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Reaching an Agreement Using Delayed Information*†

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***Abstract*— This paper studies a modified version of the Vicsek's problem, also known as the "consensus problem." Vicsek *et al.* consider a discrete-time model consisting of n autonomous agents all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of the headings of its "neighbors." We consider a modified version of the Vicsek's problem in which integer valued delays occur in sensing the values of headings which are available to agents. By appealing to the concept of *graph composition*, we side-step most issues involving products of stochastic matrices and present a variety of graph theoretic results which explains how convergence to a common heading is achieved.**

I. INTRODUCTION

Current interest in cooperative control of groups of mobile autonomous agents has led to the rapid increase in the application of graph theoretic ideas together with more familiar dynamical systems concepts to problems of analyzing and synthesizing a variety of desired group behaviors such as maintaining a formation, swarming, rendezvousing, or reaching a consensus. While this in-depth assault on group coordination using a combination of graph theory and system theory is in its early stages, it is likely to significantly expand in the years to come. One line of research which "graphically" illustrates the combined use of these concepts, is the recent theoretical work by a number of individuals [1], [2], [3], [4], [5], [6] which successfully explains the heading synchronization phenomenon observed in simulation by Vicsek [7], Reynolds [8] and others more than a decade ago. Vicsek and co-authors consider a simple discrete-time model consisting of n autonomous agents or particles all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading plus the current headings of its "neighbors." Agent i's *neighbors* at time t, are those agents, including itself, which are either in or on a circle of pre-specified radius r_i centered at agent i's current position. In their paper, Vicsek *et al.* provide a variety of interesting simulation results which demonstrate that the nearest neighbor rule they are studying can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors can change with time. A theoretical explanation for this observed behavior has recently been given in [1]. The explanation exploits ideas from graph theory [9] and from the theory of non-homogeneous Markov chains [10], [11], [12]. With the benefit of hindsight it is now reasonably clear that it is more the graph theory than the Markov chains which will prove key as this line of research advances. An illustration of this is the recent extension of the findings of [1] which explain the behavior of Reynolds' full nonlinear "boid" system [6]. In this paper, by appealing to the concept of *graph composition*, we side-step most issues involving products of stochastic matrices and present a variety of graph theoretic results which explain how convergence to a common heading is achieved.

In the past few years many important papers have appeared [2], [3], [4], [5], [13] which expand the results obtained in [1] and extend the Vicsek model in many directions. For example, in a recent paper [5] a modified version of the Vicsek problem is considered in which integer valued delays occur in sensing the values of headings which are available to agents. The aim of this paper is to consider the same problem, but more from a graph theoretic point of view. This enables us to relax the conditions stated in [5] under which consensus is achieved.

The rest of the paper is organized as follows. In section 2, the main convergence result is introduced. In section 3, we present the state space model of the modified version of Vicsek's problem with measurement delays. In sections 4 and 5, several classes of graphs with special structures are defined and the properties of their composition graphs are studied. In section 6, we give the proof of our main convergence result.

II. COORDINATION FACING MEASUREMENT DELAYS

We consider a modified version of the Vicsek's problem. More precisely we suppose that at each time $t \in \{0, 1, 2, \ldots\}$, the value of neighboring agent j's headings which agent i may sense is $\theta_j(t - d_{ij}(t))$ where $d_{ij}(t)$ is a delay whose value at t is some integer between 0 and $m_j - 1$; here m_j is a pre-specified positive integer. While well established principles of feedback control would suggest that

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delays should be dealt with using dynamic compensation, in this paper we will consider the situation in which the delayed value of agent j's heading sensed by agent i at time t is the value which will be used in the heading update law for agent i. Let $\mathcal{N}_i(t)$ and $n_i(t)$ denote the set of labels and the number of agent i's neighbors at time t respectively. Thus

$$\theta_i(t+1) = \frac{1}{n_i(t)} \left(\sum_{j\in\mathcal{N}_i(t)} \theta_j(t - d_{ij}(t)) \right) \qquad (1)$$

where $d_{ij}(t) \in \{0, 1, \ldots, (m_j - 1)\}$ if $j \neq i$ and $d_{ij}(t) = 0$ if $i = j$.

In the delay-free version of the problem treated in [1], $d_{ij} = 0$ for $i \in \{1, \ldots, n\}$ and $j \in \mathcal{N}_i(t)$. Thus in this case each agent's heading update equation can be written as

$$\theta_i(t+1) = \frac{1}{n_i(t)} \left(\sum_{j\in\mathcal{N}_i(t)} \theta_j(t) \right) \qquad (2)$$

The explicit forms of the update equations determined by (1) and (2) respectively depend on the relationships between neighbors which exist at time t. These relationships can be conveniently described by a directed graph $\mathbb{N}(t)$ with vertex set $\mathcal{V} = \{1, 2, \ldots n\}$ and arc set $\mathcal{A}(\mathbb{N}(t)) \subset \mathcal{V} \times \mathcal{V}$ which is defined in such a way so that (i, j) is an *arc* or directed edge from i to j just in case agent i is a neighbor of agent j at time t. Thus $\mathbb{N}(t)$ is a directed graph on n vertices with at most one arc connecting each ordered pair of distinct vertices and with exactly one self-arc at each vertex. We write $\mathcal{G}_{sa}$ for the set of all such graphs.

