Proceedings of the 17th International Symposium on Mathematical
Theory of Networks and Systems, Kyoto, Japan, July 24-28, 2006

Use of Meta-Formations for Cooperative Control

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***Abstract*— This paper reviews a number of very recent results in rigid graph theory and their extension for directed graphs to persistence theory, with an application focus on the cooperative control of formations. Particular attention is paid to issues related to the merging of formations, where the internal structure of each of the individually merging formations is, to the maximum extent possible, downplayed in the calculation. The meta-formation framework is then introduced in light of merging process for its construction. The ideas also have application to sensor network localization, where there is potential to make great computational saving.**

***Keywords*— Graph Theory, Multi-agent Formation Control**

I. The problem domain

This paper is a contribution to the development of a general theory for understanding rigidity of formations: a formation of agents such as autonomous airborne vehicles, underwater autonomous vehicles or robots, moving in a two-dimensional or three-dimensional space, can be termed a rigid formation when, while the overall formation moves, the distance between every pair of agents remains constant. This can be secured by certain sensing, communication and control architectures; these prescribe that certain quantities are measured, and certain quantities are maintained constant. For example, if enough of the inter-agent distances are measured and controlled to be constant, intuitively one can understand that all inter-agent distances in the formation will be maintained. A secondary area of application of these ideas is to sensor networks, in particular the problem of sensor network localization, which is one of determining the sensor locations (Euclidean coordinates) of all sensors in a network from the given locations of a subset of the sensors and a set of inter-sensor distance measurements.

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The main thrust of this paper is to study properties of interconnections and other operations on formations, where it becomes possible to aggregate many of the agents in viewing the interconnection. To understand this further, we first devote almost the first half of the paper to a review of work on formations where their properties are reviewed in terms of the individual agents and their associated distance constraints.

II. Formations and undirected rigid graphs

Rigid graph theory is a tool which has been used to analyse the property of formation rigidity, see [1], [2], [7], [15], [17], [21]. Agents are modelled as points. Agent pairs for which the inter-agent distance is actively constrained to be constant can be joint by bars depicting the inter-agent distance constraints. The system can be therefore modelled by a graph where vertices represents point-like agents and inter-agent distance constraints are abstracted as edges. (Naturally, one can contemplate other constraints than distance, e.g. those involving angle, or angle and distance; however the theory begins with distance constraints, and we will restrict discussion to this case). Rigid graph theory is concerned with stating properties of graphs which ensure that the formation being modelled by the graph will be rigid; formal definitions are available of course, but roughly speaking, a rigid formation is one in which the only smooth motions are those corresponding to translation or rotation of the whole formation.

Two key tools of analysis appear. The first is a tool of linear algebra. Given knowledge of the positions of the agents at any one time, one can construct a matrix, the so-called *rigidity matrix*, and the dimensions and rank of this matrix allow one to conclude that the formation is or is not rigid. The dimensions and rank are the same for almost all positions of the agents. This means that rigidity matrices formed from two formations differing from each other only in terms of the values for the constrained distances will have the same rank, except for very special sets of the agent positions. It proves possible in two dimensions to also characterize rigidity in purely combinatorial terms, i.e. counting-type conditions related to the graph (discarding therefore the agent coordinates) can be used to conclude the rigidity or otherwise of a generic formation corresponding to the graph. This is the celebrated Laman's Theorem [16], for which no three-dimensional equivalent exists. In three dimensions, differing necessity and sufficiency conditions are known for a graph to correspond to a formation which will be rigid for generic values of the constrained inter-agent distances [23].

One major result concerns the *construction* of rigid formations. As for the characterisation of rigidity, the theory for two-dimensional formations is better developed than that for three-dimensional formations. Before describing the result, we flag the concept of *minimal rigidity*. A formation is minimally rigid if it is rigid and if no single inter-agent distance constraint can be removed without losing rigidity. A graph is minimally rigid if almost all formations to which the graph corresponds are minimally rigid. Minimal rigidity is easily described in two and three dimensions with the rigidity matrix, characterisable in two dimensions with Laman's Theorem, and the subject of some necessary conditions in three dimensions on the graph determined by a formation. Necessary conditions in two and three dimensions are that $|E| = 2|V| - 3$ and $|E| = 3|V| - 6$ respectively, where $|E|$ and $|V|$ are the numbers of edges and vertices of the graph.

