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Principles to Control Autonomous Formation Merging

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Abstract— **This paper provides a complete description of possible scenarios of merging two minimally rigid or globally rigid formations to obtain a single minimally rigid or globally rigid formation, respectively, both in $\Re^2$ and $\Re^3$. A strategy is developed based on simplification of the merging problem to a problem of growing a minimally or globally rigid graph. Based on this strategy, three principles are provided to control the merging efficiently and optimally, in the sense of minimizing the number of added edges and the number of vertices incident to these edges. Any possible scenario of merging two given minimally or globally rigid formations can be handled using a combination of these three principles. Although the main results are provided for merging two formations, application of the proposed techniques to merging of more than two formations is briefly discussed as well.**

I. INTRODUCTION

Multi-agent systems have attracted considerable attention recently as witnessed by the explosion of papers in the area (see for example [3], [5], [6], [9], [12], [13], [15], [16]). Agents, abstracted as vertices of graphs in this paper (following [6], [10], [19]), can be thought as any autonomous agents including combat robots, underwater vehicles, unmanned aerial vehicles, and ground vehicles.

Many control tasks for point-agent systems involve maintenance of the distance between nominated pairs of agents. For such tasks, a graph can naturally be used to depict the control architecture as follows: To each agent corresponds a vertex, and for each agent (vertex) pair i, j there is an undirected edge (i, j) if there is a constraint on the distance they must jointly actively maintain. If by explicitly maintaining some distances, the distance between any pair of agents is consequentially held fixed, then a formation of these agents is said to be *rigid*, and such a rigid formation can thus move as a cohesive whole. (Formal definitions of a rigid graph and other terms appear in the next section.) A rigid formation with a fixed number of agents is called *minimally rigid* if the number of explicitly maintained distances is minimal. If there is a unique rigid formation that models a given set of distances that are to be explicitly maintained during continuous move, the formation is said to be *globally rigid*. (Though it may not be obvious, it is possible to have 2 non-congruent but rigid formations with the same explicitly maintained distances. Thus rigidity is not the same as global rigidity). From now on, we use edges to represent these explicitly maintained distances, which in practice often require sensing and communication links.

Olfati-Saber and Murray [14] introduced an algebra that consists of performing some basic operations on (minimally rigid) formations which are useful in performing and representing rejoin/split maneuvers of distributed formations in a distributed fashion. Later in [6], operations on rigid formations are classified into problems of closing ranks, splitting formations, and merging formations. A rigid formation merging problem is one of creating a single post-merged rigid formation by adding new edges (i.e., new sensing and communication links) between two or more pre-merge rigid sub-formations. Whiteley [18] gives a detailed explanation of this problem, using rigidity theory. Such a problem, for example, may occur in a scenario when the formation has split into two formations to pass around an obstacle in the planned path and then must be reestablished as a single formation. One could therefore expect the goal of the strategy to be one of minimizing the total number of new edges required to preserve rigidity.

Two important subclasses of the rigid formation merging problem are that for minimally rigid formations and that for globally rigid formations. In [14], rejoining (merging) of two minimally rigid formations in two scenarios is investigated: the two formations have a common edge, or are disjoint. In [6], splitting and merging of minimally rigid formations are studied by solving the associated ”minimal cover problem”. Eren et al [7] also presented brief results on merging two globally rigid sub-formations in $\Re^2$ and $\Re^3$, considering the following two scenarios: (i) There is no shared agent (vertex) between (minimally or globally) rigid sub-formations G_1 and G_2; (ii) There are $d + 1$ vertices common to two globally rigid sub-formations G_1 and G_2 in $\Re^d$ $(d \in \{2, 3\})$.

We note that the two scenarios presented in [6], [7] *do not cover all possible cases of merging of two formations*. For example, suppose that G_1 and G_2 are merged into a single rigid formation in $\Re^3$, and there are 3 critical vertices which the previous merging is based on (e.g. say, by sharing of these 3 vertices); however, 2 of the 3 vertices (representing agents/robots) malfunction or are lost, then only 1 agent is common to G_1 and G_2, which is not considered in the above scenarios. Moreover, the problem of merging more than two sub-formations is not discussed either.

To motivate our discussion further, we note that scenarios describing sharing of vertices by globally rigid formations often occur in *sensor network localization* problems [2], [8], where the distributed process of localization may start from

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different locations, typically triples of connected nodes, and progressively more nodes are localized in an iterative process building from each start location until eventually localized sub-networks overlap each other; the localized nodes in the overlapped zones are best represented using shared vertices. If there are two or more locations where the localization process starts, a problem of merging multiple globally rigid formations occurs.

