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Synthesis of parameter-dependent controllers yielding affine-in-parameters characteristic polynomials

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Abstract— **The problem considered is that of designing a parameter-dependent linear controller for a parameter-dependent linear plant, where the parameter may be time-varying. The set of controllers considered is restricted to those which ensure the closed-loop characteristic polynomial is affine in the parameter. This allows stability results for the case when the parameter is time-varying to be inferred from stability properties applying for time-frozen values of the parameter. The proposed design procedures and conditions for a parameter-dependent controller design are illustrated on a practical design problem.**

I. INTRODUCTION

It is often appropriate to model a physical plant with a collection of linear models, where the collection is obtained by varying some parameter appearing in a transfer function description. The parameter may correspond to a physical parameter, or capture a notionally steady-state operating point used for a nonlinear plant, so that the linear model represents a linearization round the operating point, and when the operating point changes, the model changes.

Here we focus on the case where the plant is single-input, single-output, and there is a single parameter, assumed to lie somewhere in a closed interval. At the limit values of the parameter, which is always assumed available for measurement, there are separate linear time-invariant (LTI) models of the plant. At intermediate values of the parameter, a model of the plant is adopted by interpolation, as indicated in a moment below. For each of the two limit LTI models, a satisfactory and certainly stabilizing LTI controller is assumed available. The main contribution of the paper is to propose a methodology for the design of a parameter-dependent linear controller stabilizing the range of plants corresponding to the interval of parameter values. In addition, the controller-plant loop should have some tolerance of time-variation of the parameter, and preferably the performance obtained for any fixed value of the parameter should not be too different from that obtained with the two extreme values.

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The assumptions behind our problem are as follows: Consider a single-input, single-output linear parameter-dependent plant model:

$$G(s,\lambda) = \frac{\lambda n_1(s) + (1-\lambda)n_0(s)}{\lambda d_1(s) + (1-\lambda)d_0(s)} = \frac{\lambda p_n(s) + n_0(s)}{\lambda p_d(s) + d_0(s)} = \frac{n(s,\lambda)}{d(s,\lambda)} \tag{1}$$

where $n_0(s)$, $n_1(s)$, $d_0(s)$, and $d_1(s)$ are some known LTI polynomials and $\lambda \in [0,1]$ is the measured variable (parameter) indicating actual operating conditions. The model can be regarded as nothing more than attempt to interpolate between two given plants, corresponding to the extreme values of λ; however, it is also true that many physical parameters give rise to plant transfer functions in the form (1), where λ corresponds to the physical parameter or its inverse, such as mass, friction coefficient, etc. See [7].

For a time-varying parameter $\lambda(t)$, equation (1) corresponds to a linear time-varying (LTV) model. Our interest is not in situations where $\lambda(t)$ varies extremely slowly, but rather situations where it could vary with a time constant longer than (but not necessarily much longer than) the time constants of the closed-loop system. It may even undergo infrequent step changes. This will become apparent in our later results. To this extent, the results constitute an improvement to those design approaches which just focus on frozen parameter values, or very slow rates of time variation.

The polynomials $p_n(s)$ and $p_d(s)$ from (1) are given by:

$$\begin{aligned} p_n(s) &= n_1(s) - n_0(s) \\ p_d(s) &= d_1(s) - d_0(s) \end{aligned} \tag{2}$$

and they are assumed to be coprime. We state in the Appendix what happens if this is not the case. The LTI plant models for the two extreme cases $\lambda = 0$, $\lambda = 1$ are:

$$G(s,0) = \frac{n_0(s)}{d_0(s)}, G(s,1) = \frac{n_1(s)}{d_1(s)} \tag{3}$$

and we suppose that there are available two closed-loop controllers $C_0(s)$ and $C_1(s)$:

$$C_0(s) = \frac{R_0(s)}{S_0(s)}, C_1(s) = \frac{R_1(s)}{S_1(s)} \tag{4}$$

stabilizing $G(s,0)$ and $G(s,1)$ respectively, where $R_0(s)$, $S_0(s)$, $R_1(s)$, and $S_1(s)$ are LTI polynomials.

The actual problem we consider is: Find a methodology to design a closed-loop parameter-dependent controller $C(s,\lambda)$ (leading to an LTV controller when the parameter λ is

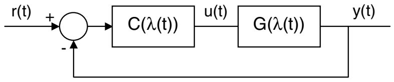

Fig. 1. Considered closed-loop configuration.

allowed to be time-varying $\lambda(t)$). The considered closed-loop configuration is given in Figure 1 and the considered controller:

$$C(s,\lambda) = \frac{R(s,\lambda)}{S(s,\lambda)} = \frac{\lambda^n r_n(s) + \lambda^{n-1} r_{n-1}(s) + \ldots + r_0(s)}{\lambda^n s_n(s) + \lambda^{n-1} s_{n-1}(s) + \ldots + s_0(s)} \quad (5)$$

(with $r_i(s), s_i(s) \mid i = 0, 1, ...n$ some LTI polynomials) stabilizes the plant model (1) for all frozen $\lambda \in [0, 1]$, and, given a bound on the average rate of variation of $\lambda(t)$, also stabilizes the given LTV plant model for all admissible time-varying $\lambda(t)$. Additionally the LTV controller should preserve in some way if possible the performance obtained with $(G(s, 0)$, $C_0(s))$ and $(G(s, 1)$, $C_1(s))$ closed loops.