The two-dimensional result is this. Suppose a minimally rigid graph corresponding to a two-dimensional formation exists. Then there are two operations on rigid formation construction known as the vertex addition operation and the edge-splitting operation which can be used to build another minimally rigid graph *with one more vertex*. Obviously, bigger and bigger minimally rigid graphs can be built this way, with the process being known as *Henneberg sequence* construction [5], [23]. What is important are the following two additional properties:

- All two-dimensional minimally rigid graphs with any number of vertices are constructible from a primitive comprising a two-vertex single-edge graph by an appropriate sequence of these operations.
- Any two-dimensional minimally rigid graph can be "deconstructed" by the inverse operations, to yield a sequence of minimally rigid graphs each with one less vertex than its predecessor in the sequence, and terminating with a two-vertex, single-edge graph.

In three dimensions, the results are not so complete and it is still a matter of conjecture that a certain set of operations is necessary and sufficient to build and "deconstruct" all minimally rigid graphs.

Before introducing the works reflecting the title of the paper, we need to digress to introduce another important graph theoretic concept, *global rigidity*. Consider a formation in which agents are labelled, and certain inter-agent distances are prescribed but the Euclidean positions of the agents are not known. One can then ask: what Euclidean positions of the agents would correspond to the data? Obviously translations and rotations must be allowed. Almost as obviously, reflections must be allowed. Thus if the agents were so located as to correspond to the data, and if then the signs of all the coordinates of every agent were changed and new agents placed at the new positions, another formation corresponding to the distance data would result, and in general it would not be obtainable from the first by translation and rotation.

A two-dimensional formation (and by extension its graph) is termed *globally rigid* if and only if any two formations corresponding to the distance data differ by at most translation, rotation and reflection. It is perhaps not immediately obvious, but global rigidity is a more demanding concept than rigidity, i.e. there exist rigid formations which are not globally rigid, and such formations can only be converted to globally rigid formations by the addition of more distance constraints. Figure 1 gives two examples of two dimensional formations which are rigid but not globally rigid, and yet correspond to the same set of distance constraints.

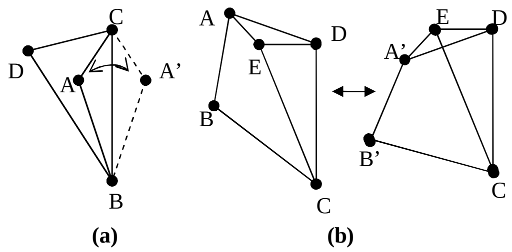

Fig. 1. Illustration of (a) Flip ambiguity: Vertex A can be flipped over the edge (B,C) to a symmetric position A' and the distances remain constraints the same. (b) Discontinuous flex ambiguity: Temporarily removing the edge (A,D), the edge triple (A,E), (A,B), (B,C) can be flexed to obtain positions A' and B', such that the edge length (A',D) equals the edge length (A,D), therefore all the distance constraints are the same.

Henneberg sequence construction is also possible for two-dimensional globally rigid graphs. Two different sets of operations have been advanced, see [5] and [14]. Recent unpublished work of some of the authors has shown that both sets of operations lead to the same graphs, in fact all globally rigid graphs can be grown this way.

The definition of global rigidity for two-dimensional formations extends obviously to three dimensions. However, though global rigidity is a generic concept in two dimensions, and thus is a concept that can be associated with a graph, it is not known whether it is a generic concept in three dimensions, although certainly some three-dimensional graphs can legitimately be termed globally rigid.