Let $\mathcal{G}$ be the set of all directed graphs with vertex set $\mathcal{V}$. Let $\mathcal{A}(\mathbb{G})$ denote the set of arcs of $\mathbb{G}$. It is natural to call a vertex i a *neighbor* of vertex j in $\mathbb{G} \in \mathcal{G}$ if (i, j) is an arc in $\mathbb{G}$. In the sequel we will call a vertex i of a directed graph $\mathbb{G}$, a *root* of $\mathbb{G}$ if for each other vertex j of $\mathbb{G}$, there is a path from i to j. Thus i is a root of $\mathbb{G}$, if it is the root of a directed spanning tree of $\mathbb{G}$. We will say that $\mathbb{G}$ is *rooted at* i if i is in fact a root. Thus $\mathbb{G}$ is rooted at i just in case each other vertex of $\mathbb{G}$ is *reachable* from vertex i along a path within the graph. $\mathbb{G}$ is *strongly rooted at* i if each other vertex of $\mathbb{G}$ is reachable from vertex i along a path of length 1. Thus $\mathbb{G}$ is strongly rooted at i if i is a neighbor of every other vertex in the graph. By a *rooted graph* $\mathbb{G} \in \mathcal{G}$ is meant a graph which possesses at least one root. A *strongly rooted graph* is a graph which has at least one vertex at which it is strongly rooted.

Here we will "combine graphs" using the notion of "graph composition" rather than the notion of "graph union" used in [1], [2], [3]. By the *composition* of graph $\mathbb{G}_{q_1} \in \mathcal{G}$ with $\mathbb{G}_{q_2} \in \mathcal{G}$, written $\mathbb{G}_{q_2} \circ \mathbb{G}_{q_1}$, is meant the directed graph with vertex set $\mathcal{V}$ and arc set defined in such a way so that (u, v) is an arc of the composition just in case there is a vertex w such that (u, w) is an arc of $\mathbb{G}_{q_1}$ and (w, v) is an arc of $\mathbb{G}_{q_2}$. We say that a finite sequence of directed graphs $\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_k}$ from $\mathcal{G}$ is *jointly rooted* if the composition $\mathbb{G}_{p_k} \circ \mathbb{G}_{p_{k-1}} \circ \cdots \circ \mathbb{G}_{p_1}$ is a rooted graph. We say that an infinite sequence of graphs $\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots,$ in $\mathcal{G}$ is *repeatedly jointly rooted* if there is a positive integer m for which each finite sequence $\mathbb{G}_{p_{m(k-1)+1}}, \ldots, \mathbb{G}_{p_{mk}}, \quad k \geq 1$ is jointly rooted.

The main result of [1] is similar to the follows.

Theorem 1: Let the $\theta_i(0)$ be fixed. For any trajectory of the system determined by (2) along which the sequence of neighbor graphs $\mathbb{N}(0), \mathbb{N}(1), \ldots$ is repeatedly jointly rooted, there is a constant θ_{ss} for which

$$\lim_{t\to\infty} \theta_i(t) = \theta_{ss} \qquad (3)$$

where the limit is approached exponentially fast.

The aim of this paper is to prove that essentially the same result holds in the face of measurement delays.

Theorem 2: Let the $\theta_i(0)$ be fixed. For any trajectory of the system determined by (1) along which the sequence of neighbor graphs $\mathbb{N}(0), \mathbb{N}(1), \ldots$ is repeatedly jointly rooted, there is a constant θ_{ss} for which

$$\lim_{t\to\infty} \theta_i(t) = \theta_{ss} \qquad (4)$$

where the limit is approached exponentially fast.

As noted in the introduction, the consensus problem with measurement delays we've been discussing has been considered previously in [5]. It is possible to compare the hypotheses of Theorem 2 with the corresponding hypotheses for exponential convergence stated in [5], namely assumptions 2 and 3 of that paper. To do this, let us agree, as before, to say that the *union* of a set of graphs $\mathbb{G}_{r_1}, \mathbb{G}_{r_2}, \ldots, \mathbb{G}_{r_k}$ with vertex set $\mathcal{V}$ is that graph with vertex set $\mathcal{V}$ and arc set consisting of the union of the arcs of all of the graphs $\mathbb{G}_{r_1}, \mathbb{G}_{r_2}, \ldots, \mathbb{G}_{r_k}$. Taken together, assumptions 2 and 3 of [5] are more or less equivalent to assuming that there are finite positive integers q and s such that the *union*

$$\mathbb{G}(k) \triangleq \mathbb{N}((k+1)q-1) \cup \mathbb{N}((k+1)q-2) \cup \cdots \cup \mathbb{N}(kq)$$

is strongly connected and independent of k for $k \geq s$. By way of comparison, the hypothesis of Theorem 2 is equivalent to assuming that there is a finite positive integer q such that the *composition*