The main idea is that we will search for a controller $C(s,\lambda)$ constraining the closed loop characteristic polynomial $P_{CL}(s,\lambda)$ *to be affine* in λ, i.e. in the form $P_{CL}(s,\lambda) = \lambda p_1(s) + p_0(s)$. This closed-loop property has several advantages: 1) Using conditions developed in [1], closed-loop stability for all constant values of $\lambda \in [0, 1]$ can be easily checked. 2) A bound on the average rate of $\lambda(t)$ variation assuring closed loop stability can be evaluated using conditions from [3]. 3) Since the system performance is given in part by its characteristic polynomial, it is likely that the time-varying closed-loop performance for slow-time variations will be a kind of linear interpolation of the closed-loop performances for the two limit LTI closed loops $(G(s, 0), C_0(s))$ and $(G(s, 1), C_1(s))$.

A similar philosophy is exploited in [2], where a general approach is presented for design of linear parameter-dependent controllers. The considered plant models in [2] are single-input, multiple-output with transfer functions where numerator and denominator depend in a multi-affine way on several parameters. Controllers of the same structure are chosen, and the closed-loop characteristic polynomial is also arranged to be multi-affine in the parameters. The proposed design procedure guarantees stability of the time-varying closed loop and gives information on acceptable rate of the variation in $\lambda(t)$.

The controller $C(s,\lambda)$ in closed loop with the plant $G(s,\lambda)$ yields the following closed-loop characteristic polynomial:

$$P_{CL}(s,\lambda) = d(s,\lambda)S(s,\lambda) + n(s,\lambda)R(s,\lambda) \quad (6)$$

To simplify notation, polynomials will be denoted by its names omitting the operator s. The characteristic polynomial becomes:

$$\begin{aligned} P_{CL}(\lambda) &= d(\lambda)S(\lambda) + n(\lambda)R(\lambda) \\ &= d_0 S(\lambda) + n_0 R(\lambda) + \lambda(p_d S(\lambda) + p_n R(\lambda)) \quad (7) \end{aligned}$$

and the corresponding closed-loop sensitivities are

$$\begin{aligned} S_{io}(\lambda) &= \tfrac{d(\lambda)S(\lambda)}{P_{CL}(\lambda)}, \quad GCS(\lambda) = \tfrac{n(\lambda)R(\lambda)}{P_{CL}(\lambda)} \\ CS(\lambda) &= \tfrac{d(\lambda)R(\lambda)}{P_{CL}(\lambda)}, \quad GS(\lambda) = \tfrac{n(\lambda)S(\lambda)}{P_{CL}(\lambda)} \end{aligned} \quad (8)$$

The rest of the paper is organized as follows. In Section 2 the design methodologies and some conditions are developed for different parameter-dependent controllers of different orders in λ-dependence. Section 3 provides an analysis of LTV closed-loop stability. An illustrative example is given in Section 4 and Section 5 gives some conclusions.

II. CONTROLLER DESIGN

A. General λ-dependent controller

We shall first consider the requirements on the controller (5) for unrestricted n which 1) assure a closed-loop characteristic polynomial affine in λ, and 2) ensure interpolation of C_0 and C_1, possibly with only stable pole-zero cancellation. Later, we will restrict n, and include the requirement for stability of the closed-loop characteristic polynomial.

Theorem 1: Consider the parameter-dependent model (1) connected in closed loop with the parameter-dependent controller (5) of order $n \geq 2$ and two LTI controllers (4) stabilizing, in closed loop, the two extreme plant models (3). Suppose there exist polynomials q_i, $i = 2, 3, ...n$ and stable polynomials β and γ satisfying the following equality

$$\left(\sum_{i=2}^{n} q_i\right)(d_0 p_n - n_0 p_d) - \beta(p_d S_0 + p_n R_0) + \gamma(p_d S_1 + p_n R_1) = 0 \quad (9)$$

Then there exists a set of stabilizing controllers (5) of order $n \geq 2$ ensuring that the characteristic polynomial (7) is affine in λ and the set interpolates C_0 and C_1, possibly after only stable pole-zero cancellations. The controller polynomials are given by:

$$\begin{aligned} r_k &= -d_0 q_{k+1} - p_d q_k \\ s_k &= n_0 q_{k+1} + p_n q_k \end{aligned} \quad (10)$$

for $k = 2, 3, \ldots, n$, where $q_{n+1} = 0$ and with

$$\begin{aligned} r_0 &= R_0 \beta \\ s_0 &= S_0 \beta \\ r_1 &= R_1 \gamma - r_n - r_{n-1} - \ldots - r_0 \\ s_1 &= S_1 \gamma - s_n - s_{n-1} - \ldots - s_0 \end{aligned} \quad (11)$$

Proof: See the Appendix. ■

Remark 1: The theorem is also valid with polynomials p_n, p_d non-coprime and with common divisor containing zeros in right-half plane (unstable). The common divisor is removed from equation (9) and the equation is solved without it. The polynomials (11) do not change in this case, see the Proof for more details.