Global rigidity is of interest in various application areas, including *sensor network localization* [6], [8], [18]. In sensor networks, there is given a set of points (like the agents in a formation but corresponding to sensors), and a set of distances between pairs of points (obtained by exchange of information between points in the network); the distances are typically available for pairs of points which are within a Euclidean distance of one another that is less than some threshold. The associated graph is termed a *unit disk graph*. The sensor network localization problem is to pass from the distance set to a set of Euclidean coordinates for the sensors consistent with the distance set. In the absence of further information, the Euclidean coordinates are only specified up to translation, rotation or reflection. That further information is normally obtained from so-called anchor nodes or sensors, the position of which are known absolutely.

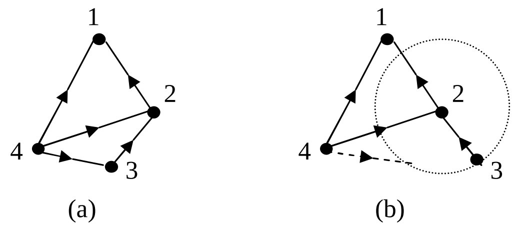

Fig. 2. Illustration of a 4-agent directed formation that is not constraint consistent

III. FORMATIONS AND DIRECTED GRAPHS

The discussion to this point about formation information and control architectures to secure rigidity has been sketchy. Let us observe now that the task of maintaining a prescribed distance between a nominated agent pair requires control action, and one can conceive that the execution of the task could be the "responsibility" of both agents, or one nominated agent of the pair. Modelling using undirected graphs is appropriate in the former case. However in the latter case, it is important to recognise the distinction by assigning a direction to all edges in the graph. A directed edge from vertex u to vertex v appears when agent u has the task of maintaining its distance constant from agent v, and agent v is unconstrained in its own motions with respect to the motion of u, i.e. it is "unconscious" of the task that agent u has to execute.

There is still interest in the basic question: what conditions ensure that the motions of a formation are restricted to translation or rotation? This question is examined in [9] for two dimensions, and in [12], [27], [28] for three dimensions. We will describe the two-dimensional result first. A notion termed *persistence* is introduced, which is an amalgam of two conditions, rigidity (as before) and a notion termed *constraint consistence*. The rigidity property says that *if* certain inter-agent distances are maintained, then all inter-agent distances are maintained. The new property, *constraint consistence*, is equivalent to the requirement that *it is possible* to maintain the nominated inter-agent distances. To illustrate this further, consider Fig. 2 above.

Suppose agents 1 and 2 are fixed, with agent 2 at its correct distance from 1. Suppose also that agent 3 is at its correct distance from agent 2 and agent 4 at its correct distance from 1, 2 and 3. Now observe that agent 3 has only one distance constraint, thus it can move, while maintaining its distance from 2, on a circle centered at agent 2. It is unconscious of the constraint which 4 is supposed to maintain on the distance between agents 3 and 4. When agent 3 moves, agent 4 then has an impossible task. There are only two possible positions where agent 4 can be in order to maintain its correct distances from agents 1 and 2; for generic allowable positions of agent 3, agent 4 will not be able to maintain the correct distance from agent 3 from either of these two positions. We describe such an arrangement as being *not constraint consistent*. Evidently, too much is being asked of one agent.

Constraint consistent formations are those where no agent is given potentially impossible constraints, in the manner of agent 4 in Fig. 2. The notion of constraint consistence can be applied and described with directed graphs. Formal definitions of constraint consistence and persistence can be found in [9], [28]. Let us simply note the following key facts;

- Any two dimensional graph which has no more than two outgoing edges from any vertex is constraint consistent (though there are constraint consistent graphs where some vertices have out-degree greater than two).
- A graph can be checked for persistence (i.e. rigidity plus constraint consistence) by testing a certain collection of subgraphs (in which the edge directions are neglected) for rigidity.

In the light of the above remarks, an important question presents itself. Suppose that a two-dimensional undirected graph is rigid. Can one assign edge directions so that it is constraint consistent and thus persistent? At the time of writing, the question in its full generality remains open. However, affirmative answers exist for minimally rigid graphs [9], and graphs with certain structures, including wheel graphs, trilateration graphs, complete graphs and power graphs of circle graphs [24].

