Summary for MTNS 2006

Reaching a Consensus in the Face of Measurement Delays*

M. Cao A. S. Morse
Yale University

B. D. O. Anderson
Australian National University
and National ICT Australia Ltd.

Current interest in cooperative control of groups of mobile autonomous agents has led to the rapid increase in the application of graph theoretic ideas together with more familiar dynamical systems concepts to problems of analyzing and synthesizing a variety of desired group behaviors such as maintaining a formation, swarming, rendezvousing, or reaching a consensus. While this in-depth assault on group coordination using a combination of graph theory and system theory is in its early stages, it is likely to significantly expand in the years to come. One line of research which "graphically" illustrates the combined use of these concepts, is the recent theoretical work by a number of individuals [1], [2], [3], [4], [5], [6] which successfully explains the heading synchronization phenomenon observed in simulation by Vicsek [7], Reynolds [8] and others more than a decade ago. Vicsek and co-authors consider a simple discrete-time model consisting of n autonomous agents or particles all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading plus the current headings of its "neighbors." Agent i's *neighbors* at time t, are those agents which are either in or on a circle of pre-specified radius r_i centered at agent i's current position. In their paper, Vicsek *et al.* provide a variety of interesting simulation results which demonstrate that the nearest neighbor rule they are studying can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors can change with time. A theoretical explanation for this observed behavior has recently been given in [1]. The explanation exploits ideas from graph theory [9] and from the theory of non-homogeneous Markov chains [10], [11], [12]. With the benefit of hindsight it is now reasonably clear that it is more the graph theory than the Markov chains which will prove key as this line of research advances. An illustration of this is the recent extension of the findings of [1] which explain the behavior of Reynolds' full nonlinear "boid" system [6].

In the past few years many important papers have appeared [2], [3], [4], [5], [13] which expand the results obtained in [1] and extend the Vicsek model in many directions. For example, in a recent paper [5] a modified version of the Vicsek problem is considered in which integer valued delays occur in sensing the values of headings which are available to agents. The aim of this paper is to consider the same problem, but more from a graph theoretic point of view. This enables us to relax the conditions stated in [5] under which consensus is achieved. We will compare results at the end of this summary.

In the sequel we suppose that at each time $t \in \{0, 1, 2, \ldots\}$, the value of neighboring agent j's headings which agent i senses is $\theta_j(t - d_{ij}(t))$ where $d_{ij}(t)$ is a delay whose value at t is some integer between 0 and $m_i - 1$; here m_i is a pre-specified positive integer. While well established principles of feedback control would suggest that delays should be dealt with using dynamic compensation, as in [5] we will consider the situation in which the delayed value of agent j's heading sensed by agent i at time t is the value which will be used in the heading update law for agent i. Let $\mathcal{N}_i(t)$ and $n_i(t)$ denote the set of labels and the number of agent i's neighbors at time t respectively. Thus

$$\theta_i(t+1) = \frac{1}{n_i(t)} \left(\sum_{j \in \mathcal{N}_i(t)} \theta_j(t - d_{ij}(t)) \right) \tag{1}$$

where $d_{ij}(t) \in \{0, 1, \ldots, (m_j - 1)\}$ if $j \neq i$ and $d_{ij}(t) = 0$ if $i = j$.

*The work of Cao and Morse was supported in part, by grants from the U.S. Army Research Office and the U.S. National Science Foundation and by a gift from the Xerox Corporation. The work of Anderson was supported by National ICT Australia, which is funded by the Australian Government's Department of Communications, Information Technology and the Arts and Australian Research Council through the Backing Australia's Ability initiative and the ICT Center of Excellence Program.

It is possible to represent this agent system using a state space model similar to the models used in [1], [2], [3]. Towards this end we will use graphs to capture the structure of (1). At each time, a directed graph will be used where each vertex corresponds to one entry of the state vector needed when writing (1) as a linear state-variable system. This means that depending on the maximum delay with which an agent's heading appears in (1), a number of vertices may be associated with that agent. The set of vertices $\mathcal{V}_i$ associated with agent i will be denoted by $\mathcal{V}_i = \{v_{i1} \dots, v_{im_i}\}$, and the vertex set of the graph will be denoted by $\bar{\mathcal{V}} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_n$. An incoming arc to vertex v_{i1} from vertex v_{jl} appears just when the state vector entry corresponding to agent i at time $t+1$ depends in part on the state variable entry corresponding to j's heading at time $t-l+1$. Since both neighbor relationships and delays in measurements may change over time, we have to consider *all* the graphs that may arise during the evolution of (1). Let $\bar{\mathcal{D}}$ denote the set of all directed graphs with vertex set $\bar{\mathcal{V}}$. Let $\bar{\mathcal{Q}}$ be an index set parameterizing $\bar{\mathcal{D}}$, i.e., $\bar{\mathcal{D}} = \{\mathbb{G}_q : q \in \bar{\mathcal{Q}}\}$. We sometimes write i for $v_{i1},\ \ i \in \{1, 2, \dots, n\}$, and $\mathcal{V}$ for the subset of vertices $\{v_{11}, v_{21}, \dots, v_{n1}\}$.