$$\bar{\mathbb{G}}(k) \triangleq \mathbb{N}((k+1)q-1) \circ \mathbb{N}((k+1)q-2) \circ \cdots \circ \mathbb{N}(kq)$$

is rooted for $k \geq 0$. The latter assumption is weaker than the former for several reasons. First, the arc set of $\mathbb{G}(k)$ is always a subset of the arc set of $\bar{\mathbb{G}}(k)$ and in some cases the containment may be strict. Second, $\bar{\mathbb{G}}(k)$ is not assumed to be independent of k, even for k sufficiently large, whereas $\mathbb{G}(k)$ is; in other words, $\bar{\mathbb{G}}(k)$ is not assumed to converge whereas $\mathbb{G}(k)$ is. Third, each $\mathbb{G}(k)$ is assumed to be strongly connected whereas each $\bar{\mathbb{G}}(k)$ need only be rooted; note that a strongly connected graph is a special type of rooted graph in which every vertex is a root. Perhaps most important about Theorem 2 and the development which justifies it, is that the underlying structural properties of the graphs involved required for consensus are explicitly determined.

III. STATE SPACE SYSTEM

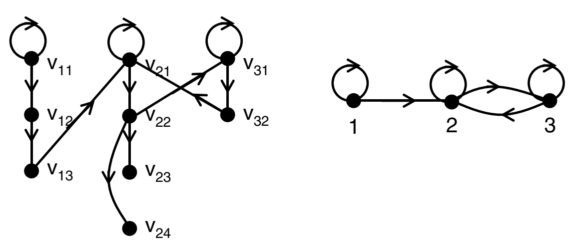
It is possible to represent the agent system defined by (1) using a state space model. Towards this end, let $\bar{\mathcal{G}}$ denote the set of all directed graphs with vertex set $\bar{\mathcal{V}} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_n$ where $\mathcal{V}_i = \{v_{i1} \ldots, v_{im_i}\}$. Here vertex v_{ij} labels the jth possible delay value of agent i, namely $j-1$. We sometimes write i for $v_{i1}, \;\; i \in \{1, 2, \ldots, n\}$, $\mathcal{V}$ for the subset of vertices $\{v_{11}, v_{21}, \ldots, v_{n1}\}$, and think of v_{i1} as an alternative label of agent i.

To take account of the fact that each agent can use its own current heading in its update formula (1), we will utilize those graphs in $\bar{\mathcal{G}}$ which have self arcs at each vertex in $\mathcal{V}$. We will also require the arc set of each such graph to have, for $i \in \{1, 2, \ldots, n\}$, an arc from each vertex $v_{ij} \in \mathcal{V}_i$ except the last, to its successor $v_{i(j+1)} \in \mathcal{V}_i$. Finally we stipulate that for each $i \in \{1, 2, \ldots, n\}$, each vertex v_{ij} with $j > 1$ has in-degree of exactly 1. In the sequel we call any such graph a *delay graph* and write $\mathcal{D}$ for the subset of all such graphs. Note that there are graphs in $\mathcal{D}$ possessing vertices without self-arcs. Nonetheless each vertex of each graph in $\mathcal{D}$ has positive in-degree.

The specific delay graph representing the sensed headings the agents use at time t to update their own headings according to (1), is that graph $\mathbb{D}(t) \in \mathcal{D}$ whose arc set contains an arc from $v_{ik} \in \mathcal{V}_i$ to $v_{j1} \in \mathcal{V}$ if agent j uses $\theta_i(t+1-k)$ to update. There is a simple relationship between $\mathbb{D}(t)$ and the neighbor graph $\mathbb{N}(t)$ defined earlier. In particular,

$$\mathbb{N}(t) = Q(\mathbb{D}(t)) \tag{5}$$

where $Q(\mathbb{D}(t))$ is the "quotient graph" of $\mathbb{D}(t)$. By the *quotient graph* of any $\mathbb{G} \in \bar{\mathcal{G}}$, written $Q(\mathbb{G})$, is meant that directed graph in $\mathcal{G}$ with vertex set $\mathcal{V}$ whose arc set consists of those arcs (i, j) for which $\mathbb{G}$ has an arc from some vertex in $\mathcal{V}_i$ to some vertex in $\mathcal{V}_j$. The quotient graph of $\mathbb{D}(t)$ thus models which headings are being used by each agent in updates at time t without describing the specific delayed headings actually being used. The following is an example of a delay graph (left) and its quotient graph (right).



