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Passive Angle Measurement Based Localization Consistency via Geometric Constraints

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Abstract

In this paper we examine the geometric relations between various measured parameters and their corresponding errors in angle-measurement based emitter localization scenarios. We derive a geometric constraint formulating the relationship among the measurement errors in such a scenario. Using this constraint, we formulate the localization task as a constrained optimization problem that can be performed on the measurements in order to provide the optimal values such that the solution is consistent with the underlying geometry.

1. Introduction

LOCALIZATION and positioning systems typically involve three types of measurements; distance, time difference of arrival (TDOA) and angle. In this paper we focus on angle-measurement based localization. Angle measuring sensors measure the direction to the target with respect to either a global or local direction [1], [2], [3], [4]. Passive surveillance radars, for example typically involve measuring only the bearings of the targets since they are more readily available in a passive scenario.

The objective of this work is to derive and examine functional relationships between the measurement errors in a passive surveillance scenario. We seek to form a constraint function that can be used in constrained optimization processes where the aim is to minimize the location estimation error for a target. Indeed the errors may be estimated such that the final solution satisfies the proposed constraint and hence is consistent with the geometry. In this paper we only consider situations in $\Re^2$.

Geometric constraints have been examined for distance based problems and recently a functional relationship between the errors in distance-based localization was derived [5] and exploited to secure improvement in localization using noisy measurements. In [5] a quadratic constraint on the distance measurement errors is given. One of the contributions of this paper is development of such a geometric constraint for angle-measurement based localization.

Furthermore, we show how this constraint on the measured angles can be used to find a consistent solution to the localization problem when no solution would traditionally be reached using only the noisy measurements that are available. Using the geometric constraint we derive, we formulate the above problem as a constrained optimization problem and show that equal or better performance results when compared to a mean estimation solution.

The rest of this paper is organized as follows. In Section 2 we formulate the problem and derive a functional relationship between the angle-measurement errors in a localization system involving three or more sensors. Section 3 outlines the inconsistency problem associated with the over-determined system of measurement equations from a geometric perspective and outlines the solution to the problem conceptually. Section 4 discusses the constrained optimization approach taken to find a consistent solution to the problem and Section 5 examines a simulation showing the algorithms performance in practice. In Section 6 we give our concluding remarks along with suggestions for future work.

2. Angle Constraints For Passive Surveillance

Consider the problem of localizing an emitter in $\Re^2$ using measurements from three or more sensors. The sensors are capable of estimating the bearings to the emitter relative to a global direction. The inter-sensor distances are known and the bearings between the sensors are accurately known relative to the global direction. The over-determined system of equations defined by the three or more measurements will not have a unique solution in the presence of noisy measurements. We consider the problem of finding a solution to the inconsistent system of equations by formulating a relationship between the

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measurement errors. Consider the scenario in Figure 1.

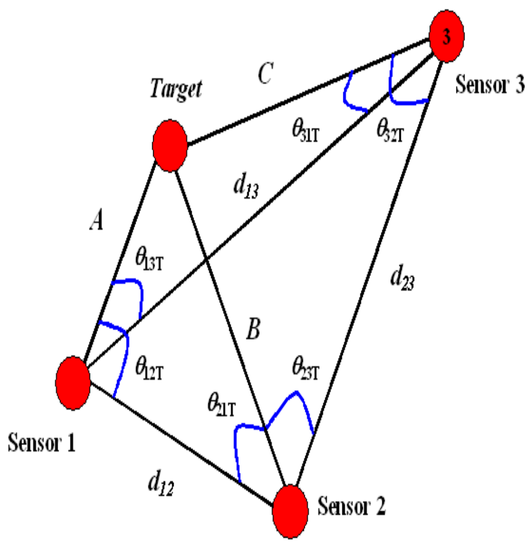

Fig. 1: Angle relationships in a sensor triangle

Let us consider the internal angles and their corresponding errors. We denote the angle subtended at sensor i by the target and sensor j as θ_{ijT}, and its noisy estimate by $\widehat{\theta}_{ijT}$. The relevant equations are,

$$\begin{aligned}\widehat{\theta}_{12T} &= \theta_{12T} - e_1\\ \widehat{\theta}_{13T} &= \theta_{13T} - e_1\\ \widehat{\theta}_{21T} &= \theta_{21T} - e_2\\ \widehat{\theta}_{23T} &= \theta_{23T} + e_2\\ \widehat{\theta}_{31T} &= \theta_{31T} - e_3\\ \widehat{\theta}_{32T} &= \theta_{32T} - e_3\end{aligned} \tag{1}$$