Equation (9) does not always have a solution for stable polynomials β, γ and some polynomials q_i. The following lemma from [4] applied in our specific case establishes an existence condition for such solution.

Lemma 1 (solution existence condition [4]): Consider the polynomials:

$$\begin{aligned} c_0 &= -(p_d S_0 + p_n R_0) \\ c_1 &= (p_d S_1 + p_n R_1) \\ c_2 &= (d_0 p_n - n_0 p_d) \end{aligned} \tag{12}$$

mutually coprime, i.e. the common divisor of c_0, c_1, and c_2 is 1. Then solutions $(\sum_{i=2}^{n} q_i)$ and stable β, γ exist if and only if the transfer function:

$$H = \frac{c_2}{c_0 c_1} = \frac{n_h}{d_h} \tag{13}$$

has the parity interlacing property defined below.

Proof: See [4]. ■

The transfer function H has the parity interlacing property if there exists $\epsilon = +1$ or -1 such that:

$$\epsilon d_h(z_{ri}) < 0 \mid \forall z_{ri} \tag{14}$$

such that z_{ri} is a real unstable zero of n_h. Equivalently one can say that $d_h(z_{ri})$ has to have the same sign $\forall z_{ri}$. The condition in Lemma 1 can also be interpreted as the following requirement: There is an even number of unstable real zeros of the polynomial d_h between any two zeros z_{ri}.

The sum of the polynomials in (9) does not introduce more freedom than one polynomial $\alpha = \sum_{i=2}^{n} q_i$. Different approaches can be used to solve the polynomial equation (9) for some α and stable β and γ:

1) An algorithm is presented in [4] giving a solution triple for the equation (9) (assuming the solvability condition is fulfilled) in which β and γ are stable and α has no real unstable roots.
2) The problem of finding a solution of (9) with β and γ stable can be considered as a simultaneous stabilization of two LTI models. Consider the diophantine polynomial equation:

$$\beta c_0 + \gamma c_1 = -\alpha c_2 \tag{15}$$

Then the set of all solutions can be described by:

$$\begin{aligned} \beta &= \alpha\beta_0 - c_1 q \\ \gamma &= \alpha\gamma_0 + c_0 q \end{aligned} \tag{16}$$

with q an arbitrary polynomial and β_0, γ_0 a specific solution-pair to the equation: $\beta c_0 + \gamma c_1 = -c_2$. Requiring β and γ in (16) to be stable is equivalent to the requirement of stabilizing two LTI models $P_1 = \beta_0/c_1$, $P_0 = \gamma_0/c_0$ with one LTI controller $C_s = \alpha/q$, or equivalently stabilizing the two LTI models $P_1' = c_1/\beta_0$, $P_0' = c_0/\gamma_0$ with one controller $C_s' = q/\alpha$. This controller design problem can be equivalently expressed as stabilizing of one LTI model with one stable controller [5]. For a constructive method achieving stable controller design, see [6].

Remark 2: Notice that there is generally an infinite set of solutions for α, β and γ and so there is a certain freedom in the construction of the controller (5) making the closed-loop characteristic polynomial affine in λ. This set of controllers can be explored to obtain the following closed-loop properties: 1) stability for all fixed $\lambda \in [0, 1]$, 2) the highest possible bound on average $\lambda(t)$ variation, consistent with retaining stability 3) a better LTV closed-loop performance. This will be discussed further in Section III.

Applying the controller (5) of any order in λ, but which makes the closed-loop characteristic polynomial affine in λ, leads to the following closed-loop characteristic polynomial:

$$\begin{aligned} P_{CL}(\lambda) &= \lambda[\gamma(d_1 S_1 + n_1 R_1) - \beta(d_0 S_0 + n_0 R_0)] \\ &\quad + \beta(d_0 S_0 + n_0 R_0) \end{aligned} \tag{17}$$

The polynomials q_i are not directly present in the characteristic polynomial (though they enter the equation indirectly via β and γ as they are solutions of (9)), but they appear in the numerators of the closed-loop sensitivities related to the higher powers of the parameter λ:

$$\begin{aligned} S_{io}(\lambda) &= \frac{\lambda^{n+1} p_d s_n + \Sigma_{i=2}^{n}[\lambda^i (d_0 s_i + p_d s_{i-1})] + d_0 s_0}{P_{CL}(\lambda)} \\ GCS(\lambda) &= \frac{\lambda^{n+1} p_n r_n + \Sigma_{i=2}^{n}[\lambda^i (n_0 r_i + p_n r_{i-1})] + n_0 r_0}{P_{CL}(\lambda)} \\ CS(\lambda) &= \frac{\lambda^{n+1} p_d r_n + \Sigma_{i=2}^{n}[\lambda^i (d_0 r_i + p_d r_{i-1})] + d_0 r_0}{P_{CL}(\lambda)} \\ GS(\lambda) &= \frac{\lambda^{n+1} p_n s_n + \Sigma_{i=2}^{n}[\lambda^i (n_0 s_i + p_n s_{i-1})] + n_0 s_0}{P_{CL}(\lambda)} \end{aligned} \tag{18}$$