To represent the fact that each agent can use its own current heading in its update formula (1) and to capture more precisely the delay structure, we will utilize a subset of the set $\bar{\mathcal{D}}$ comprising those graphs with three properties: (i) there is a self-arcs at each vertex in $\mathcal{V}$; (ii) for each $i \in \{1, 2, \dots, n\}$, there is an arc from each vertex $v_{ij} \in \mathcal{V}_i$ except the last, to its successor $v_{i(j+1)} \in \mathcal{V}_i$; and (iii) for each $i \in \{1, 2, \dots, n\}$, each vertex v_{ij} with $j > 1$ has in-degree of exactly 1. In the sequel we write $\mathcal{D}$ for the subset of all such graphs, and use the symbol $\mathcal{Q}$ to denote that subset of $\bar{\mathcal{Q}}$ for which $\mathcal{D} = \{\mathbb{G}_q : q \in \mathcal{Q}\}$. Note that each vertex of each graph in $\mathcal{D}$ has positive in-degree.

In addition to $\mathbb{G}_q \in \mathcal{D}$, we will make use of another graph called the "quotient" of $\mathbb{G}_q$ where by the *quotient graph* of any graph $\mathbb{G} \in \bar{\mathcal{D}}$, we mean that directed graph with vertex set $\mathcal{V}$ whose arc set consists of those arcs (i, j) for which $\mathbb{G}$ has an arc from some vertex in $\mathcal{V}_i$ to some vertex in $\mathcal{V}_j$. Note that the quotient of $\mathbb{G}_q$ models which headings are being used by each agent in updates at time t without describing the specific delayed headings actually being used.

The set of agent heading update rules defined by (1) can now be written in state form. Towards this end define $\theta(t)$ to be that $(m_1 + m_2 + \cdots + m_i)$ vector whose first m_1 elements are $\theta_1(t)$ to $\theta_1(t+1-m_1)$, whose next m_2 elements are $\theta_2(t)$ to $\theta_2(t+1-m_2)$ and so on. Order the vertices of $\bar{\mathcal{V}}$ as $v_{11}, \dots, v_{1m_1}, v_{21}, \dots, v_{2m_2}, \dots, v_{n1}, \dots, v_{nm_n}$ and with respect to this ordering define

$$F_q = D_q^{-1} A_q', \ \ q \in \mathcal{Q} \tag{2}$$

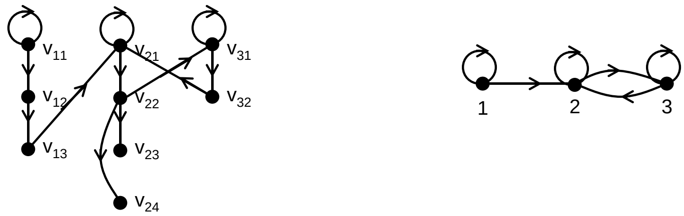
where A_q' is the transpose of the adjacency matrix of $\mathbb{G}_q \in \mathcal{D}$ and D_q the diagonal matrix whose ijth diagonal element is the in-degree of vertex v_{ij} within the graph. Then

$$\theta(t+1) = F_{\sigma(t)}\theta(t), \quad t \in \{0, 1, 2, \dots\} \tag{3}$$

where $\sigma : \{0, 1, \dots\} \to \mathcal{Q}$ is a switching signal whose value at time t is the index of the graph representing which headings the agents use at time t to update their own headings according to (1). Our goal is to characterize switching signals for which all entries of $\theta(t)$ converge to a common steady state value.