Notice that the set of the measurements only involves three independent errors (e_i for $i \in \{1,2,3\}$). The reasoning for this relates directly to the way the internal angles are determined. Each sensor measures the bearing to the target relative to North. Call those bearing estimates $\widehat{\phi}_{1T}$, $\widehat{\phi}_{2T}$ and $\widehat{\phi}_{3T}$. These have errors e_i such that the true bearings ϕ_{iT} obey $\widehat{\phi}_{iT} = \phi_{iT} + e_i$. Also, we have available exactly the angles ϕ_{12} and ϕ_{13}, see Figure 2. Now it is easily seen that $\widehat{\theta}_{12T} = \phi_{12} - \widehat{\phi}_{1T} = \phi_{12} - \phi_{1T} - e_1 = \theta_{12T} - e_1$ and so on. This argument will explain the sign difference in the equation for $\widehat{\theta}_{23T}$ in (1).

For simplicity we shall assume the internal angles are determined from the bearings and hence we omit any further reference to the actual bearing measurements.

Theorem 1: Consider the arrangement of three sensors and a target all located in the plane shown in Figure 1, with the three sensors not collinear and no sensor collocated with the target. The sensor angles in the triangle formed by the sensors are precisely known and the bearings from each sensor (relative to North) to the target are approximately known. Suppose that θ_{ijT} denotes the angle subtended at sensor i by sensor j and the target, and suppose that noisy measurements are available for the associated six quantities as indicated in (1). Then in terms of the known quantities $\widehat{\theta}_{ijT}$, the unknown errors e_i satisfy the following trigonometric constraint,

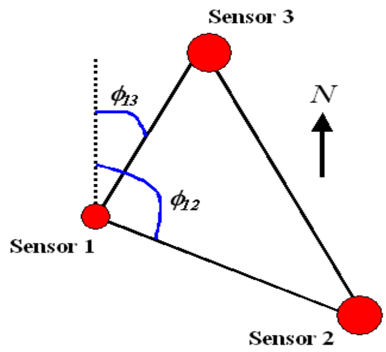

Fig. 2: Internal angle determination from bearings

$$\begin{aligned}&sin(\widehat{\theta}_{12T} + e_1)sin(\widehat{\theta}_{23T} - e_2)sin(\widehat{\theta}_{31T} + e_3) - \\ &sin(\widehat{\theta}_{21T} + e_2)sin(\widehat{\theta}_{32T} + e_3)sin(\widehat{\theta}_{13T} + e_1) = 0\end{aligned} \tag{2}$$

Proof: Referring to Figure 1, a system of equations can be given as follows

$$Asin(\theta_{12T}) - Bsin(\theta_{21T}) = 0 \tag{3}$$
$$Bsin(\theta_{23T}) - Csin(\theta_{32T}) = 0 \tag{4}$$
$$Asin(\theta_{13T}) - Csin(\theta_{31T}) = 0 \tag{5}$$

We would like to eliminate A, B, and C from the above system of equations. Solving (3) for A and substituting this into (5),

$$A = \frac{Bsin(\theta_{21T})}{sin(\theta_{12T})} \quad \Rightarrow \quad C = \frac{Bsin(\theta_{21T})sin(\theta_{13T})}{sin(\theta_{12T})sin(\theta_{31T})} \tag{6}$$

Substituting the result for C (6) into (4),

$$B = \frac{Bsin(\theta_{21T})sin(\theta_{32T})sin(\theta_{13T})}{sin(\theta_{12T})sin(\theta_{23T})sin(\theta_{31T})} \tag{7}$$

Hence,

$$\begin{aligned}&sin(\theta_{12T})sin(\theta_{23T})sin(\theta_{31T}) - \\ &sin(\theta_{21T})sin(\theta_{32T})sin(\theta_{13T}) = 0\end{aligned} \tag{8}$$

Therefore, (8) is a functional relationship between the internal angles of the triangle system given in Figure 1. By adding the corresponding error term to both sides of each equation in (1) and substituting into (8) we obtain the constraint equation (2). ■

Remark 1: The constraint (2) appears to have no absolutely known term, i.e a quantity such as ϕ_{12}. However, although (2) does not have any explicitly defined absolutely known value, it does have some implicitly defined. In the determination of the measured internal angles we use the bearings to known sensor nodes and hence we assume these bearings are known absolutely. Hence, the absolute bearing values are implicitly represented in the constraint (2).