Much the same ideas arise in three dimensions; one can certainly define constraint consistence and persistence. However, there is a subtle twist. In effect, one needs the equivalent of constraint consistence for all subsets of vertices, as opposed to just each individual vertex considered one at a time. For three and indeed higher dimensions, a concept termed structural persistence is required [27], [28], and in three dimensions, it is very easy to check structural persistence given persistence. Here are some salient points.

- Any three dimensional graph which has no more than three outgoing edges from any vertex is constraint consistent.
- A graph can be checked for persistence, (i.e. rigidity plus constraint consistence) by testing a certain collection of subgraphs (in which the edge directions are neglected) for rigidity.

Next, if a directed graph is persistent,

- The graph can be checked for *structural persistence*, which is now the necessary and sufficient condition to be able to provide enough interagent distance controls to ensure the formation behaves as a cohesive whole; for example, structural persistence of a three-dimensional formation can be verified by checking its persistence *and* verifying there is at most one vertex of the graph with no outwardly directed edges.
- A three-dimensional persistent graph is always structurally persistent if it is cycle-free.
- It is provable that all persistent graphs in R^2 and R^1 are also structural persistent.
- A generalized check for structural persistence can be

executed based on the following theorem: a persistent graph in $\Re^d$ ($d \geq 1$) is structurally persistent if and only if every one of its closed subgraphs [1] with less than d vertices is persistent.

Given the existence of Henneberg sequence theory for undirected graphs, it is logical to ask whether it can be applied to directed graphs. The topic is treated in [10] in this conference and in a wider context in [11] by the authors. The broad conclusion is that it can be applied, so long as the primitive operations are modified to allow directed edges in the graphs and also a further primitive operation is introduced. There is more than one possible operation, but the simplest possible operation is edge-reversal, i.e., reversing the direction of one edge arriving at a vertex with a degree of freedom (DOF)[2].

IV. Operations with formations

In [4], several operations involving formations were introduced. In particular, the concepts of merging, splitting and closing ranks were defined, for formations which were modelled using undirected graphs. We give more details:

- *Merging*. Consider two rigid formations. How can additional distance constraints be determined, with one agent in each formation, such that the union of the agents of the two formations, and the union of the distance constraints in the original formations and the new distance constraints, will describe a single rigid formation? Of course, the problem can be expressed using graph theoretic terms as well. Additionally, one can consider variations assuming the starting formations are minimally rigid, one can consider two and three dimensional versions, and one can consider directed graph versions. One can also consider questions of algorithm complexity, and the possibility of posing computational constraints on individual agents if there is a wish to perform calculations on a decentralised basis.
- *Splitting*. Consider a single rigid formation. A splitting literately means that its agents are divided into two subsets, and that distance constraints between agents in the different subsets are suppressed. In graph theory terms, after this step, there are two separate (sub-)graphs, neither of which may be rigid. How can one introduce additional distance constraints in the separate subformations to ensure rigidity of them both? Again, the problem can be cast in graph theory language, and the same variations apply as for merging.
- *Closing ranks*. Consider a single rigid formation. Suppose that one agent is removed, and, consequentially, any distance constraints that applied between this agent and the remaining agents of the formation. Where should new distance constraints be inserted, in order that the formation can be re-rendered rigid? The same extending remarks apply here also. In addition, the closing ranks problem can be generalised to contemplate formations in which *more than one* agent is simultaneously removed, with the associated distance constraints.

[1] $G' = (V', E')$ is a closed subgraph of G if there is no directed path in G starting from V' and containing either a vertex or an edge that does not belong to G'.

[2] In $\Re^2$, a vertex has two, one or zero degree(s) of freedom if it has no, one, or at least two outgoing edges; each outgoing edge uses up one DOF. A minor variation applies in three dimensions.