There are a number of differences between the situation under consideration here and the delay-free situation considered in [1], [2], [3]. For example, here we use directed graphs to represent neighbor relations whereas in [1] undirected graphs are used. Here we will "combine graphs" using the notion of "graph composition" rather than the notion of "graph union" used in [1], [2], [3]. By the *composition* of graph $\mathbb{G}_{q_1} \in \bar{\mathcal{D}}$ with $\mathbb{G}_{q_2} \in \bar{\mathcal{D}}$, written $\mathbb{G}_{q_2} \circ \mathbb{G}_{q_1}$, is meant the directed graph with vertex set $\bar{\mathcal{V}}$ and arc set defined in such a way so that (u, v) is an arc of the composition just in case there is a vertex w such that (u, w) is an arc of $\mathbb{G}_{q_1}$ and (w, v) is an arc of $\mathbb{G}_{q_2}$. Simple examples show that the set of graphs used to model the system under consideration, namely $\mathcal{D}$, is not closed under composition except in the special case when all of the delays are at most 1; i.e., when all of the $m_i \leq 2$. In order to characterize the smallest subset of $\bar{\mathcal{D}}$ containing $\mathcal{D}$ which is closed under composition and to state our main result, we will need several new concepts.

A vertex v of a directed graph $\mathbb{G}$ is called a *root* if for each other vertex in the graph, there is a path from v to u. We say that $\mathbb{G}$ is *rooted* if it has least one root. A rooted graph $\mathbb{G}$ is a *hierarchical graph* with *hierarchy* $\{u_1, u_2, \dots, u_k\}$ if it is possible to re-label its vertices as $u_1, u_2, \dots u_k$ in such a way so that u_1 is a root of $\mathbb{G}$ with a self-arc and for $i > 1$, u_i has a neighbor u_j "lower" in the hierarchy where by *lower* we mean $j < i$. Thus, a hierarchical graph is similar to a *topological sort* [14] except that the root of a hierarchical graph must have a self-arc, whereas the root of a topological sort cannot. A graph $\mathbb{G} \in \bar{\mathcal{D}}$ is said to be a *delay graph* if for each $i \in \{1, 2, \dots, n\}$, (i) every neighbor of $\mathcal{V}_i$ (i.e., every vertex with an outgoing arc terminating at a vertex of $\mathcal{V}_i$) which is not in $\mathcal{V}_i$ is a neighbor of v_{i1} and (ii) the subgraph of $\mathbb{G}$ induced by $\mathcal{V}_i$ has $\{v_{i1} \dots, v_{im_i}\}$ as a hierarchy. The following is an example of a delay graph and its quotient graph; note that the delay graph in question is not in the set $\mathcal{D}$, but only the set $\bar{\mathcal{D}}$.



Example: A delay graph (left) and its quotient graph (right)

It is easy to see that every graph in $\mathcal{D}$ is a delay graph. What's more, the set of all delay graphs can be shown to be closed under composition.

Let us agree to say that a finite sequence of graphs $\mathbb{G}_{q_1}, \mathbb{G}_{q_2}, \ldots, \mathbb{G}_{q_k}$ in $\mathcal{D}$ is *jointly quotient rooted* if the quotient of the composition $\mathbb{G}_{q_k} \circ \mathbb{G}_{q_{(k-1)}} \circ \cdots \circ \mathbb{G}_{q_1}$ is rooted. We say that an infinite sequence of graphs $\mathbb{G}_{q_1}, \mathbb{G}_{q_2}, \ldots,$ in $\mathcal{D}$ is *repeatedly jointly quotient rooted* if there is a positive integer m for which each finite sequence $\mathbb{G}_{q_{m(k-1)+1}}, \ \ldots, \mathbb{G}_{q_{mk}}, \quad k \geq 1$ is jointly quotient rooted. Our main result on leaderless coordination with measurement delays is as follows.

Theorem 1: Let $\theta(0)$ be fixed and with respect to (3), let $\sigma : [0, 1, 2, \ldots) \rightarrow \bar{\mathcal{Q}}$ be a switching signal for which the infinite sequence of graphs $\mathbb{G}_{\sigma(0)}, \mathbb{G}_{\sigma(1)}, \ldots$ in $\mathcal{D}$ is repeatedly jointly quotient rooted. Then there is a constant steady state heading θ_{ss}, depending only on $\theta(0)$ and σ, for which

$$\lim_{t \to \infty} \theta(t) = \theta_{ss}\mathbf{1} \tag{4}$$

where the limit is approached exponentially fast.