Remark 2: The constraint on the errors given by (2) is invalid for scenarios involving three collinear sensors. Referring to Figure 1 it is clear that θ_{12T} will equal θ_{13T} and θ_{31T} will equal θ_{32T} when the three sensors are collinear. Furthermore, it is straightforward to find that $sin(\theta_{21T})$ equals $sin(\theta_{23T})$ when all three sensors are collinear. Hence, the relationship given by (8) is identically satisfied in all cases, i.e. including scenarios when the system of angles (1) and the corresponding rays do not intersect at a single point. The constraint (2) has the advantage however, of involving only the measured angle values as parameters.

For completeness, we provide a general constraint equation that is valid for all geometries including the case involving three collinear sensors, but that involves some inter-sensor distances. We assume that no sensor is collocated with the target. Consider the scenario depicted in Figure 1 again.

Theorem 2: Assume the same hypothesis as for Theorem 1, save that the three sensors may be collinear with each other. Let d_{ij} denote the distance between sensors i and j. Then in terms of the known quantities $\widehat{\theta}_{ijT}$ and d_{ij}, the unknown errors e_i satisfy the following trigonometric constraint,

$$\begin{aligned} &d_{12}sin(\widehat{\theta}_{12T}+e_1)sin(\widehat{\theta}_{23T}-e_2+\widehat{\theta}_{32T}+e_3)- \\ &d_{23}sin(\widehat{\theta}_{32T}+e_3)sin(\widehat{\theta}_{12T}+e_1+\widehat{\theta}_{21T}+e_2)=0 \end{aligned} \quad (9)$$

Proof: Referring to Figure 1, a system of equations can be given as follows

$$B=\frac{d_{12}sin(\theta_{12T})}{sin(\theta_{12T}+\theta_{21T})} \quad (10)$$
$$B=\frac{d_{23}sin(\theta_{32T})}{sin(\theta_{23T}+\theta_{32T})} \quad (11)$$

This clearly leads directly to,

$$\begin{aligned} &d_{12}sin(\theta_{12T})sin(\theta_{23T}+\theta_{32T})- \\ &d_{23}sin(\theta_{32T})sin(\theta_{12T}+\theta_{21T})=0 \end{aligned} \quad (12)$$

Therefore, (12) is a trigonometric relationship between the internal angles of the triangle system given in Figure 1. By adding the corresponding error term to both sides of each equation in (1) and substituting into (12) we obtain the constraint equation (9). ■

Similarly to the method proposed in [5], we can derive a system of functional relationships for the cases involving more than three sensors. If the system involves $k > 3$ measurements then we have $k-2$ independent constraints by creating constraints of the form,

$$c_{i-2}(e_1, e_2, e_i) = 0, \quad for\ i = 3, 4, \ldots, k \quad (13)$$

These equations are obtained by considering the relations for the angles involving the target and a sensor triangle including sensors 1, 2 and i, for $i = 3, 4, \ldots, k$. In the remainder of this paper we shall focus on the case involving three sensors only.

3. Geometric Interpretations

If we attempt to localize a single target in $\Re^2$ using two angle measurements from two sensors we have a well defined system. That is, we have two measurement equations and two unknowns (i.e. the x and y coordinates), and it so happens that there is a unique solution. If we involve another sensor and hence another measurement we have an overdetermined system of equations. In an ideal case there will be a unique solution to this overdetermined system of equations. In practice, i.e. in the presence of noise, it will generally be the case that there will be no (exact) solution to the system. One approach that may be followed in the presence of noisy measurements is to do pair-wise localization from the three pairs of well defined systems. The result is depicted in Figure 3 where we can clearly see that three intersections result in three possible target estimations.