These challenges motivate us to complete the scenarios of merging minimally rigid or globally rigid formations, categorized by the number of shared or common vertices (agents), for both $\Re^2$ and $\Re^3$. The main results are obtained for merging two formations, while the strategies for merging more than two formations will only be briefly discussed.

The paper is organized as follows: In Section 2, we review the notions and properties of rigidity, minimal rigidity and global rigidity. In Section 3, we formally characterize the problem of merging, give a strategy that can be used to solve this problem and provide three principles to control the merging efficiently and optimally. In Section 4, we seek to provide complete results on merging of two minimally or two globally rigid sub-formations, where we require a minimal number of new edges (links) and a minimal number of vertices (agents) incident to these edges; problems of merging three or more sub-formations are also briefly discussed with preliminary results in a meta-formation view. We end the paper with conclusions and future works.

II. RIGIDITY

In this section, we review the the notions and some properties of rigidity, minimal rigidity and global rigidity. Our description will be largely using graphs. The graph modelling a formation is what is obtained when vertex position information and edge length information is thrown away. Noncongruent formations can have the same graph. Most properties to do with rigidity are properties shared by almost all formations with the same graph, and so we shall use properties such as rigidity in talking both about graphs and formations, largely following convention. A rigid (minimally rigid, globally rigid) graph is one for which for almost all choices of edge lengths and vertex positions for which a formation exists, the corresponding formation is rigid (minimally rigid, globally rigid) [19]. In $\Re^d$ ($d \in \{2,3\}$), a *representation* of an undirected graph $G=(V,E)$ with vector set V and edge set E is a function $p : V \to \Re^d$. We say that $p(i) \in \Re^d$ is the *position* of the vertex i, and define the distance between two representations p_1 and p_2 of the same graph by

$$d(p_1,p_2) = \max_{i \in V} ||p_1(i) - p_2(i)|| \, .$$

A *distance set* $\bar{d}$ for G is a set of distances $d_{ij} > 0$, defined for all edges $(i,j) \in E$. A distance set is *realizable* if there exists a representation p of the graph for which $||p(i) - p(j)|| = d_{ij}$ for all $(i,j) \in E$. Such a representation is then called a *realization*. Note that each representation p of a graph induces a realizable distance set (defined by $d_{ij} = ||p(i) - p(j)||$ for all $(i,j) \in E$), of which it is a realization.

A representation p is *rigid* if there exists $\epsilon > 0$ such that for all realizations p' of the distance set induced by p and satisfying $d(p,p') < \epsilon$, there holds $||p'(i) - p'(j)|| = ||p(i) - p(j)||$ for all $i,j \in V$. (We say in this case that p and p' are *congruent*). A graph is said to be *generically rigid* if almost all its representations are rigid. (A graph has a generic property if almost all its representations have the property.) Some discussions concerning the meaning of the terms "generic" and "almost all" can be found in [10], [17], [19]. One reason for using these terms in $\Re^d$ ($d \in \{2,3\}$) is to avoid the problems arising from having $d+1$ or more vertices lying on a d_1-dimensional hyper-surface where $d_1 \leq d-1$. In the sequel however, we shall frequently assume the "generic" qualifier applies without explicit use of the word, when no misunderstanding is likely to occur.

A. Minimal Rigidity

In some scenarios of multi-agent formation control, an information structure with a minimum number of communication links (or distance constraints) is to be exploited while preserving the rigidity of the formation. (Rigidity of the formation means rigidity of the representation it induces, with the associated graph.) This leads to a notion that is widely used in rigidity analysis, *minimal rigidity*. A graph is called *minimally rigid* if it is rigid and if there exists no rigid graph with the same number of vertices and a smaller number of edges. Equivalently, a graph is *minimally rigid* if it is rigid and if no single edge can be removed without losing rigidity. These two equivalent definitions of *minimal rigidity* lead to the following theorems, versions of which are presented in [18].

Theorem 1: If a graph $G=(V,E)$ in $\Re^d$ ($d \in \{2,3\}$) with at least d vertices is rigid, then there exists a subset E' of edges such that $G'=(V,E')$ is minimally rigid. This also implies the following:

- $|E'| = d|V| - d(d+1)/2$.
- Any subgraph $G''=(V'',E'')$ of G' with at least d vertices satisfies $|E''| \leq d|V''| - d(d+1)/2$.

Theorem 2: Let $G=(V,E)$ be a minimally rigid graph in $\Re^d$ ($d \in \{2,3\}$) and $G'=(V',E')$ a subgraph of G such that $|E'| = d|V'| - d(d+1)/2$. Then, G' is minimally rigid.