The initial way to solve these problems revolved round finding a significant modification of the Henneberg sequence concept. In [3], a so-called *minimal cover* problem was introduced and solved: in the minimal cover problem, a graph is presented which is not minimally rigid. One is required to determine a minimal set of edges (minimality being in the sense of the actual number) which when added to the graph will render it generically rigid. The solution of the minimal cover problem can be applied to solve each of the problems of formatting merging, splitting and closing ranks. Additionally, it has been observed that:

- The splitting problem is actually a particular case of the closing ranks problem. One subformation can regard the agents of the other subformation as the lost agents.
- The closing ranks problem (in graph theoretic language) can always be solved by introducing new edges between former neighbours of the lost vertices of the graph, i.e. by performing a local repair. In connection with the splitting problem, this means that any new edges can be restricted to connecting pairs of those vertices in one subformation graph that were previously neighbours of vertices which ended up in the other subformation graph.

The above formation operations can also be contemplated for directed graphs. However, little work has so far been done.

V. Meta-vertices, bodies and meta-formations

In merging two formations, it is obvious that much internal structure is largely irrelevant. As it turns out for example, if two rigid formations are to be merged in two dimensions, this can always be done by introducing three distance constraints with one agent in each of the two formations, and ensuring that in each of the two formations, at least two agents are involved in the distance constraints [25]. There is obviously some kind of a general rule that is operative here. Our interest is in establishing the general rules concerning the connection of formations to form larger formations, particularly ensuring preservation of rigidity; with the view however that the internal connections of the individual formations are unimportant, we shall term the larger formation a meta-formation [26]. In this connection, we shall first note two streams of work.

A. Rigidity and two-dimensional formations of formations

The papers [19], [22] investigate what are sometimes termed body-bar systems. A body is like a generalization of a point agent. Any rigid formation of agents can be replaced by a body, a rigid object that in two dimensions has three degrees of freedom, two displacements and one rotation. (In contrast, a point agent in two dimensions has two degrees of freedom, both translational). Each body can be deemed to have a set of connection points on its surface, with the property that distances can be constrained between two connection points in different bodies. One can imagine a formation comprising a set of bodies, which might also be termed meta-vertices or meta-agents, with certain distance constraints between them (usually more than one connection point on the surface of a body is used; for if only one connection point were used, the body or the meta-vertex could rotate about it). The term meta-vertex is however probably best restricted to applying to the graph equivalent of a body. One can also pose the question: when will such a formation be rigid? Of course, it is desired to answer this question taking no account of the internal structure of the bodies.

The question was answered for meta-formations of bodies in [19], [22], using both a generalization of the rigidity matrix, and a generalization of Laman's Theorem for the two-dimensional case. Recall that Laman's Theorem provides necessary and sufficient conditions for generic rigidity of a graph corresponding to a formation of point agents, and the conditions are of a "counting" form; a simple adjustment of certain numbers appearing in the statement of Laman's Theorem converts it to a theorem concerning generic rigidity of a graph corresponding to a two-dimensional bar-body framework. As for checking rigidity of a normal graph in $\Re^3$, the available counting condition for three-dimensional body-bar framework are necessary for it to be rigid but not sufficient. The rigidity matrix ideas work in three dimensions (where the bodies are three dimensional and thus have six degrees of freedom, three translational ones and three rotational ones) [22].

Interconnection of two formations is a matter of interconnection of two bodies, and the Laman's Theorem extension easily provides the result that three distance constraints between connection points on each of the two bodies, with at least two connection points involved for each body, serves to give rigidity of the overall formation. This idea could be extended in that of merging more than two formations (meta-vertices) and agents (vertices) [26]. This type of result of course exists for directed graphs, some results are available in [13].

B. More on formation merging

A recent paper [25] by some of the authors considers the problem described above of connecting two formations in two dimensions as well as other problems:

- Connecting (via insertion of additional edges) two formations in three dimensions to secure minimal rigidity.
- Connecting two formations in two or three dimensions to secure global rigidity.
- The two formations being connected are not disjoint, i.e., they are permitted to have a limited number of common vertices and/or a limited number of common edges.

By appealing to various results on rigidity and global rigidity, a series of conditions are established to solve these problems. The conditions are generally of the form: make m connections, involve at least n vertices of one formation, and at least p vertices of the second formation. We give two examples, to make more concrete the form of the results.