B. Specific cases

1) 1st-order in λ controller: We will now study the two specific cases $n = 1$ and $n = 2$. Consider first the simplest 1st-order in λ controller:

$$C(\lambda) = \frac{\lambda r_1 + r_0}{\lambda s_1 + s_0} \tag{19}$$

with r_1, r_0, s_1, s_0 some polynomials. The next Theorem is closely related to Theorem 1.

Theorem 2: Consider the parameter-dependent model (1) connected in closed loop with the parameter-dependent controller (19) and two LTI controllers (4) stabilizing in closed loop the two extreme models (3). There exist a stabilizing controller (19) with only stable pole-zero cancellations making the characteristic polynomial (7) affine in λ if and only if the two polynomials

$$\begin{aligned} \beta &= (p_d S_1 + p_n R_1) \\ \gamma &= (p_d S_0 + p_n R_0) \end{aligned} \tag{20}$$

have all their zeros in the left-half plane (stable). Then the corresponding polynomials of the controller (19) are:

$$\begin{aligned} r_0 &= R_0\beta \\ s_0 &= S_0\beta \\ r_1 &= R_1\gamma - R_0\beta \\ s_1 &= S_1\gamma - S_0\beta \end{aligned} \tag{21}$$

Proof: See the Appendix. ■

Remark 3: Observe from (20) that to guarantee the stability of β and γ, the initial controllers C_0 and C_1 must both stabilize the following LTI model:

$$G_{nd} = \frac{p_n}{p_d} = \frac{n_1 - n_0}{d_1 - d_0} \quad (22)$$

Remark 4: If the condition of Theorem 2 is satisfied, then that of Lemma 1 is also satisfied. The condition of Theorem 2 is more restrictive than that of Lemma 1.

2) 2nd-order in λ controller: Consider now the 2nd-order λ-dependent controller obtained by specializing the general parameter-dependent controller (5):

$$C(\lambda) = \frac{\lambda^2 r_2 + \lambda r_1 + r_0}{\lambda^2 s_2 + \lambda s_1 + s_0} \quad (23)$$

From Theorem 1 we can observe that the controller has all properties of the general controller (5) and that the polynomial equation (9) to be solved is now:

$$q_2(d_0 p_n - n_0 p_d) - \beta(p_d S_0 + p_n R_0) + \gamma(p_d S_1 + p_n R_1) = 0 \quad (24)$$

Remark 5: The polynomial $(\sum_{i=2}^{n} q_i)$ in (9) plays the same role as the polynomial q_2 in the 2nd-order in λ controller equation (24). The sum of polynomials does not provide any additional freedom for the solution of the equation. Consequently, the set of achievable affine closed-loop characteristic polynomials, see (17), is no greater for controllers of the form (5) for $n > 2$ than it is for controllers of the form (23) with $n = 2$. However the closed-loop performance for the 2nd-order in λ controller is different from the performance of the general controller (with $n > 2$). This is because some additional higher-power λ terms appear in closed-loop sensitivity numerators (18) for the general parameter-dependent controller.

III. CLOSED-LOOP STABILITY ANALYSIS

A. Stability for all frozen values of λ

In the previous section it has been shown that the controller (5) can make the closed loop characteristic polynomial $P_{CL}(\lambda)$ affine in λ. We have seen that this polynomial has always the same structure shown in (17). For extreme values of λ the polynomial becomes:

$$\begin{aligned} P_{CL}(\lambda = 0) &= \beta(d_0 S_0 + n_0 R_0) \\ P_{CL}(\lambda = 1) &= \gamma(d_1 S_1 + n_1 R_1) \end{aligned} \quad (25)$$

where the β and γ polynomials are common factors of the numerator and denominator of $C(\lambda)$ when the extreme values of λ are adopted in $C(\lambda)$. In [1], necessary and sufficient conditions for stability of an affine polynomial set are proposed. Applying them in our case we can establish the following stability theorem:

Theorem 3 ([1]): Consider the λ-dependent closed-loop polynomial $P_{CL}(\lambda)$ as in (17) and suppose that the two polynomials $P_{CL}(1)$ and $P_{CL}(0)$ have the same degree. Then the closed-loop characteristic polynomial $P_{CL}(\lambda)$ is stable for all frozen values $\lambda \in [0, 1]$ if and only if the inequality:

$$|arg[P_{CL}(1)] - arg[P_{CL}(0)]| < \pi \quad (26)$$

holds at all finite points on the imaginary axis (for all ω with laplace operator $s = j\omega$ in the polynomials $P_{CL}(0)$, $P_{CL}(1)$).