B. Global Rigidity

In some scenarios of multi-agent formation control [7] and problems of sensor network localization [2], [8], global rigidity is of interest. A graph is *generically globally rigid* if, for almost all distance sets, any two realizations of the one distance set are congruent, i.e., differ at most by translation, rotation or reflection. Likewise, one can define global rigidity of a graph with a prescribed realizable distance set. We refer the reader to [4], [11] for source material on global rigidity. We summarize a few results about globally rigid formations below.

In two dimensions, a graph is generically globally rigid if and only if it is 3-connected and generically redundantly rigid [4], [11], where a rigid graph is called generically *redundantly rigid* if it preserves generic rigidity after deletion of any single edge.

In three-dimensional space, it is necessary that a graph be 4-connected and generically redundantly rigid for the graph to be generically globally rigid. However, these conditions are known to be insufficient. No necessary and sufficient conditions for generic global rigidity are known, and it is not clear that such conditions have to exist, in contrast to the two dimensional case. That is, there may be examples of three dimensional graphs for which specification of a set of lengths confined to certain intervals for each length always guarantees global rigidity, while specification of the lengths for the same sensor pairs but confined to other intervals for each length results in lack of global rigidity.

III. PRINCIPLES TO CONTROL MERGING

A rigid formation merging problem is one of creating a single post-merged rigid formation by addition of new edges (i.e., new sensing and communication links) between two or more pre-merge rigid sub-formations. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be underlying rigid graphs of two rigid formations. The *merging problem* is to find a set of new edges E_{new} such that the resulting graph $G' = (V', E')$, where $V' = V_1 \cup V_2$ and $E' = E_1 \cup E_2 \cup E_{new}$, is rigid. Note that E_{new} can be the empty set. We further note that the above definition can be easily generalized to encompass minimal rigidity and globally rigidity. An *optimal procedure* that can solve the merging problem is one which minimizes both $|E_{new}|$ *and* the number of vertices in V' incident to the edges in E_{new}. Having defined the problem and the goal of deriving optimal procedure for merging, we will develop our techniques based on the following lemmas.

Lemma 1: [18] If two rigid graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $\Re^d$ ($d \in \{2, 3\}$) satisfy $|V_c| \geq d$, where $V_c = V_1 \cap V_2$, then the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ is rigid.

Lemma 2: [7] If two globally rigid graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $\Re^d$ $d \in \{2, 3\}$ satisfy $|V_c| \geq d + 1$, where $V_c = V_1 \cap V_2$, then the graph $G_1 \cup G_2$ is globally rigid.

The following analogous result for merging of minimally rigid formations can be verified using the definition of minimal rigidity, Theorem 1 and Lemma 1.

Lemma 3: If two minimally rigid graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $\Re^d$ $d \in \{2, 3\}$ satisfy $|V_c| \geq d$ and $|E_c| = d|V_c| - d(d + 1)/2$, where $V_c = V_1 \cap V_2$ and $E_c = E_1 \cap E_2$, then the graph $G_1 \cup G_2$ is minimally rigid.

Remark 1: It follows from Theorem 2 that the graph $G_c = (V_c, E_c)$ in Lemma 3 is minimally rigid.

Based on Lemmas 2 and 3, we develop an iterative strategy to simplify the rigid merging problem and to form a basis for implementation of an *optimal procedure* for merging of globally rigid and minimally rigid formations. In development of the particular strategy for merging globally rigid formations, we employ the following lemma, which is a simple consequence of Lemma 2.

Lemma 4: Consider two globally rigid graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $\Re^d$ $d \in \{2, 3\}$ for which $|V_c| \leq d$, where $V_c = V_1 \cap V_2$. Let $n_{new} = d + 1 - |V_c|$ and $V_{new} \subseteq V_2 \backslash V_1$ be a set of vertices satisfying $|V_{new}| = n_{new}$ and E_{new} a set of edges connecting V_{new} to G_1. If the graph $G'_1 = (V'_1, E'_1)$ where $V'_1 = V_1 \cup V_{new}$ and $E'_1 = E_1 \cup E_{new}$ is globally rigid, then the graph $G' = (V', E')$ where $V' = V_1 \cup V_2$ and $E' = E_1 \cup E_2 \cup E_{new}$ is globally rigid as well.

