} & 0 \end{bmatrix}$$

The losslessness of the network defined by (6) follows from the fact that F_L and J_L are skew symmetric and $G_L = H_L$.

<u>Theorem 2</u> Let $Z(s)$ be symmetric, positive real with $Z(\infty) < \infty$ and $\{F_1, G_1, H_1, J\}$ be a minimum realization. Suppose an $(n+2r)$-port lossless network is defined by (6). If the last $2r$ ports of this network are terminated in unit resistors, then the impedance $Z(s)$ is seen at the remaining n ports.

The third synthesis procedure [3] may be said to be a state-space version of the Bayard's synthesis. It depends on a Gauss Factorization procedure discussed extensively in [4]. An outline of the synthesis procedure is as follows.

Given any rational p.r. symmetric $Z(s)$ with $Z(\infty) < \infty$ and $Z(\infty)$ nonsingular, a Gauss Factorization is performed to obtain a spectral factor of $Z(s) + Z'(-s)$ as given by (4). A realization $\{\hat{F}_w, \hat{G}_w, \hat{L}_w, W_0\}$ is constructed [3] for the spectral matrix $W(s)$ (which has even numerators of all elements because of reciprocity) such that $[\hat{F}_w, \hat{L}_w]$ is completely observable, i.e. the matrix $[\hat{L}_w : \hat{F}_w'\hat{L}_w : \dots (\hat{F}_w')^{p-1}L_w]$ is nonsingular. Define $P_w = P_w' > 0$ as a unique solution of

$$P_w\hat{F}_w + \hat{F}_w'P_w = -\hat{L}_w\hat{L}_w'$$

and let T be any matrix such that $T'T = P_W$; then using T as a basis transformation, a new realization $\{F_W, G_W, L_W, W_0\}$ of $W(s)$ given by $\{T\hat{F}_W T^{-1}, T\hat{G}_W, (T^{-1})'\hat{L}_W, W_0\}$ is formed. Finally a matrix H_W is computed by $H_W = G_W + L_W W_0$. Then the quadruple $\{F_W, G_W, H_W, J = Z(\infty)\}$ is a realization, nonminimal in general, of $Z(s)$ which has nice properties as given in the following theorem.

<u>Theorem 3</u> The realization $\{F_W, G_W, H_W, J = Z(\infty)\}$, given above for a symmetric p.r $Z(s)$ with $Z(\infty) < \infty$, and $Z(\infty)$ nonsingular, defines a passive network synthesis of $Z(s)$; an orthogonal transformation of the coordinate basis of the realization $\{F_W, G_W, H_W, Z(\infty)\}$ yields a new realization which defines a passive reciprocal synthesis. Further, the number of resistors required in the sy thesis is a minimum. In addition, the orthogonal transformation T required to give a reciprocal passive network realization is obtained via factoring a matrix P as $T'\Sigma T$ where P is defined uniquely by

$$F_W P = P F_W' \qquad \text{and} \qquad P L_W = L_W$$

and Σ is a diagonal matrix with +1 and -1 entries only.

In the above theorem, the constraint that $Z(\infty)$ be nonsingular does not actually present any restriction at all, because if $Z(\infty)$ is singular, then by carrying a finite number of inversions and extraction of poles at infinity, a $\hat{Z}(s)$ is obtained for which $\hat{Z}(\infty)$ is nonsingular, (see [3]).

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[2] T. Koga, "Synthesis of Finite Passive n-Ports with Prescribed Positive Real Matrices of Several Variables," <u>IEEE Trans. on Circuit Theory</u>, Vol.CT-15, No.1, March, 1968, pp. 2-23.

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[4] R.W. Newcomb, <u>Linear Multiport Synthesis</u>, McGraw Hill, New York, 1966.