Proof: See [1]. ∎

The key insight to be obtained from this theorem is that, no matter what order in λ controller is chosen in accord with the scheme of Theorem 1, it will be stabilizing for all λ in the relevant range if and only if the designs of the two extreme controllers C_0, C_1 achieve a certain relation between the two closed-loop characteristic polynomials. Either all controllers obtained via the procedure of Theorem 1 will be stabilizing for each frozen value of λ, or none of them will be stable for all frozen values of λ.

The condition requiring the same degree of the two polynomials $P_{CL}(1)$ and $P_{CL}(0)$ leads to some restrictions concerning the degrees of numerators and denominators of the models $G(0)$, $G(1)$, the controllers C_0, C_1, and the polynomials β, γ. For example, for a 1st-order in λ controller it is sufficient that the following equalities on polynomial degrees hold: $deg(n_1) = deg(n_0)$, $deg(d_1) = deg(d_0)$, $deg(S_1) = deg(S_0)$, $deg(R_1) = deg(R_0)$. These equalities can be interpreted as the following two constraints: 1) The structure of the plant model does not change in the entire range of the considered operational conditions, i.e. the number of poles and zeros is the same for all $\lambda \in [0, 1]$; 2) The controllers C_0 and C_1 have the same number of zeros and the same number of poles. If the first condition on plant model degree is fulfilled, the second condition can be ensured by applying the same design technique for design of both controllers.

Applying (25) inequality (26) becomes:

$$|\arg[\gamma(d_1 S_1 + n_1 R_1)] - \arg[\beta(d_0 S_0 + n_0 R_0)]| < \pi \mid \forall\omega \quad (27)$$

For the 1st-order in λ controller (19), the polynomials β and γ are known from (51). Using this fact and some basic operations the inequality becomes:

$$\left| \begin{array}{c} \arg(p_d S_0 + p_n R_0) - \arg(p_d S_1 + p_n R_1) + \\ + \arg(d_1 S_1 + n_1 R_1) - \arg(d_0 S_0 + n_0 R_0) \end{array} \right| < \pi \mid \forall\omega \quad (28)$$

The stability condition (28) is satisfied, if these two requirements are satisfied simultaneously: 1) The plant models $G(0)$ and $G(1)$ have their zeros sufficiently close as well as their poles; 2) The controllers C_0 and C_1 have their zeros sufficiently close as well as their poles. Then zeros of the characteristic polynomials in the inequality (28) are also similar and consequently the angles related to these zeros. Thus the two angle differences in (28) are small.

B. Stability for time-varying $\lambda(t)$

The previous section shows that the controller (5) of any λ-order can, under certain conditions, provide an affine stable closed-loop characteristic polynomial $P_{CL}(\lambda)$ for frozen λ. For a certain degree of stability of $P_{CL}(\lambda)$ affine in λ, a bound on the average rate of parameter variation can be determined [3], which indicates how fast the parameter $\lambda(t)$ can vary in time without destabilizing the closed loop. The bound gives us a sufficient condition for stability.

The degree of stability σ of the polynomial $P_{CL}(\lambda)$ is defined as follows: All zeros of $P_{CL}(\lambda)$ are in the left half-plane for all frozen $\lambda \in [0,1]$ and their distance from the imaginary axis is at least σ for all $\lambda \in [0,1]$. A minor variant on Theorem 3 shows how to check that the polynomial has this degree of stability for all frozen values λ by examining the polynomials at the extreme values.

To define a measure on the average rate of acceptable parameter variation we use the result from [3]. We get the following theorem adjusted for our problem:

Theorem 4 ([3]): Consider a parameter-dependent controller and parameter-dependent plant model making the closed loop characteristic polynomial $P_{CL}(\lambda)$ affine in λ and stabilizing the closed loop for all frozen $\lambda \in [0,1]$ with a stability degree σ. Then the the closed loop system is exponentially asymptotically stable with a stability degree of $\delta < \sigma$ for the time-varying parameter

$$\lambda(t) \in (0,1) \tag{29}$$

varying in time, if for some $T > 0$ and $\forall t \geq 0$ the following inequality holds:

$$\sup_t \frac{1}{T} \int_t^{t+T} \left| \frac{d}{d\tau} \ln \left(\frac{\lambda(\tau) - 0}{1 - \lambda(\tau)} \right) \right| d\tau < 4(\sigma - \delta) \tag{30}$$

Proof: See [3]. ■

Remark 6: Because (30) restricts the average rate of variation, there can be step changes in $\lambda(t)$ which do not violate (30).

Remark 7: Notice that using condition (30), a minimal required degree of stability σ can be evaluated for any signal type $\lambda(t)$. This minimal degree value guarantees for $\delta \to 0$ the system stability. Consider, in our case, a sinusoidal signal $\lambda(t) = 0.5 + 0.499 \sin(\omega t)$, then for each frequency ω a corresponding minimal degree of stability can be evaluated. Figure 2 shows an approximate graph of the required minimal degree of stability for the frequency ω of the sinusoidal signal.