We are now ready to present our iterative strategy. We formulate this strategy for globally rigid graphs, however this formulation can be easily extended for minimally rigid graphs. Consider two globally rigid graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $\Re^d$ $d \in \{2, 3\}$. If $V_c = V_1 \cap V_2$ satisfies $|V_c| \geq d + 1$, Lemma 2 indicates that $G_1 \cup G_2$ is already globally rigid and we do not need to add any more edges. Therefore, we only need to consider the case where $|V_c| \leq d$ and hence $n_{new} = d + 1 - |V_c| > 0$. In this case, we apply Lemma 4 to outline our strategy, i.e., we look for a *systematic* way for *optimal* selection of a vertex set $V_{new} \subseteq V_2 \backslash V_1$ satisfying $|V_{new}| = n_{new}$ and an edge set E_{new} such that $G'_1 = (V'_1, E'_1) = (V_1 \cup V_{new}, E_1 \cup E_{new})$ is globally rigid.

In our strategy, we grow G_1 to G'_1 in n_{new} steps, where in each step we add a single vertex and a certain set of edges incident on this vertex. Let us denote the resultant graph in step m ($m \in \{1, ..., n_{new}\}$) by $\bar{G}_1(m)$, e.g., $\bar{G}_1(n_{new}) = G'_1$ and let $\bar{G}_1(0) = G_1$. In order to produce the graphs $\bar{G}_1(m)$ ($m \in \{1, ..., n_{new}\}$) and eventually G'_1, we follow a well-known procedure in the literature which is called the Henneberg construction (HC) [6] and its extensions.

The HC is a systematic way of constructing a minimally rigid graph $G' = (V', E')$ from a given rigid graph $G = (V, E)$ in $\Re^d$, ($d \in \{2, 3\}$, $V \subset V'$) in $m' = |V'| - |V|$ steps. The HC produces a sequence of graphs $\bar{G}(m) = (\bar{V}(m), \bar{E}(m))$ ($m \in \{0, ..., m'\}$), where $\bar{G}(0) = G$ and $\bar{G}(m') = G'$, which is called a Henneberg sequence (HS). Each $\bar{G}(m)$ ($m \in \{1, ..., n'\}$) is obtained in step m of the HC and is proven to be minimally rigid once one of the following two HC operations is used at each step m [6]:

- Vertex addition: Adding a new vertex i and d edges between i and d other vertices in $\bar{V}(m - 1)$.
- Edge splitting: Removing an edge $(j, k) \in \bar{E}(m - 1)$ and then adding a new vertex i together with $d+1$ edges incident on i, two of which are (i, j) and (i, k).

The HC procedure above can be extended to grow globally rigid graphs as follows: Given a globally rigid graph $G = (V, E)$ in $\Re^d$, ($d \in \{2, 3\}$), consider the sequence $\bar{G}(m) = (\bar{V}(m), \bar{E}(m))$ ($m \in \{0, 1, 2, \ldots\}$), where $\bar{G}(0) = G$ and each $\bar{G}(m)$ ($m \in \{1, ..., n'\}$) is obtained in step m of the procedure applying one of the following two operations:

- Extended vertex addition (or Trilateration in $\Re^2$): Adding a new vertex i with $d + 1$ edges between i and $d + 1$ other vertices in $\bar{V}(m - 1)$.
- Edge splitting: Procedure identical with the above original HC step.

That edge splitting preserves global rigidity in $\Re^2$ is established in [11]; the same strategy can be used for proving the generalization to $\Re^3$, and it can also be easily shown that global rigidity is also preserved by extended vertex addition. In the sequel, we will refer to this extended procedure as extended HC (EHC) and the sequence produced by each EHC as an extended HS (EHS).

Next, we present three principles that would ensure achievability of an *optimal procedure* of the above type as well as enabling one to derive all other (possibly non-optimal) procedures that emerge in an arbitrary scenario.

A. First Principle

Lemma 4 and the EHC procedure above imply that given two globally rigid graphs (formations) G_1 and G_2 in $\Re^d$ ($d \in \{2,3\}$) as described in Lemma 4, selecting n_{new} vertices in $V_2 \backslash V_1$ and adding at most $(d+1)n_{new}$ edges incident on these vertices, we can guarantee that the resultant post-merged graph is also globally rigid. But actually, a saving in edge number is possible.

This saving will be described in the first instance for the problem of (arbitrary) rigid formation merging in $\Re^2$ (the arguments are essentially the same for minimally rigid or globally rigid formation merging.)