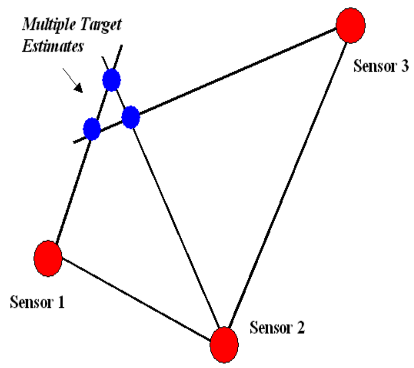

Fig. 3: Inconsistent system

We can estimate the mean position of the three estimates as follows,

$$\begin{aligned} x_{mean} &= (x_{1:2}+x_{1:3}+x_{2:3})/3 \\ y_{mean} &= (y_{1:2}+y_{1:3}+y_{2:3})/3 \end{aligned} \quad (14)$$

where, $x_{i:j}$ and $y_{i:j}$ are the x and y estimates for the well defined system formed between sensors i and j. This is an ad-hoc way of localizing, and it might well transpire that the errors associated with this estimate fail to satisfy (2) or (9). Given the existence of the constraint on the errors in the system, we should seek to estimate a position for which the associated errors are consistent with (2) and (9). That is, if we take pair-wise systems of the resulting consistent equations, the three points should all fall on the same point. We will force this constraint by estimating the errors through a constrained optimization process analogous to that proposed in [5].

4. The Constrained Optimization Approach

The constraints on the measurement errors discussed in the previous sections can be used in finding an optimal solution to the localization problem defined in Section 2 after formulating the localization problem as a constrained optimization problem [6], [7]. The constraints force the solution to be consistent with the underlying geometry such as the requirement that all the nodes lie on a plane.

There exist various approaches in the literature to solve a general constrained optimization problem. The least squares estimation approach [8] with linear [9] and quadratic [10], [11] constraints has been well documented in the literature. Furthermore, other optimization techniques such as linear and quadratic programming [12], [13] are available for optimization subject to linear and quadratic constraints. Other nonlinear constraints can be used in the optimization algorithms by employing nonlinear programming [14], sequential quadratic programming and numerical techniques. We use the optimization toolbox in Matlab v7 (R14) to perform the numerical optimization in this paper [15].

In this paper we formulate the optimization problem as follows: consider the following objective function,

$$f(e_1, e_2, e_3) = e_1^2 + e_2^2 + e_3^2 \tag{15}$$

We want to minimize the cost function in (15) subject to the constraint that $c(e_1, e_2, e_3) = 0$ (e.g. equation (2) or (12)). We use a sequential quadratic programming method to perform constrained nonlinear optimization numerically using the $fmincon$ function in the Matlab Optimization Toolbox (see [15]).

Sequential quadratic programming is widely used for solving nonlinear optimization problems where the objective and constraint functions are continuously differentiable [16] and is discussed in detail in [6], [17], [15]. We note, that in this paper we employ an optimization bound on the measurement errors such that $-15^o < e_i < 15^o$ for $i \in \{1, 2, 3\}$ in order to improve the convergence of the optimization algorithm. We also begin the algorithm with an initial estimate of 0^o.

The formula (15) can be thought of as coming up with a maximum likelihood estimation of the errors, given that they satisfy the particular constraints and that the errors have the same variances. If the variances are different and given an $a\ priori$ estimate of the variances then each squared error in (15) should be weighted by the inverse of its $a\ priori$ variance.

Following the convergence of the optimization algorithm we have an estimate of the errors $\epsilon = [e_1,\ e_2,\ e_3]$. According to the constraints ((2) or (12)) and (1), it is this error value plus the corresponding $measured$ angle value that results in system consistency along with hopefully estimating the true angle values. Therefore, we add the appropriate error to the appropriate $measured$ angle (e.g. in (1)) and localize the target in the standard way.

5. Numerical Simulation

In order to illustrate the purpose and effectiveness of the constraint based localization we examine a typical localization scenario in detail. We consider three sensors and a single target. The sensors can measure the direction to the target relative to a global direction. Although the results are applicable to cases with non-identical error distributions, we consider here a case with identical error distributions. The measurement errors obey a zero-mean Gaussian distribution with standard deviation, σ_θ, of 5^o.

A. No Collinear Sensors

We examine the performance of the constraint given in Theorem 1 first. The scenario involving no constrained optimization is given in Figure 4.