The set of agent heading update rules defined by (1) can be written in state form. Towards this end define $\theta(t)$ to be that $(m_1 + m_2 + \cdots + m_i)$ vector whose first m_1 elements are $\theta_1(t)$ to $\theta_1(t+1-m_1)$, whose next m_2 elements are $\theta_2(t)$ to $\theta_2(t+1-m_2)$ and so on. Order the vertices of $\bar{\mathcal{V}}$ as $v_{11}, \ldots, v_{1m_1}, v_{21}, \ldots, v_{2m_2}, \ldots, v_{n1}, \ldots, v_{nm_n}$ and with respect to this ordering define for each graph $\mathbb{D} \in \mathcal{D}$, the flocking matrix

$$F = D^{-1}A' \tag{6}$$

where A' is the transpose of the adjacency matrix of $\mathbb{D}$ and D the diagonal matrix whose ijth diagonal element is the in-degree of vertex v_{ij} within the graph. Any $n \times n$ stochastic matrix S determines a directed graph $\gamma(S)$ with vertex set $\{1, 2, \ldots, n\}$ and arc set defined is such a way so that (i, j) is an arc of $\gamma(S)$ from i to j just in case the jith entry of S is non-zero. It is known [14] that for a set of $n \times n$ stochastic matrices $S_1, S_2, \ldots, S_p$

$$\gamma(S_p \cdots S_2 S_1) = \gamma(S_p) \circ \cdots \circ \gamma(S_2) \circ \gamma(S_1) \tag{7}$$

One can check that $\gamma(F) = \mathbb{D}$ and

$$\theta(t+1) = F(t)\theta(t), \quad t \in \{0, 1, 2, \ldots\} \tag{8}$$

Let $\bar{\mathcal{F}}$ denote the set of all such F. As before our goal is to characterize the sequences of neighbor graphs $\mathbb{N}(0), \mathbb{N}(1), \ldots$ for which all entries of $\theta(t)$ converge to a common steady state value.

There are a number of similarities and a number of differences between the situation under consideration here and the delay-free situation considered in [14]. For example, the notion of graph composition defined earlier can be defined in the obvious way for graphs in $\bar{\mathcal{G}}$. On the other hand, unlike the situation in the delay-free case, the set of graphs used to model the system under consideration, namely the set of delay graphs $\mathcal{D}$, is not closed under composition except in the special case when all of the delays are at most 1; i.e., when all of the $m_i \leq 2$. In order to characterize the smallest subset of $\bar{\mathcal{G}}$ containing $\mathcal{D}$ which is closed under composition, we will need several new concepts.

IV. HIERARCHICAL GRAPHS

As before, let $\mathcal{G}$ be the set of all directed graphs with vertex set $\mathcal{V} = \{1, 2, \ldots n\}$. Let us agree to say that a rooted graph $\mathbb{G} \in \mathcal{G}$ is a *hierarchical graph* with *hierarchy* $\{v_1, v_2, \ldots, v_n\}$ if it is possible to re-label the vertices in $\mathcal{V}$ as $v_1, v_2, \ldots v_n$ in such a way so that v_1 is a root of $\mathbb{G}$ with a self-arc and for $i > 1$, v_i has a neighbor v_j "lower " in the hierarchy where by *lower* we mean $j < i$. It is clear that any graph in $\mathcal{G}$ with a root possessing a self-arc is hierarchical. Note that a graph may have more than one hierarchy and two graphs with the same hierarchy need not be equal. Note also that even though rooted graphs with the same hierarchy share a common root, examples show that the composition of hierarchical graphs in $\mathcal{G}$ need not be hierarchical or even rooted. On the other hand the composition of two rooted graphs in $\mathcal{G}$ with the same hierarchy is always a graph with the same hierarchy. To understand why this is so, consider two graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ in $\mathcal{G}$ with the same hierarchy $\{v_1, v_2, \ldots, v_n\}$. Note first that v_1 has a self -arc in $\mathbb{G}_2 \circ \mathbb{G}_1$ because v_1 has self arcs in $\mathbb{G}_1$ and $\mathbb{G}_2$. Next pick any vertex v_i in $\mathcal{V}$ other than v_1. By definition, there must exist vertex v_j lower in the hierarchy than v_i such that (v_j, v_i) is an arc of $\mathbb{G}_2$. If $v_j = v_1$, then (v_1, v_i) is an arc in $\mathbb{G}_2 \circ \mathbb{G}_1$ because v_1 has a self-arc in $\mathbb{G}_1$. On the other hand, if $v_j \neq v_1$, then there must exist a vertex v_k lower in the hierarchy than v_j such that (v_k, v_j) is an arc of $\mathbb{G}_1$. It follows from the definition of composition that in this case (v_k, v_i) is an arc in $\mathbb{G}_2 \circ \mathbb{G}_1$. Thus v_i has a neighbor in $\mathbb{G}_2 \circ \mathbb{G}_1$ which is lower in the hierarchy than v_i. Since this is true for all v_i, $\mathbb{G}_2 \circ \mathbb{G}_1$ must have the same hierarchy as $\mathbb{G}_1$ and $\mathbb{G}_2$. This proves the claim that composition of two rooted graphs with the same hierarchy is a graph with the same hierarchy.

Our objective is to show that the composition of a sufficiently large number of graphs in $\mathcal{G}$ with the same hierarchy is strongly rooted. Note that the fact that the composition of $(n-1)^2$ rooted graphs in $\mathcal{G}_{sa}$ is strongly rooted [14], cannot be used to reach this conclusion because the v_i in the graphs under consideration here do not all necessarily have self-arcs.