Consider indeed the case in Figure 1(a) depicting a vertex addition of $i \in G_2 \backslash G_1$ to G_1 by inserting edges (l,i) and (m,i) where $l, i \in G_1 \backslash G_2$. Note that the vertex $k \in G_c = G_1 \cap G_2$ is not used. Consider in contrast the case in Figure 1(b), a new edge (l,i) is inserted and (k,i) is to be added. By the fact that both $i, k \in G_2$ and G_2 is rigid, there is already an explicit or implicit edge (i,k) (denoted by the dashed line) which holds the distance between i and k fixed. This suggests that one in fact does not need to add an explicit edge (i,k) while one still can conclude the rigidity of the resulting graph $G_1 \cup G_2 \cup \{(l,i)\}$, for the same reasons as used for (HC) vertex addition. We then see that only a single edge (l,i) is required by exploiting the existence of the explicit or implicit edges between the common vertex k and vertex i, and this saves one edge from the case described in 1(a) which applies the (HC) vertex addition. Using such a modified HC/EHC operation, one minimizes the number of new edges to be added for a rigid merging of two rigid formations. We summarize the above discussions in the following principle:

Principle 1: One can minimize the number of new edges required for a rigid merging of two rigid formations by exploiting, wherever possible, the existence of explicit and implicit edges incident on any common vertex and new vertex introduced in an iterative growing process of one of the pre-merged sub-formations.

In the sequel, we use Principle 1 and the associated modified HS/EHS to find the minimal number of new edges required (equivalent to the minimal number of distance measurements to be performed) in all different scenarios for merging two formations.

B. Second Principle

We remark that both vertex addition and edge splitting of HC for growing (minimally) rigid graphs in $\Re^d$ introduce a net addition of d edges when the new vertex is added.

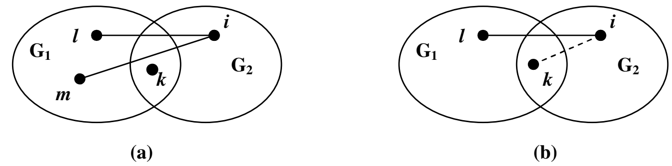

Fig. 1. Illustration of the Principle 1. The vertex k is common to G_1 and G_2, which are each minimally rigid and in $\Re^2$.

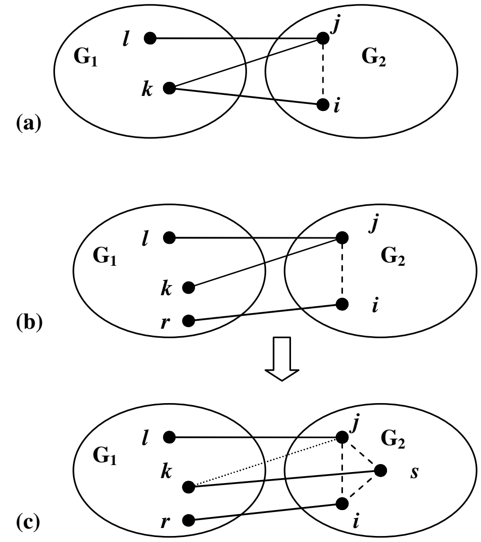

Fig. 2. Application of Principle 3 to merge two minimally rigid sub-formations in $\Re^2$ that do not share any agent, in various ways.

However, for the EHC used for growing a globally rigid graph, the extended vertex addition introduces $(d+1)$ new edges at each step when a new vertex is added; edge splitting still only requires d new edges, so where possible, we prefer to use edge splitting to add a new vertex to keep the number of new edges to be added at a minimum. This leads to our second principle:

Principle 2: One can minimize the number of new edges required for merging globally rigid formations by applying, wherever possible, an edge splitting operation in preference of an extended vertex addition.

An illustration of Principle 2 is given in Section IV with the aid of Figure 3.

C. Third Principle

As defined in the beginning of this section, an optimal procedure that can solve the merging problem is one which minimizes both $|E_{new}|$ and the number of vertices in V' on which the edges in E_{new} are incident. Let V_1^{new} and V_2^{new} denote the subsets of V_1 and V_2, respectively, on which edges of E_{new} are incident.

Consider as in Figure 2 merging two minimally rigid formations G_1 and G_2 in $\Re^2$, where V_c is the empty set. In constructing G_1', first one adds j by inserting (l,j) and (k,j). Then, applying Principle 1, one adds i into G_1' by inserting (k,i) and exploiting the existence of an explicit or implicit edge (j, i). This is indeed an optimal procedure,as shown in Figure 2(a), which has $|V_1^{new}| = |V_2^{new}| = 2$ and $|E_{new}| = 3$.