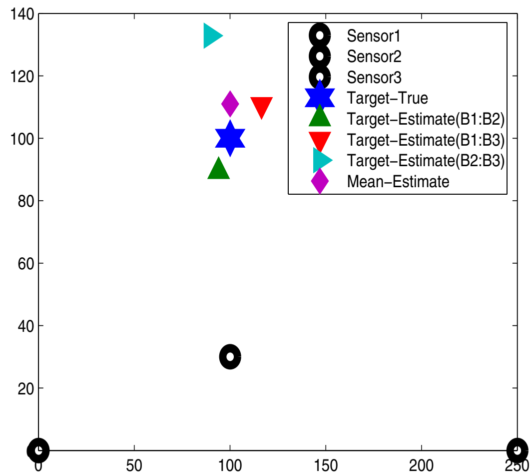

Fig. 4: Simulation Example 1 - Estimation ignoring constraint consistency

We can clearly see that there are three distinct estimates corresponding to the three pair-wise well defined localization systems. We also note the position of the mean estimate of these three pair-wise estimates. Following the constrained optimization procedure we obtain an estimate of the measurement errors that will also force solution consistency. The result is given in Figure 5 along with the original mean position estimate.

The three pair-wise estimates fall on the same point and hence we see the consistency enforced through the constrained

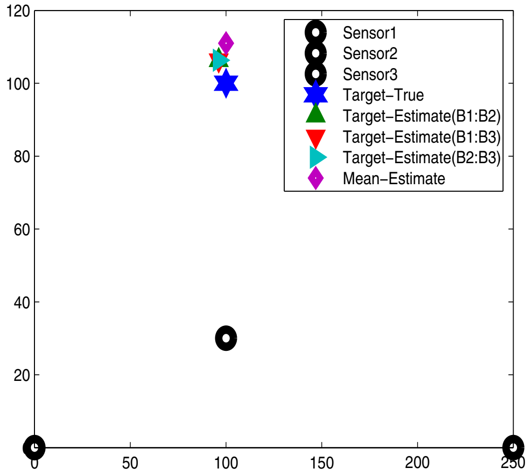

Fig. 5: Simulation Example 1 - Estimation considering constraint consistency

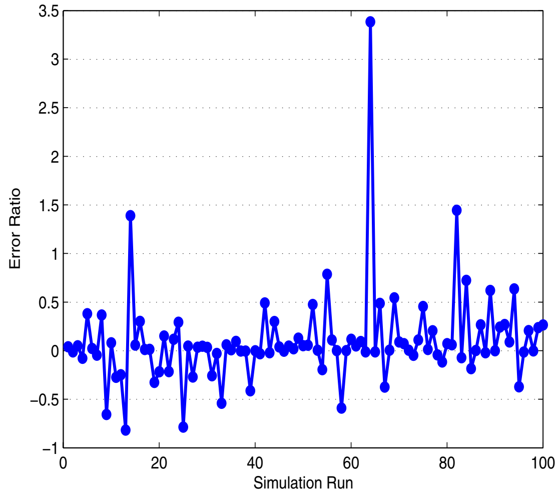

Fig. 6: Simulation Example 1 - Error comparison

optimization process. The mean estimate and the constrained optimization estimate can be shown to closely follow each other. An analysis of the absolute position error for the mean estimate and the constrained optimization estimate is given over 100 simulation runs in Figure 6. In Figure 6 we plot the following ratio,

$$ratio = \frac{m_e - co_e}{co_e} \tag{16}$$

where, m_e is the mean estimate error and co_e is the constrained optimization estimate error. The mean estimate error is defined as,

$$m_e = \sqrt{(x_{true} - x_{mean})^2 + (y_{true} - y_{mean})^2}$$

while the constrained optimization estimate error can be defined as,

$$co_e = \sqrt{(x_{true} - x_{co})^2 + (y_{true} - y_{co})^2}$$

where x_{co} and y_{co} are the x and y coordinates estimated using the constrained optimization technique.

Referring to Figure 6 we see that the ratio in (16) is more likely to be positive and *hence the constrained optimization estimate is closer to the target more often than the mean estimate of the original three pair-wise estimates.*

B. Collinear Sensors

We now examine the performance of the constraint given in Theorem 2. The scenario involving no constrained optimization is given in Figure 7.