The following proposition says that the composition of a sufficiently large number of graphs in $\mathcal{G}$ with the same hierarchy is strongly rooted.

Proposition 1: Let $\mathbb{G}_1, \mathbb{G}_2, \dots \mathbb{G}_m$ denote a set of rooted graphs in $\mathcal{G}$ which all have the same hierarchy. If $m \geq n-1$ then $\mathbb{G}_m \circ \cdots \mathbb{G}_2 \circ \mathbb{G}_1$ is strongly rooted.
The proof of this proposition appears in [15].

V. The Closure of $\mathcal{D}$

We now return to the study of the graphs in $\mathcal{D}$. As before $\mathcal{D}$ is the subset of $\bar{\mathcal{G}}$ consisting of those graphs which (i) have self arcs at each vertex in $\mathcal{V} = \{v_{11}, v_{21}, \dots, v_{n1}\}$, (ii) for each $i \in \{1, 2, \dots, n\}$, have an arc from each vertex $v_{ij} \in \mathcal{V}_i$ except the last, to its successor $v_{i(j+1)} \in \mathcal{V}_i$, and (iii) for each $i \in \{1, 2, \dots, n\}$, each vertex v_{ij} with $j > 1$ has in-degree of exactly 1. It can easily be shown by example that $\mathcal{D}$ is not closed under composition. We deal with this problem as follows. First, let us agree to say that a vertex v in a graph $\mathbb{G} \in \bar{\mathcal{G}}$ is a *neighbor* of a subset of $\mathbb{G}$'s vertices $\mathcal{U}$, if v is a neighbor of at least one vertex in $\mathcal{U}$. Next, we say that a graph $\mathbb{G} \in \bar{\mathcal{G}}$ is an *extended delay graph* if for each $i \in \{1, 2, \dots, n\}$, (i) every neighbor of $\mathcal{V}_i$ which is not in $\mathcal{V}_i$ is a neighbor of v_{i1} and (ii) the subgraph of $\mathbb{G}$ induced by $\mathcal{V}_i$ has $\{v_{i1} \dots, v_{im_i}\}$ as a hierarchy. We write $\bar{\mathcal{D}}$ for the set of all extended delay graphs in $\bar{\mathcal{G}}$. It is easy to see that every delay graph is an extended delay graph. The converse however is not true. The set of extended delay graphs has the following property.

Proposition 2: $\bar{\mathcal{D}}$ is closed under composition.
In the light of this proposition it is natural to call $\bar{\mathcal{D}}$ the *closure* of $\mathcal{D}$. To prove the proposition, we will need the following fact.

Lemma 1: Let $\mathbb{G}_1, \mathbb{G}_2, \dots, \mathbb{G}_q$ be any sequence of $q > 1$ directed graphs with vertex set $\mathcal{V}$. For $i \in \{1, 2, \dots, q\}$, let $\bar{\mathbb{G}}_i$ be the subgraph of $\mathbb{G}_i$ induced by $\mathcal{U} \subset \mathcal{V}$. Then $\bar{\mathbb{G}}_q \circ \cdots \circ \bar{\mathbb{G}}_2 \circ \bar{\mathbb{G}}_1$ is contained in the subgraph of $\mathbb{G}_q \circ \cdots \circ \mathbb{G}_2 \circ \mathbb{G}_1$ induced by $\mathcal{U}$.
The proof of Lemma 1 appears in [15].

Proof of Proposition 2: Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two extended delay graphs in $\bar{\mathcal{D}}$. It will first be shown that for each $i \in \{1, 2, \dots, n\}$, every neighbor of $\mathcal{V}_i$ which is not in $\mathcal{V}_i$ is a neighbor of v_{i1} in $\mathbb{G}_2 \circ \mathbb{G}_1$. Fix $i \in \{1, 2, \dots, n\}$ and let v be a neighbor of $\mathcal{V}_i$ in $\mathbb{G}_2 \circ \mathbb{G}_1$ which is not in $\mathcal{V}_i$. Then $(v, k) \in \mathcal{A}(\mathbb{G}_2 \circ \mathbb{G}_1)$ for some $k \in \mathcal{V}_i$. Thus there is a $s \in \bar{\mathcal{V}}$ such that $(v, s) \in \mathcal{A}(\mathbb{G}_1)$ and $(s, k) \in \mathcal{A}(\mathbb{G}_2)$. If $s \notin \mathcal{V}_i$, then $(s, v_{i1}) \in \mathcal{A}(\mathbb{G}_2)$ because $\mathbb{G}_2$ is an extended delay graph. Thus in this case $(v, v_{i1}) \in \mathcal{A}(\mathbb{G}_2 \circ \mathbb{G}_1)$ because of the definition of composition. If, on the other hand, $s \in \mathcal{V}_i$, then $(v, v_{i1}) \in \mathcal{A}(\mathbb{G}_1)$ because $\mathbb{G}_1$ is an extended delay graph. Thus in this case $(v, v_{i1}) \in \mathcal{A}(\mathbb{G}_2 \circ \mathbb{G}_1)$ because v_{i1} has a self-arc in $\mathbb{G}_2$. This proves that every neighbor of $\mathcal{V}_i$ which is not in $\mathcal{V}_i$ is a neighbor of v_{i1} in $\mathbb{G}_2 \circ \mathbb{G}_1$. Since this must be true for each $i \in \{1, 2, \dots, n\}$, $\mathbb{G}_2 \circ \mathbb{G}_1$ has the first property defining extended delay graphs in $\bar{\mathcal{D}}$.