It is then trivial to obtain all other possible variations of an optimal procedure, which simply use different sets of vertices of interest. Moreover, it is also possible to obtain all non-optimal procedures (as opposed to the *optimal procedure* we have defined) using a combination of a number of following operations:

- To only increase $|V_1^{new}|$ by one: At the second step where i is added into G_1', select a different vertex r, from k, which is used for adding vertex j at the first step. (Refer to Figure 2(b)).
- To alone increase $|V_2^{new}|$ by one: Referring to Figure 2(c), one applies an edge splitting to add a vertex $s \in V_2$ into G_1', (k, j) (denoted by the dotted line) is removed and only (k, s) is inserted in light of Principle 1, which preserves the total number of new edges (links) required.
- To increase the number of edges: Simply add explicit edges to a post-merged rigid formation.

We remark that the first two operations above preserve the (minimal or global) rigidity, while the last one only preserves global rigidity and destroys minimal rigidity since extra edges are introduced. Moreover, all the operations can be easily generalized to $\Re^3$, as well as the merging of globally rigid formations. These are summarized as our third principle:

Principle 3: With the above operations, one can obtain all possible optimal and non-optimal merging procedures from an optimal procedure that utilizes Principles 1 and 2.

This powerful principle reduces the tedious work of verifying if the results shown in the next section necessarily cover all possible merging scenarios and procedures that likely exist. Only optimal merging procedures thus need to be explored for different scenarios.

IV. ANALYSIS OF MERGING SCENARIOS

With the three principles to control formation merging introduced in Section III, we are ready to characterize the *optimal procedures* that can emerge in all possible scenarios. The results are categorized by the number of common vertices of the pre-merge formations and summarized into two tables for merging of two globally rigid, and two minimally rigid formations, respectively. For completeness, the results that were already presented in [6], [7] are also included. We also provide detailed illustrations and proofs of our merging strategies, one in each subsection. Full illustrations and proofs of all the new results will be included in an extended version of this paper.

A. Globally Rigid Formation Merging

Using our strategy together with Principles 1 and 2, we have derived the minimal numbers for $|E_{new}|$, $|V_1^{new}|$ and $|V_2^{new}|$ that are required for each possible merging scenario categorized in terms of the number of common vertices ($|V_c|$). The results are summarized in Table I.

TABLE I
OPTIMAL MERGING OF TWO GLOBALLY RIGID FORMATIONS

Dim.	$\|V_c\|$	$\|E_{new}\|$	$\|V_1^{new}\| = \|V_2^{new}\|$
$\Re^2$	0	4	3
$\Re^2$	1	2	2
$\Re^2$	2	1	1
$\Re^2$	3 or more	0	0
$\Re^3$	0	7	4
$\Re^3$	1	4	3
$\Re^3$	2	2	2
$\Re^3$	3	1	1
$\Re^3$	4 or more	0	0

As an example, Figure 3 illustrates the formal procedure of applying Principles 1 and 2 to obtain an optimal merging strategy for two 3-dimensional globally rigid formations, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, sharing a single agent. For this example, in the framework of Lemma 4, we have $|V_c| = 1$ and $n_{new} = 3$. Hence, we look for an *optimal procedure* to select a set V_{new} of $n_{new} = 3$ vertices in $G_2 \backslash G_1$ and a set E_{new} of new edges to join these three vertices to G_1 applying a certain $n_{new} = 3$-step (modified) EHS in light of Principles 1 and 2. Following Principle 1, the nature of the modification is to exploit the existence of explicit or implicit edges in G_2 in order to reduce the number of new edges that are introduced, while still relying on the HC or EHC steps. Having performed the first step (as shown in Fig 3(a)) which only utilizes Principle 1, in the subsequent steps, Principle 2 can be applied together with Principle 1 because there always is new edge(s) that can be removed via (modified) edge splitting to reduce the total number of new edges added. Using Lemma 4 and Principle 1, one can easily verify that the merged graph $G' = (V_1 \cup V_2, E_1 \cup E_2 \cup E_{new})$ is globally rigid.

B. Minimally Rigid Formation Merging

Application of the three principles of Section III to the minimally rigid formation merging problem is very similar to the globally rigid case. However, one needs to pay attention to the constraints on the relations between the number of edges and the number of vertices of certain graphs posed by Theorem 2 and Lemma 3.