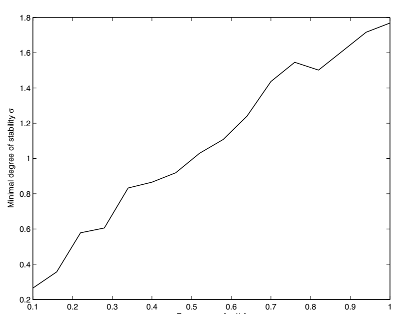

Fig. 2. Minimal degree of stability for the frequency ω of the sinusoidal signal $\lambda(t) = 0.5 + 0.499 \sin(\omega t)$.

IV. EXAMPLE

The plant to be controlled is a pick-and-place machine constructed at Flanders' MECHATRONICS Technology Centre (FMTC). The system consists of a carriage driven by a linear motor in a horizontal direction and a vertical beam mounted on the carriage and actuated by a DC motor in a vertical direction. The model for horizontal motion is shown in Figure 3 where $m_1 = 18$kg, $m_2 = 1.2$kg are the masses of the carriage and the beam end-point respectively and $k(p) \in [k_{min}, k_{max}]$, $k_{min} = 4.32 \cdot 10^4$, $k_{max} = 1.92 \cdot 10^5$ is a parameter of stiffness varying in time according to the length p of the beam. The variable F is the force applied on the carriage, and x_1, x_2 are the horizontal positions of the carriage and the beam end-point respectively. The viscous friction is represented by two constants $c_1 = 750$N.s/m and $c_2 = 6.9$N.s/m, and the high-frequency dynamics of the frame are neglected.

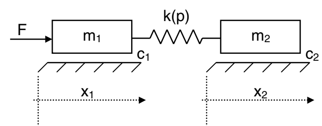

Fig. 3. Physical model of pick-and-place machine.

The plant is modelled as a linear parameter-dependent transfer function $x_2 = G(s, \lambda)F$:

$$\begin{aligned} G(s,\lambda) &= \frac{\lambda d_k + k_{min}}{\lambda p_d(s) + d_0(s)} \\ p_d(s) &= d_k(m_1+m_2)s^2 + d_k(c_1+c_2)s \\ d_0(s) &= s^4 + (c_1 m_2 + c_2 m_1)s^3 + \\ &+ (c_1 c_2 + k_{min}(m_1+m_2))s^2 + k_{min}(c_1+c_2)s \end{aligned} \tag{31}$$

where the force F is the input signal and the position of the beam end-point x_2 is the output signal. The varying stiffness $k(p)$ is characterized with a parameter λ: $k(p) = \lambda d_k + k_{min}, d_k = k_{max} - k_{min}$ to normalize the variation of the parameter from 0 to 1. The two extreme LTI transfer functions for $\lambda = 0$ and $\lambda = 1$ are $G(s,0)$, $G(s,1)$ and they have the frequency responses shown in Figure 4.

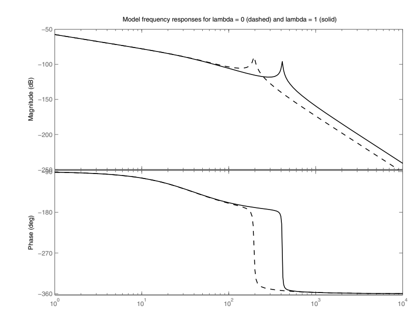

Fig. 4. Frequency responses of $G(s,0)$ (dashed) and $G(s,1)$ (solid).

Two stabilizing controllers $C_0(s)$, $C_1(s)$ are designed by the pole-placement method for the models $G(s,0)$ and $G(s,1)$ respectively. The controllers give very fast closed-loop step responses and their transfer functions are bi-proper, with order 3. Concerning the robustness $C_0(s)$ does not stabilize the model $G(s,1)$ and $C_1(s)$ does not stabilize the model $G(s,0)$. Thus neither of the designed controllers can be used to control the entire range of operational conditions.

We proceed with a design of parameter-dependent controller. To be able to design the 1st-order in λ controller, one additional condition has to be satisfied (see Section II-B.1): The two controllers $C_0(s)$ and $C_1(s)$ must both stabilize the plant in (22) and it can be verified that that this condition holds in our case. Thus there exists an admissible 1st-order in λ controller (19) making the closed-loop characteristic polynomial affine in λ. The inequality (28) is used to check if the 1st-order controller stabilizes the closed loop with model (31) for all fixed values of $\lambda \in [0, 1]$. The left-side of the inequality for frequencies from 10rad/s to 10000rad/s gives the maximum angle difference of -178.2deg, i.e. -3.11rad, i.e. the inequality holds.

Finally we can check the LTV closed-loop system stability with respect to parameter $\lambda(t)$ variation. The inequality (30) will be used, but first a degree of stability σ of the LTV closed loop has to be evaluated. To do so we use the stability condition (28). The laplace operator (s) is substituted with its shifted version $(s - \sigma)$ in this condition and the largest σ providing satisfaction of the condition is searched. The degree of stability of the LTV closed-loop for the 1st-order in λ controller is $\sigma = 1.069$.