To establish the second property, we exploit the fact that the composition of two graphs with the same hierarchy is a graph with the same hierarchy. Thus for any integer $i \in \{1, 2, \dots, n\}$, the composition of the subgraphs of $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively induced by $\mathcal{V}_i$ must have the hierarchy $\{v_{i1}, v_{i2}, \dots, v_{im_i}\}$. But by Lemma 1, for any integer $i \in \{1, 2, \dots, n\}$, the composition of the subgraphs of $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively induced by $\mathcal{V}_i$, is contained in the subgraph of the composition of $\mathbb{G}_1$ and $\mathbb{G}_2$ induced by $\mathcal{V}_i$. This implies that for $i \in \{1, 2, \dots, n\}$, the subgraph of the composition of $\mathbb{G}_1$ and $\mathbb{G}_2$ induced by $\mathcal{V}_i$ has $\{v_{i1}, v_{i2}, \dots, v_{im_i}\}$ as a hierarchy. ■

Our main result regarding extended delay graphs is as follows.

Proposition 3: Let m be the largest integer in the set $\{m_1, m_2, \dots, m_n\}$. The composition of any set of at least $m(n-1)^2 + m - 1$ extended delay graphs will be strongly rooted if the quotient graph of each of the graphs in the composition is rooted.

To prove this proposition we will need several more concepts. Let us agree to say that a extended delay graph $\mathbb{G} \in \bar{\mathcal{D}}$ has *strongly rooted hierarchies* if for each $i \in \mathcal{V}$, the subgraph of $\mathbb{G}$ induced by $\mathcal{V}_i$ is strongly rooted. Proposition 1 states that a hierarchical graph on m_i vertices will be strongly rooted if it is the composition of at least $m_i - 1$ rooted graphs with the same hierarchy. This and Lemma 1 imply that the subgraph of the composition of at least $m_i - 1$ extended delay graphs induced by $\mathcal{V}_i$ will be strongly rooted. We are led to the following lemma.

Lemma 2: Any composition of at least $m - 1$ extended delay graphs in $\bar{\mathcal{D}}$ has strongly rooted hierarchies.

To proceed we will need one more type of graph which is uniquely determined by a given graph in $\bar{\mathcal{G}}$. By the *agent subgraph* of $\mathbb{G} \in \bar{\mathcal{G}}$ is meant the subgraph of $\mathbb{G}$ induced by $\mathcal{V}$. Note that while the quotient graph of $\mathbb{G}$ describes relations between distinct agent hierarchies, the agent subgraph of $\mathbb{G}$ only captures the relationships between the roots of the hierarchies. Note in addition that both the agent subgraph of $\mathbb{G}$ and the quotient graph of $\mathbb{G}$ are graphs in $\mathcal{G}_{sa}$ because all n vertices of $\mathbb{G}$ in $\mathcal{V}$ have self arcs.

Lemma 3: Let $\mathbb{G}_p$ and $\mathbb{G}_q$ be extended delay graphs in $\bar{\mathcal{D}}$. If $\mathbb{G}_p$ has a strongly rooted agent subgraph and $\mathbb{G}_q$ has strongly rooted hierarchies, then the composition $\mathbb{G}_q \circ \mathbb{G}_p$ is strongly rooted.

Lemma 4: The agent subgraph of any composition of at least $(n-1)^2$ extended delay graphs in $\bar{\mathcal{D}}$ will be strongly rooted if the agent subgraph of each of the graphs in the composition is rooted.

Lemma 5: Let $\mathbb{G}_p$ and $\mathbb{G}_q$ be extended delay graphs in $\bar{\mathcal{D}}$. If $\mathbb{G}_p$ has strongly rooted hierarchies and $\mathbb{G}_q$ has a rooted quotient graph, then the agent subgraph of the composition $\mathbb{G}_q \circ \mathbb{G}_p$ is rooted.

The proofs of lemmas 3, 4 and 5 appear in [15].