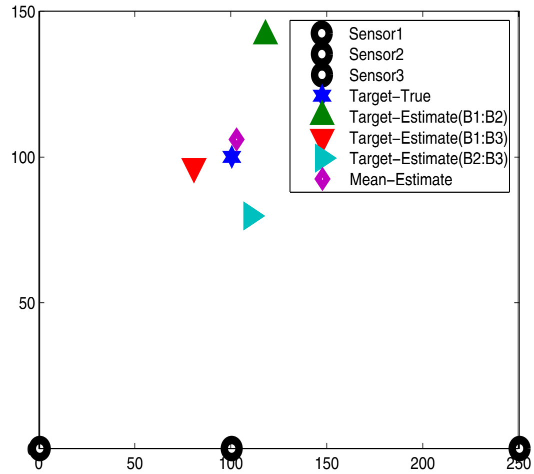

Fig. 7: Simulation Example 2 - Estimation ignoring constraint consistency

We can clearly see that there are three distinct estimates corresponding to the three pair-wise well defined localization systems. We also note the position of the mean estimate of these three pair-wise estimates. Following the constrained optimization procedure we obtain an estimate of the measurement errors that will also force solution consistency. The result is given in Figure 8 along with the original mean position estimate.

The three pair-wise estimates fall on the same point and hence we see the consistency enforced through the constrained optimization process. An analysis of the absolute position error for the mean estimate and the constrained optimization estimate is given over 100 simulation runs in Figure 9. In Figure 9 we plot ratio (16).

Referring to Figure 9 we see that the ratio in (16) is more likely to be positive and *hence the constrained optimization estimate*

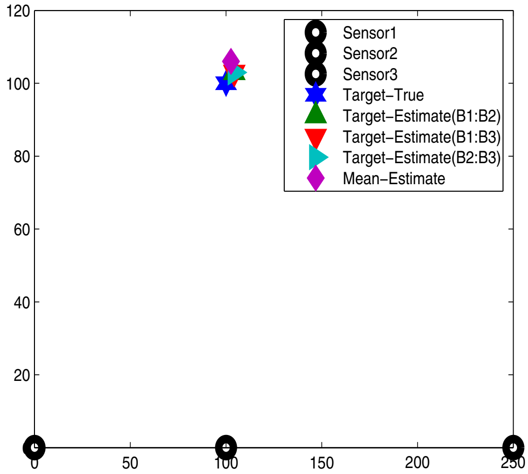

Fig. 8: Simulation Example 2 - Estimation considering constraint consistency

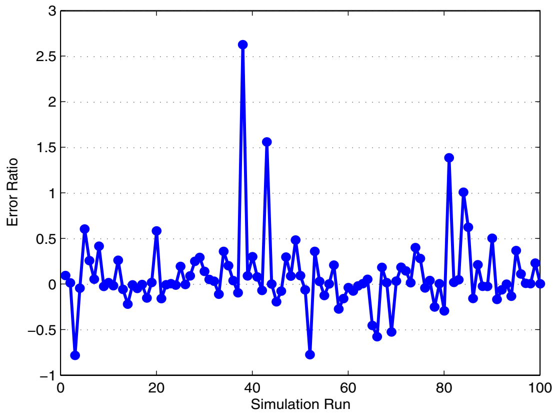

Fig. 9: Simulation Example 2 - Error comparison

is closer to the target more often than the mean estimate of the original three pair-wise estimates.

C. Discussion

From our simulations we note that the constrained optimization estimate is generally a better choice than the mean position estimate. The constrained optimization estimate is also forcing a consistent solution which more insightfully depicts the ideal mathematical scenario where the overdetermined system will still yield a single unique solution. There are also more benefits to constraint based localization than just forcing system consistency. A distance constraint on the inter-node distances in a sensor network has been recently shown to be a novel method to estimate the path-loss exponent in a signal strength based localization scheme [18]. Therefore, these lines of research warrant further investigation and reporting on.

Remark 3: Finally, it is important to note that although we employed a least squares objective function for illustrative purposes, we are by no means restricted to this. Indeed, the least squares cost function can be thought of as simply a particular incarnation of our algorithm.

6. Conclusion

This paper introduced a constraint on the errors in passive angle-measurement based localization systems. The constraint introduced allows us to estimate the errors such that the system is consistent with the geometrical requirements. The problem is formulated as a constrained optimization problem and the resulting location estimate is compared with an average estimate of the multiple inconsistent pair-wise estimates. The comparison shows that the proposed method in general performs equally well or better. This method takes advantage of the underlying dependency between the measurement errors.

The future direction of this work involves analyzing, in more detail, the optimization process and examining the potential of various objective functions. The extension of this work to localization in $\Re^3$ space may also be examined.

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