- Consider two globally rigid two-dimensional graphs, with one vertex in common. Then by adding two new edges with one vertex in each formation, and such that in at least one of the formations the edges are incident on two different vertices, a globally rigid graph results. (Note that the two new edges added cannot be incident on the vertex common to the two initially given graphs)
- Consider two minimally rigid three dimensional graphs, with no vertices in common. Then after addition of six new edges, incident in each graph on at least three vertices, and with no more than two edges incident to a single vertex, there results a minimally rigid graph. A related result is that if the two three dimensional graphs are rigid but not necessarily minimally rigid, addition of six new edges using the same incidence rules will result in a rigid graph, but fewer than six edges cannot. We might regard such an interconnection as minimally rigid from the meta-formation point of view, since the issue of whether or not the individual meta-vertices (themselves formations) are minimally rigid (they must of course be rigid) is considered irrelevant.

Directed versions of these results have in part been obtained [13]. It still remains the case of counting that the new edges are adequate. Here are some examples of conclusions which can be established:

- In order to merge two minimally persistent graphs (in $\Re^3$) into a larger minimally persistent graph, one needs to add six directed interconnection edges that leave vertices with some degrees of freedom (one DOF for each out-going edge) but that can arrive at any vertices. Not every selection of interconnection edges leads to a persistent merged graph, but it is always possible to find a set of interconnection edges that makes the merged graph structurally persistent, even when the initial graphs are not structurally persistent.
- When the merged graph needs to be persistent and not necessarily minimally persistent, one still needs to add six directed edges leaving vertices with some degrees of freedom. Other edges (possibly leaving vertices without DOF) can under some conditions also be added, but they can always be avoided. As a consequence, at least six degrees of freedom must be available in the two initial graphs; otherwise the

two graphs cannot be merged.
- If one of the two initial graphs has no degree of freedom and if simultaneously the other one is not structurally persistent, then they cannot be merged into a persistent graph. However, in every other case, if six degrees of freedom are available, it is always possible to choose six directed interconnection edges to make the merged graph persistent and even structurally persistent.

We have further defined two types of persistent merging:

- (Meta or Macro) *Leader-Follower merging*, in which all newly added directed edges leave from a persistent formation G_1 (the "meta-follower") and arrives at G_2 (the "meta-leader"). These directed meta-edges therefore remove all DOFs of G_1 and the merged persistent formation retains the same DOF allocation pattern of G_2.
- *Collaborative merging*, in which each formation will have some newly added directed edges leaving one or more of their vertices with positive DOF, and arriving at the other formation. Evidently, both formations will lose some DOF, and as a result, the DOF allocation pattern of the post-merged persistent formation may be different from that of either G_1 or G_2.

Some results related to the above characterizations of merging persistent formations have been noted and some are given below:

- Collaborative merging in the three-dimensional case may produce a formation that is not structurally persistent, even if the two pre-merging formations are structurally persistent. This happens when both initial formations contain a leader, and when none of the edges added during the merging process leaves either of those two leaders.
- In the case of a leader-follower merging, where the pre-merging formations contain more than just a single agent, the formation obtained is always structurally persistent.

C. Towards a more systematic theory

Given that one can find a version of Laman's Theorem describing the rigidity of a meta-formation, obtained by connecting together meta-vertices or meta-agents, one can also ask: is there a concept of a Henneberg sequence for meta-formations? Such a sequence could start with a single meta-vertex, or rigid formation, and involve the successive addition of meta-agents to the meta-formation. Each addition would result in a meta-formation that had the minimal number of edges between meta-vertices so as to guarantee rigidity of the overall meta-formation. Indeed, that is the case. Analogs to vertex addition and edge splitting, termed *meta-vertex addition* and *meta-edge splitting* respectively, can be constructed, see for example Fig. 3. The process can also be described by building on the results described in the previous section, and was set out in detail in our recent work [26].