Proof of Proposition 3: Let $\mathbb{G}_1, \mathbb{G}_2, \dots \mathbb{G}_s$ be a sequence of at least $m(n-1)^2 + m - 1$ extended delay graphs with rooted quotient graphs. The graph $\mathbb{G}_s \circ \cdots \mathbb{G}_{(m(n-1)^2+1)}$ is composed of at least $m-1$ extended delay graphs. Therefore $\mathbb{G}_s \circ \cdots \mathbb{G}_{(m(n-1)^2+1)}$ must have strongly rooted hierarchies because of Lemma 2. In view of Lemma 3, to complete the proof it is enough to show that $\mathbb{G}_{m(n-1)^2} \circ \cdots \circ \mathbb{G}_1$ has a strongly rooted agent subgraph. But $\mathbb{G}_{m(n-1)^2} \circ \cdots \circ \mathbb{G}_1$ is the composition of $(n-1)^2$ graphs, each itself a composition of m extended delay graphs with rooted quotient graphs. In view of Lemma 4, to complete the proof it is enough to show that the agent subgraph of any composition of m extended delay graphs is rooted if each quotient graph of each extended delay graph in the composition is rooted. Let $\mathbb{H}_1, \mathbb{H}_2, \dots, \mathbb{H}_m$ be such a family of extended delay graphs. By assumption, $\mathbb{H}_m$ has a rooted quotient graph. In view of Lemma 5, the agent subgraph of $\mathbb{H}_m \circ \mathbb{H}_{m-1} \circ \cdots \circ \mathbb{H}_1$ will be rooted if $\mathbb{H}_{m-1} \circ \cdots \circ \mathbb{H}_1$ has strongly rooted hierarchies. But $\mathbb{H}_{m-1} \circ \cdots \circ \mathbb{H}_1$ has this property because of Lemma 2. ■

Finally we will need the following fact.

Proposition 4: Let $\mathbb{G}_1, \dots, \mathbb{G}_r$ be a sequence of extended delay graphs in $\bar{\mathcal{D}}$. If the composition $Q(\mathbb{G}_r) \circ \cdots \circ Q(\mathbb{G}_1)$ is rooted then so is the quotient graph $Q(\mathbb{G}_r \circ \cdots \circ \mathbb{G}_1)$.

This proposition is a direct consequence of the following lemma.

Lemma 6: Let $\mathbb{G}_p, \mathbb{G}_q$ be two extended delay graphs in $\bar{\mathcal{D}}$. For each arc (i, j) in the composition $Q(\mathbb{G}_q) \circ Q(\mathbb{G}_p)$, there is a path from i to j in the quotient graph $Q(\mathbb{G}_q \circ \mathbb{G}_p)$.

The proof of Lemma 6 appears in [15].

Proof of Proposition 4: To prove the proposition it is enough to show that if $Q(\mathbb{G}_r) \circ \cdots \circ Q(\mathbb{G}_1)$ contains a path from some $i \in \mathcal{V}$ to some $j \in \mathcal{V}$, then $Q(\mathbb{G}_r \circ \cdots \circ \mathbb{G}_1)$ also contains a path from i to j. As a first step towards this end, we claim that if $\mathbb{G}_p, \mathbb{G}_q$ are graphs in $\bar{\mathcal{D}}$ for which $Q(\mathbb{G}_q) \circ Q(\mathbb{G}_p)$ contains a path from u to v, for some $u, v \in \mathcal{V}$, then $Q(\mathbb{G}_q \circ \mathbb{G}_p)$ also contains a path from u to v. To prove that this is so, fix $u, v \in \mathcal{V}$ and $\mathbb{G}_p, \mathbb{G}_q \in \bar{\mathcal{D}}$ and suppose that $Q(\mathbb{G}_q) \circ Q(\mathbb{G}_p)$ contains a path from u to v. Then there must be a positive integer s and vertices $k_1, k_2, \dots, k_s$ ending at $k_s = v$, for which $(u, k_1),\ (k_1, k_2),\ \dots,\ (k_{s-1}, k_s)$ are arcs in $Q(\mathbb{G}_q) \circ Q(\mathbb{G}_p)$. In view of Lemma 6, there must be paths in $Q(\mathbb{G}_q \circ \mathbb{G}_p)$ from i to k_1, k_1 to k_2,..., and k_{s-1} to k_s. It follows that there must be a path in $Q(\mathbb{G}_q \circ \mathbb{G}_p)$ from i to j. Thus the claim is established.

It will now be shown by induction for each $s \in \{2, \dots, m\}$ that if $Q(\mathbb{G}_s) \circ \cdots \circ Q(\mathbb{G}_1)$ contains a path from i to some $j_s \in \mathcal{V}$, then $Q(\mathbb{G}_r \circ \cdots \circ \mathbb{G}_1)$ also contains a path from i to j_s. In view of the claim just proved above, the assertion is true if $s = 2$. Suppose the assertion is true for all $s \in \{2, 3, \dots, t\}$ where t is some integer in $\{2, \dots, r-1\}$. Suppose that $Q(\mathbb{G}_{t+1}) \circ \cdots \circ Q(\mathbb{G}_1)$ contains a path from i to j_{t+1}. Then there must be an integer k such that $Q(\mathbb{G}_t) \circ \cdots \circ Q(\mathbb{G}_1)$ contains a path from i to k and $Q(\mathbb{G}_{t+1})$ contains a path from k to j_{t+1}. In view of the inductive hypothesis, $Q(\mathbb{G}_t \circ \cdots \circ \mathbb{G}_1)$ contains a path from i to k. Therefore $Q(\mathbb{G}_{t+1}) \circ Q(\mathbb{G}_t \circ \cdots \circ \mathbb{G}_1)$ has a path from i to j_{t+1}. Hence the claim established at the beginning of this proof applies and it can be concluded that $Q(\mathbb{G}_{t+1} \circ \mathbb{G}_t \circ \cdots \circ \mathbb{G}_1)$ has a path from i to j_{t+1}. Therefore by induction the aforementioned assertion is true. ■

VI. Proof of Convergence

Our aim is to make use of the properties of extended delay graphs just derived to prove Theorem 2. We will also need the following result from [14].