It is possible to compare the hypotheses of Theorem 1 with the corresponding hypotheses for exponential convergence stated in [5], namely assumptions 2 and 3. To do this, let us agree to say that the *union* of a set of directed graphs $\mathbb{G}_{r_1}, \mathbb{G}_{r_2}, \ldots, \mathbb{G}_{r_k}$ with vertex set $\mathcal{V}$ is that graph with vertex set $\mathcal{V}$ and arc set consisting of the union of the arcs of all of the graphs $\mathbb{G}_{r_1}, \mathbb{G}_{r_2}, \ldots, \mathbb{G}_{r_k}$. Let $q(\mathbb{G})$ denote the quotient of $\mathbb{G} \in \bar{\mathcal{D}}$. Taken together, assumptions 2 and 3 of [5] are more or less equivalent to assuming that there are finite positive integers m and s such that the graph

$$\mathbb{G}(k) \stackrel{\Delta}{=} q(\mathbb{G}_{\sigma(km+m)}) \cup q(\mathbb{G}_{\sigma(km+m-1)}) \cup \cdots \cup q(\mathbb{G}_{\sigma(km+1)})$$

is strongly connected and independent of k for $k \geq s$. By way of comparison, the hypothesis of Theorem 1 is equivalent to assuming that there is finite positive integer m such that

$$\bar{\mathbb{G}}(k) \stackrel{\Delta}{=} q(\mathbb{G}_{\sigma(km+m)} \circ \mathbb{G}_{\sigma(km+m-1)} \circ \cdots \circ \mathbb{G}_{\sigma(km+1)})$$

is rooted for $k \geq 0$. The latter assumption is weaker than the former for several reasons. First, the arc set of $\mathbb{G}(k)$ is always a subset of the arc set of $\bar{\mathbb{G}}(k)$ and in some cases the containment may be strict. Second, $\bar{\mathbb{G}}(k)$ is not assumed to be independent of k, even for k sufficiently large, whereas $\mathbb{G}(k)$ is. Third, each $\mathbb{G}(k)$ is assumed to be strongly connected whereas each $\bar{\mathbb{G}}(k)$ need only be rooted; note that a strongly connected graph is a special type of rooted graph in which every vertex is a root.

Perhaps most important about Theorem 1 and the development which justifies it, is that the underlying structural properties of the graphs involved required for consensus are explicitly determined. These properties and a proof of Theorem 1 will appear elsewhere at a later date.

A related topic that will be studied in the future is the effect on convergence of rate of changes in delays. This is not an issue in our current setting since all agents are assumed to update their headings synchronously on the set of integer valued time instances. However, if all agents update their headings asynchronously or if a continuous-time model is adopted, then an extremely high rate of change in delays may lead to divergence.

References

[1] A. Jababaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48:988–1001, 2003. Also in Proc. 2002 IEEE CDC, pages 2953 - 2958.

[2] L. Moreau. Stability of multi-agent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50:169–182, 2005.

[3] W. Ren and R. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50:655–661, 2005.

[4] D. Angeli and P. A. Bliman. Extension of a result by Moreau on stability of leaderless multi-agent systems. In *Proc. of the 44th IEEE CDC*, pages 759–764, 2005.
[5] V. D. Blondel, J. M. Hendrichx, A. Olshevsky, and J. N. Tsitsiklis. Convergence in multiagent coordination, consensus, and flocking. In *Proc. of the 44th IEEE CDC*, pages 2996–3000, 2005.
[6] H. Tanner, A. Jadbabaie, and G. Pappas. Distributed coordination strategies for groups of mobile autonomous agents. Technical report, ESE Department, University of Pennsylvania, December 2002.
[7] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, pages 1226–1229, 1995.
[8] C. Reynolds. Flocks, birds, and schools: a distributed behavioral model. *Computer Graphics*, 21:25–34, 1987.
[9] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer Graduate Texts in Mathematics # 207, New York, 2001.
[10] E. Seneta. *Non-negative Matrices and Markov Chains*. Springer-Verlag, New York, 1981.
[11] J. Wolfowitz. Products of indecomposable, aperiodic, stochastic matrices. *Proceedings of the American Mathematical Society*, 15:733–736, 1963.
[12] D. J. Hartfiel. *Markov set-chains*. Springer, Berlin;New York, 1998.
[13] Z. Lin, B. Francis, and M. Brouche. Local control strategies for groups of mobile autonomous agents. *IEEE Transactions on Automatic Control*, 49:622–629, 2004.
[14] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. *Introduction to Algorithms*, pages 549–552. The MIT Press, 2nd edition, 2001.