Consider that the time-varying parameter is a sinusoidal signal $\lambda(t) = 0.5 + 0.499 \sin(\omega t)$. Using the inequality (30) we can obtain the highest acceptable frequency ω guaranteeing LTV closed-loop stability. We can use Figure 2 from which we get that the LTV closed-loop stability is in our case guaranteed for ω up to 0.5rad/s. We know that the condition is only sufficient and hence the system may remain stable even for higher frequencies ω. This is the case here; in simulating the closed-loop system with the higher frequencies, the system was found to remain stable up to approximately $\omega = 20$rad/s. Figure 5 shows an example of system response on a pulse reference with sinusoidal $\lambda(t)$ of 5rad/s. Apparently the closed loop remains stable above the guaranteed stability bound. If the obtained performance of 1st-order λ-dependent controller is judged to be unsatisfactory, we can either modify the controllers $C_0(s)$, $C_1(s)$ and re-design the controller, or proceed to the design of the 2nd-order in λ controller.

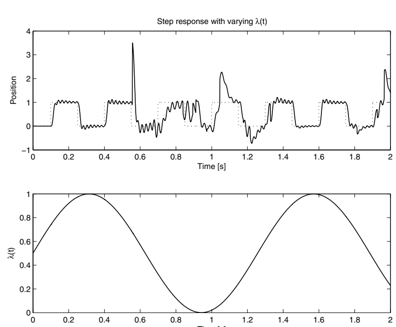

Fig. 5. Step responses of closed-loop LTV system with sinusoid $\lambda(t)$ of 5rad/s (upper axis: actual position - solid, reference - dotted).

V. CONCLUSIONS

Under certain restrictive conditions on the initial controllers $C_0(s)$ and $C_1(s)$, there exists a unique 1st-order in λ controller making the polynomial $P_{CL}(\lambda)$ affine in λ. The controller is defined by the models $G(s, 0)$, $G(s, 1)$ and the controllers $C_0(s)$, $C_1(s)$. Nothing guarantees that this controller provides LTV closed-loop stability. If the controller performance is not satisfactory, either the controllers $C_0(s)$, $C_1(s)$ are to be adjusted, or a 2nd-order in λ controller can be designed instead.

Under some mild conditions on controllers $C_0(s)$, $C_1(s)$ and models $G(s, 0)$, $G(s, 1)$, there exists a set of nth-order in λ controllers $(n \geq 2)$ making the polynomial $P_{CL}(s, \lambda)$ affine in λ. The characteristic polynomial of the closed loop is the same for any order $n \geq 2$ of these controllers. On the other hand the numerators of sensitivities are different since the higher powers of the λ parameter appear there.

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APPENDIX

Proof: (for Theorem 1) Consider the general controller structure (5). The following conditions have to be satisfied to guarantee that the characteristic polynomial (7) is affine in λ:

- The controller $C(\lambda)$ from (5) has to be equal to the controller C_0 from (4) for $\lambda = 0$, i.e.

$$\frac{r_0}{s_0} = \frac{R_0}{S_0} \tag{32}$$

- The controller $C(\lambda)$ from (5) has to be equal to the controller C_1 from (4) for $\lambda = 1$, i.e.

$$\frac{r_n + r_{n-1} + \ldots + r_0}{s_n + s_{n-1} + \ldots + s_0} = \frac{R_1}{S_1} \tag{33}$$

- The polynomials related to λ^{n+1}, λ^n, ... λ^2 in the characteristic polynomial (7) have to be zero:

$$\lambda^{n+1} \quad : \quad p_d s_n + p_n r_n = 0 \tag{34}$$

$$\lambda^{n} \quad : \quad d_0 s_n + n_0 r_n + p_d s_{n-1} + p_n r_{n-1} = 0 \tag{35}$$

$$\vdots$$

$$\lambda^{3} \quad : \quad d_0 s_3 + n_0 r_3 + p_d s_2 + p_n r_2 = 0 \tag{36}$$

$$\lambda^{2} \quad : \quad d_0 s_2 + n_0 r_2 + p_d s_1 + p_n r_1 = 0 \tag{37}$$

From (32) we have

$$\frac{r_0}{s_0} = \frac{R_0\beta}{S_0\beta} \tag{38}$$

with β some stable polynomial. (The stability of the polynomial β is necessary, since it guarantees that for $\lambda \to 0$ there will be no cancellation of unstable poles/zeros in the controller (5)).