Proposition 5: Let $\mathcal{S}_{sr}$ be any closed set of stochastic matrices which are all of the same size and whose graphs $\gamma(S),\ S \in \mathcal{S}_{sr}$ are all strongly rooted. As $j \to \infty$, any product $S_j \cdots S_1$ of matrices from $\mathcal{S}_{sr}$ converges exponentially fast to a matrix of the form $\mathbf{1}c$ at a rate no slower than λ, where c is a non-negative row vector depending on the sequence and λ is a non-negative constant less than 1 depending only on $\mathcal{S}_{sr}$.

Proof of Theorem 2: In view of (8), $\theta(t) = F(t-1) \cdots F(0)\theta(0)$. Thus to prove the theorem it suffices to prove that as $t \to \infty$ the matrix product $F(t) \cdots F(0)$ converges exponentially fast to a matrix of the form $\mathbf{1}c$.

By hypothesis, the sequence of neighbor graphs $\mathbb{N}(0), \mathbb{N}(1), \dots,$ is repeatedly jointly rooted by subsequences of length q. This means that each of the sequences $\mathbb{N}(kq), \dots, \mathbb{N}((k+1)q-1),\ k \geq 0$, is jointly rooted. Let $\mathbb{D}(t) = \gamma(F(t)),\ t \geq 0$. In view of (5), $\mathbb{N}(t) = Q(\mathbb{D}(t)),\ t \geq 0$. Thus each of the sequences $Q(\mathbb{D}(kq)), \dots, Q(\mathbb{D}((k+1)q-1)),\ k \geq 0$, is jointly rooted, so each composition $Q(\mathbb{D}((k+1)q-1)) \circ \cdots \circ Q(\mathbb{D}(kq))$ is a rooted graph. In view of Proposition 4, each graph $Q(\mathbb{D}((k+1)q-1) \circ \cdots \circ \mathbb{D}(kq)),\ k \geq 0$ is also rooted.

Set $p = (m(n-1)^2 + m - 1)q$ where m is the largest integer in the set $\{m_1, m_2, \dots, m_n\}$. In view of Proposition 3, each of the graphs $\mathbb{D}((k+1)p-1) \circ \cdots \circ \mathbb{D}(kp)),\ k \geq 0$ is strongly rooted. Let $\mathcal{F}(p)$ denote the set of all products of p matrices from $\bar{\mathcal{F}}$ which have the additional property that each such product has a strongly rooted graph. Then $\mathcal{F}(p)$ is finite and therefore compact, because $\bar{\mathcal{F}}$ is.

For $k \geq 0$, define

$$S(k) = F((k+1)p-1) \cdots F(kp) \tag{9}$$

In view of (7) and the fact that $\gamma(F(t)) = \mathbb{D}(t),\quad t \geq 0$, it must be true that $\gamma(S(k)) = \mathbb{D}((k+1)p-1) \circ \cdots \circ \mathbb{D}(kp),\ k \geq 0$. Thus each $S(k)$ has a strongly rooted graph. Moreover, each such $S(k)$ is the product of p matrices from $\bar{\mathcal{F}}$. Therefore $S(k) \in \mathcal{F}(p),\quad k \geq 0$. Therefore Proposition 5 applies with $\mathcal{S}_{sr} = \mathcal{F}(p)$ so it can be concluded that the matrix product $S(k) \cdots S(0)$ converges exponentially fast as $k \to \infty$ to a matrix of the form $\mathbf{1}c$ as $k \to \infty$.

In view of the definition of $S(k)$ it is clear that for any t, there is an integer $k(t)$ and a stochastic matrix $\widehat{S}(t)$

composed of the product of at most $p-1$ matrices from $\bar{\mathcal{F}}$ such that

$$F(t)\cdots F(1) = \widehat{S}(t)S(k(t))\cdots S(0)$$

Moreover $t \mapsto k(t)$ must be an unbounded, strictly increasing function; because of this the product $S(k(t))\cdots S(0)$ must converge exponentially fast as $t \to \infty$ to a limit of the form $\mathbf{1}c$. Since $\widehat{S}(t)\mathbf{1}c = \mathbf{1}c,\ t \geq 0$, the product $F(t)\cdots F(1)$ must also converge exponentially fast as $t \to \infty$ to the same limit $\mathbf{1}c$. ■

VII. Concluding remarks

A related topic that will be studied in the future is the effect on convergence of the rate of changes in delays. This is not an issue in our current setting since all agents are assumed to update their headings synchronously on the set of integer valued time instances. However, if all agents update their headings asynchronously or if a continuous-time model is adopted, then an extremely high rate of change in delays may lead to divergence.

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