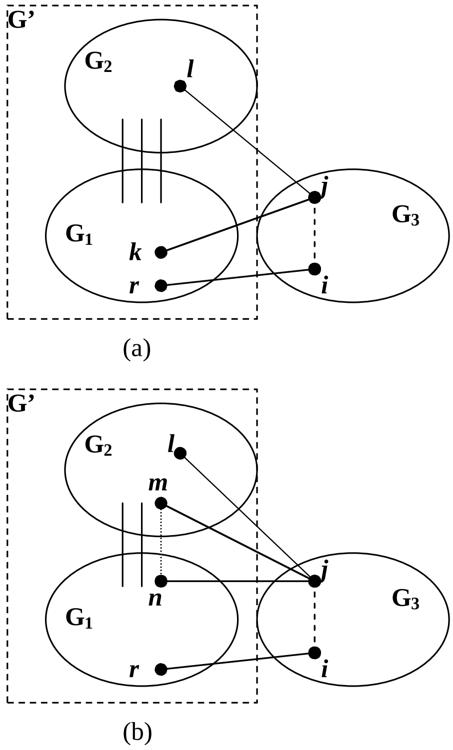

Fig. 3. Illustration of a scenario when one can perform meta-Henneberg operations: Meta-vertices G_1 and G_2 are merged into G' with three meta-edges, meta-vertex G_3 is merged using (a) meta-vertex addition and (b) meta-edge splitting.

VI. Meta-formations and sensor network localization

Meta-formation ideas have a potentially very important role to play in sensor network localization. To understand why this is so, we note that, though the sensor network localization problem for a sensor network with a globally rigid graph in its general form is NP-hard [20], there exist however particular sensor networks for which sensor network localization can be undertaken in linear time and even on a decentralized basis [6]. Such networks in two dimensions include those with so-called trilateration graphs. (A graph is termed a trilateration graph if there exists an ordering of the vertices, $v_1, v_2, \ldots v_N$ such that v_1, v_2, and v_3 are all neighbours and for all j greater than or equal to 3, v_j is a neighbour of at least three vertices in the subset $v_1, v_2, \ldots v_{j-1}$. Localization, apart from resolution of a translation, rotation or reflection of all vertices, is achievable by working through the sequence). The resolution of this remaining uncertainty is achieved by requiring that at least three of the sensors be anchor nodes, which are sensors equipped with an additional functionality that yields their precise position in Euclidean coordinates. It is not hard to see that if there are three or more anchor nodes, the uncertainty can be removed.

A trilateration graph is certainly globally rigid; in fact the graph can normally shed some edges and remain globally rigid. It is the higher connectivity of the graph that converts the normal exponential complexity of the localization problem to linear complexity one.

Now many sensor networks may be generated by a physical process which, to some extent, randomly locates the sensors. It follows that as one moves across the network, there may be parts which are more highly connected than others. In particular, there may be "islands" of the network for which the associated graph, a subgraph of the graph of the whole network, is a trilateration subgraph, while the graph of the whole network is not. Suppose also that there are edges in the network's graph which connect the trilateration subgraphs. Then one could conceive of the trilateration subgraphs as meta-vertices, (with a higher "degree" of rigidity than considered up to now), and the localization task for the whole network becomes one of localizing the meta-vertices relative to one another (which means localizing their notional center of mass and their orientations). This thinking may allow a reduction in the computational complexity of the sensor network localization problem, for a possibly widely occurring class of sensor networks.

VII. Current and Future work

Current or near-term work is looking at a number of problems which constitute obvious directions in which to build out the results reported above. These include filling out many details indicated in outline in the preceding sections, as well as:

- Meta-formations with global rigidity properties.
- Operations for growing/merging multiple persistent formations to form a single (minimally and/or structurally) persistent meta-formation.
- Studying the formation splitting and closing ranks problems for formations for which the corresponding graphs are directed.
- Understanding the connection between the rigidity matrix of a meta-formation formed without regard for the interior form of each meta-vertex, the rigidity matrices of the interiors of the meta-vertices, and the rigidity matrix of the whole meta-formation regarded as a normal formation, and thus reflecting the interior structure of each meta-vertex.

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