From (34) we have

$$\frac{r_n}{s_n} = -\frac{p_d q_n}{p_n q_n} \tag{39}$$

with q_n an arbitrary polynomial. Let us temporarily assume the polynomials p_n, p_d to be coprime. Since the polynomials r_n, s_n are known, equation (35) becomes a standard polynomial diophantine equation:

$$p_d s_{n-1} + p_n r_{n-1} = -d_0 s_n - n_0 r_n \tag{40}$$

where r_{n-1} and s_{n-1} are some unknown polynomials to be found and where the right side of the equation is known. With p_n, p_d coprime the solution always exists and the set of all solutions is given by:

$$\begin{aligned} r_{n-1} &= r_{n-1}^0 - p_d q_{n-1} \\ s_{n-1} &= s_{n-1}^0 + p_n q_{n-1} \end{aligned} \tag{41}$$

with r_{n-1}^0 and s_{n-1}^0 some particular solution of (40) and q_{n-1} an arbitrary polynomial. Now if the expressions for the polynomials r_n, s_n from (39) are used in the equation (40) one obvious solution of this equation is:

$$\begin{aligned} r_{n-1}^0 &= -d_0 q_n \\ s_{n-1}^0 &= n_0 q_n \end{aligned} \tag{42}$$

Using the technique of the previous paragraph we solve all conditions from (35) to (36) recursively with the following resulting polynomial sets (for p_n, p_d coprime):

$$\begin{aligned} r_i &= r_i^0 - p_d q_i \\ s_i &= s_i^0 + p_n q_i \end{aligned} \tag{43}$$

for $i = n-1, n-2, \ldots, 2$ and with the q_i arbitrary polynomials. The polynomials r_i^0 and s_i^0 are particular solutions of the corresponding diophantine equation and one possible solution is always:

$$\begin{aligned} r_i^0 &= -d_0 q_{i+1} \\ s_i^0 &= n_0 q_{i+1} \end{aligned} \tag{44}$$

All polynomials of the controller (5) are now fixed except r_1, s_1. Now use condition (33) to get:

$$\begin{aligned} r_1 &= R_1\gamma - r_n - r_{n-1} - \ldots - r_0 \\ s_1 &= S_1\gamma - s_n - s_{n-1} - \ldots - s_0 \end{aligned} \tag{45}$$

with γ some stable polynomial. (The stability property guarantees that for $\lambda \to 1$ there will be no cancellation of unstable poles/zeros in the controller (5)).

The last condition to be satisfied is (37). Use the above expressions for the polynomials r_i, s_i, for $i = 0, \ldots n$ and r_i^0, s_i^0 for $i = 2, \ldots n-1$ in this condition and it becomes:

$$\left(\sum_{i=2}^{n} q_i\right)(d_0 p_n - n_0 p_d) - \beta(p_d S_0 + p_n R_0) + \gamma(p_d S_1 + p_n R_1) = 0 \tag{46}$$

Hence we need to solve a special polynomial equation with three unknown polynomials: β, γ, and $(\sum_{i=2}^{n} q_i)$.

It remains to show what happens when the polynomials p_n, p_d are not coprime. Consider that the polynomials p_n, p_d have a common divisor d_{nd}:

$$\begin{aligned} p_n &= d_{nd} p_n' \\ p_d &= d_{nd} p_d' \end{aligned} \tag{47}$$

The proof is the same up to equation (39). We then still insist on the polynomials $r_n = -p_d q_n$, $s_n = p_n q_n$ resulting from (39) and they have to both contain the common divisor d_{nd}. This is so that polynomial equation (40) can be solved by (41) and (42). We again insist on all solutions (41) to contain p_n and p_d rather than p_n' and p_d' so that in the recursive process, the diophantine equations remains easily solvable. All polynomials r_i, s_i for $i = n-1, ..., 2$ can be solved in that manner.

Finally polynomials r_1, and s_1 result from (45) and equation (46) is to be solved. The common divisor d_{nd} appears only in the polynomials p_n and p_d in equation (46). Hence it can be removed from the equation before a solution is sought. The polynomial equation to be solved becomes:

$$\left(\sum_{i=2}^{n} q_i\right)(d_0 p_n' - n_0 p_d') - \beta(p_d' S_0 + p_n' R_0) + \gamma(p_d' S_1 + p_n' R_1) = 0 \tag{48}$$

The common divisor d_{nd} can contains also unstable zeros, since it appears only in the polynomials r_i and s_i, $i = 2, 3, \ldots n$ and these polynomials never appear alone in the parameter-dependent controller transfer function $C(\lambda)$. ■

Proof: (for Theorem 2) The proof is obtained similarly to the proof of Theorem 1. In the case of the 1st-order in λ controller conditions (32), (33), and (34) has to be satisfied. The index n is equal to $n = 1$ and consequently the polynomial solutions (43) do not appear in this case.

Applying the obvious solutions of the conditions (32) and (33), which are:

$$\begin{aligned} r_0 &= R_0\beta, \quad s_0 = S_0\beta \\ r_1 &= R_1\gamma - r_0, \quad s_1 = S_1\gamma - s_0 \end{aligned} \tag{49}$$

to the last condition (34) leads to the following reduced polynomial equation

$$-\beta(p_d S_0 + p_n R_0) + \gamma(p_d S_1 + p_n R_1) = 0 \tag{50}$$

A solution of this polynomial equation is

$$\begin{aligned} \beta &= p_d S_1 + p_n R_1 \\ \gamma &= p_d S_0 + p_n R_0 \end{aligned} \tag{51}$$

and since the polynomials β and γ need to be stable (as was earlier explained) we get the stability condition: The polynomials $(p_d S_1 + p_n R_1)$ and $(p_d S_0 + p_n R_0)$ have to be stable. ■