Given two minimally rigid formations $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $\Re^d$ ($d \in \{2, 3\}$) to be merged, because of Theorem 2, we have

$$|E_1| = d|V_1| - d(d+1)/2 \quad (1)$$

$$|E_2| = d|V_2| - d(d+1)/2 \quad (2)$$

Using notation of Section III, we also require the post-merged graph $G' = (V', E')$, where $V' = V_1 \cup V_2$ and $E' = E_1 \cup E_2 \cup E_{new}$, to be minimally rigid and hence satisfy

$$|E'| = d|V'| - d(d+1)/2 \quad (3)$$

Combining Equations (1)-(3) and noting that $|V'| = |V_1| + |V_2| - |V_c|$ and $|E'| = |E_1| + |E_2| + |E_{new}| - |E_c|$, one needs

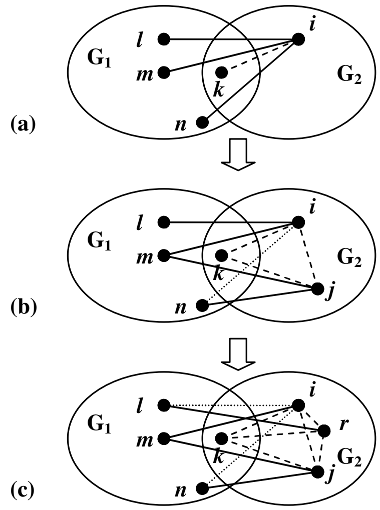

Fig. 3. Merging of two globally rigid formations in $\Re^3$ sharing a single agent to form a single globally rigid post-merged formation in light of Principles 1 and 2: (a) (First Step) Observing the existence of an implicit or explicit edge (i,k), Principle 1 is applied; (modified) extended vertex addition is used to add the edges $(i,l),(i,m),(i,n)$. (b) (Second step) Observing the existence of implicit or explicit edges $(j,k),(i,j)$, Principles 1 and 2 are applied; (modified) edge splitting is used to remove edge (i,n), which is added in the first step, and add the edges $(j,m),(j,n)$. (c) (Third step) Observing the existence of implicit or explicit edges $(i,r),(j,r),(k,r)$, Principles 1 and 2 are applied; (modified) edge splitting is used to remove the edge (i,l), which is added in the first step, and add the edge (l,r).

to satisfy

$$|E_{new}| = |E_c| - d|V_c| + d(d+1)/2 \tag{4}$$

Remark 2: Lemma 3 and Equation (4) indicate that in $\Re^d$ ($d \in \{2,3\}$), if for the two pre-merge minimally rigid formations G_1 and G_2 it holds that $|E_c| < d|V_c| - d(d+1)/2$ then G_1 and G_2 cannot be merged to form a *minimally* rigid formation[1]. One can actually verify that, in this case, $G_1 \cup G_2$ is non-minimally rigid, and removal of one or more selected edges will yield a minimally rigid graph because of Theorem 1.

Remark 3: Another special case in $\Re^3$ where two minimally rigid formations G_1 and G_2 cannot be merged to form a single post-merged minimally rigid formation (without removing any edge in $E_1 \cup E_2$) is the one with $|V_c| = 2$ and $|E_c| = 0$. In this case, Equation (4) indicates that one needs to satisfy $|E_{new}| = 0$. However, one can easily see that, given that $|V_c| = 2$ and $|E_c| = 0$, $G_1 \cup G_2$ can not be rigid. An example of such a situation is depicted in Figure 4.[2]

Equation (4) implies that the number $|E_{new}|$ of edges to be added in minimally rigid merging for any given scenario with a certain pair of $(|V_c|, |E_c|)$ (if feasible) is uniquely determined by $|V_c|$ and $|E_c|$. It also implies that in categorization of the possible scenarios of minimally rigid merging we need to consider the value of $|E_c|$ as well as $|V_c|$, in contrast to globally rigid merging where $|E_{new}|$ is uniquely determined by $|V_c|$ only and consideration of $|E_c|$ is not needed. Despite this fact, Remark 2 and Remark 3 indicate that in $\Re^2$ and $\Re^3$, for any given value of $|V_c| \in \{0,1,2,\ldots\}$, minimally rigid merging is feasible only for a *unique value* of $|E_c|$, which depends on $|V_c|$.

Using the strategy in Section III and applying only Principle 1 with cognizance of the above remarks, we have arrived at the results summarized in Table II, which is categorized in terms of the number of common vertices $|V_c|$, noting that there is only one feasible value of $|E_c|$ for each $|V_c|$.

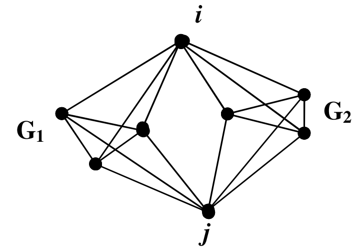

Fig. 4. A pair of minimally rigid formations in $\Re^3$ having two common vertices i and j, which cannot be merged to form a single post-merged minimally rigid formation.

TABLE II
OPTIMAL MERGING OF TWO MINIMALLY RIGID FORMATIONS

Dim.	$\|V_c\|$	$\|E_c\|$	$\|E_{new}\|$	$\|V_1^{new}\| = \|V_2^{new}\|$
$\Re^2$	0	0	3	2
$\Re^2$	1	0	1	1
$\Re^2$	2 or more	$2\|V_c\| - 3$	0	0
$\Re^3$	0	0	6	3
$\Re^3$	1	0	3	2
$\Re^3$	2	1	1	1
$\Re^3$	3 or more	$3\|V_c\| - 6$	0	0

C. Merging of Three or More Formations

It is easy to see that the strategy and the principles described in Section III could be applied to merging three or more sub-formations, by progressively performing merging of two formations. However, we note the following scenarios may need other merging strategies:

- Suppose that three (or more) sub-formations have to be merged simultaneously, as operational constraints do not allow progressive merging. In this case, one need to develop strategies and/or operations that construct all possible ways of merging multiple sub-formations.
- Consider the scenario that there are common vertices shared by more than two formations. This formulation suggests that distributed merging may not give an optimal procedure (best solution), even when a globally optimized merging procedure may exist.

[1]The case $|E_c| > d|V_c| - d(d+1)/2$ does not exist due to Theorem 1 and G_1 and G_2 being minimally rigid.

[2]For completeness, we note that for this example, if a new edge were inserted between a vertex of G_1 and a vertex of G_2, but not vertex i or vertex j, a non-minimally rigid graph results. There will then be a subset of the edges (not including that just added) such that any one can be removed and this will yield a minimally rigid graph.

Yet another problem is exemplified by the following situation. Suppose G_1, G_2, and G_3 are three minimally rigid formations in $\Re^2$ with no common vertices. Then, a merging of G_1 and G_2 to produce a minimally rigid graph G_1' will require 3 edges; a further 3 edges between G_3 and G_1' will merge G_1, G_2, and G_3 in a minimally rigid graph. But it is clearly reasonable to conjecture that 6 edges could be used with two joining G_1 and G_2, two joining G_1 and G_3 and two joining G_2 and G_3. If this is correct, one needs an approach to make transparent such a fact (and indeed ones like it, for global rigidity, $\Re^3$, and different vertex-sharing situations.)

To this end, we have commenced work on viewing such problems as meta-formation problems [1]. Each of G_1, G_2, and G_3 above is a meta-vertex; more formally, a meta-formation has a set of meta-vertices, each of which is a single formation, that is possibly minimally rigid, rigid, or globally rigid. The meta vertices are joined with meta-edges (each of which may have a certain weight reflecting the actual number of normal edges, and which must obey certain connection rules within each meta-vertex, principally to rule out having more than one normal edge joining the same pair of (non-meta) vertices.) One might say that the joining of the meta-vertices with meta-edges constitutes a meta-formation, for which minimal rigidity, rigidity, global rigidity, etc. are of interest. For this analysis, each (non-trivial) meta-vertex corresponding to a rigid graph with more than d vertices in $\Re^d$ can probably be equivalently represented by the complete graphs K_3 in $\Re^2$ and K_4 in $\Re^3$, while in case of analysis of global rigidity, one would use K_4 in $\Re^2$ and K_5 in $\Re^3$.

V. CONCLUSIONS AND FUTURE WORK

We have provided a complete description of possible scenarios of merging two minimally rigid or globally rigid formations to obtain a single minimally rigid or globally rigid post-merged formation, respectively, for both $\Re^2$ and $\Re^3$. We have categorized these scenarios by the number of shared or common vertices (agents). We have developed an iterative strategy to simplify the rigid merging problem and to form a basis for implementation of an *optimal procedure* for merging of globally rigid and minimally rigid formations, minimizing both the number of new edges to be added during the merging process and the number of vertices incident to these new edges in the post-merged single formation. In our iterative strategy, the merging problem is simplified to a problem of growing a (minimally or globally) rigid graph. We have provided three principles to control the merging efficiently and guarantee optimality in the above sense. These three principles can be used in a systematic way for deriving an optimal solution of the merging two minimally rigid or globally rigid formations problem for any possible scenario. The results of the applications of the three principles to a complete list of categories of merging scenarios are also provided in the paper.

Our study of merging two formations leads to the natural question on merging of more than two formations, which often emerges in the sensor network localization problem where the distributed localization process starts with more than three triples of anchor sensors. We have briefly discussed this problem and proposed a meta-formation framework to study this problem. Another potential problem worth considering is one of merging formations with leader-follower structure (or unilateral distance constraints), often studied using notions of "rigidity of directed graph" or more recently developed "persistence" [10], [19].

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