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Stability of Adaptive Delta Modulators with a Forgetting Factor and Constant Inputs

Sandra Hala Dandach, Soura Dasgupta
Department of Electrical
and Computer Engineering
The University of Iowa
Iowa City, IA-52242, USA.
Email: sdandach, dasgupta@engineering.uiowa.edu.

Brian D. O. Anderson
Research School of Information Sciences and Engineering
Australian National University
and National ICT Australia Ltd, Locked Bag 8001
Canberra ACT 2601 Australia
e-mail: Brian.Anderson@nicta.com.au

Abstract— **Motivated by applications to feedback control over communication networks where the actuation and feedback signals are transmitted over communication channels, we study the stability of Adaptive Delta Modulators (ADM) when the coded signal is a constant. The importance of such a setting arises because a common control task is to track a dc input. In an earlier paper we had shown that a standard accumulator based adaptive delta modulator has the following characteristic: that virtually all combinations of the algorithm parameters result in 4-cycles, that the avoidance of 4-cycles requires a nongeneric initialization, and that steady state oscillations that generically arise in the course of these cycles can have amplitudes that can be arbitrarily close to the initial error. Consequently, in this paper we study the use of a forgetting factor in the ADM loop, and provide a detailed stability analysis and design guidelines.**

I. Introduction

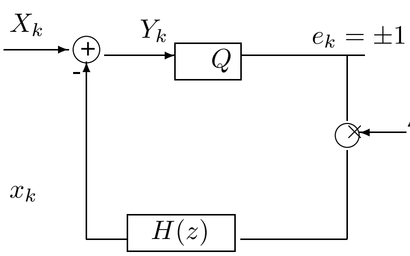

Fig. 1. A Delta Modulator at the transmitter

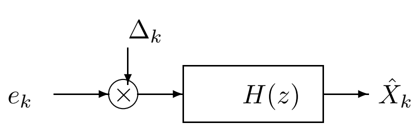

Fig. 2. A Delta Modulator at the receiver

Adaptive Delta Modulators (ADM) are used in signal processing and communications for signal quantization with variable step-size. They increase the dynamic range of the signals that can be tracked while using binary coding. Among their variations, [1] -[3], our focus is the simplest, described in [1] and depicted in fig. 1, 2. The structures in fig. 1 and fig. 2 are at the transmitter and receiver, respectively. The signal X_k is coded into the binary sequence e_k, taking values from $\{-1, 1\}$. It is e_k that is transmitted. The quantity Δ_k represents the variable step size which is increased or decreased according to the sign pattern in e_k. Consequently, if the signal at the receiver input is identical to the transmitted value of e_k, and Δ_0 is known at the receiver, then for all $k \geq 0$, Δ_k is known to the receiver. This also guarantees that the signal $\hat{X}_k$ at the receiver is identical to x_k, the output of $H(z)$ at the transmitter, if $x_0 = \hat{X}_0$. Thus should x_k approach X_k, so also would $\hat{X}_k$. An algorithm for updating Δ_k is described in [1]. In [1] $H(z)$ is an *accumulator*: i.e. with $\alpha = 1$,

$$H(z) = \frac{1}{1 - \alpha z^{-1}}. \tag{I.1}$$

The agreed upon values of Δ_0, and x_0 between the transmitter and receiver, are part of the communication protocol. In [5] we have analyzed this ADM when the signal $X_k = x$ is constant and $\alpha = 1$. We have shown that: (A) Either $\hat{X}_k$ converges to x or enters into 4-cycles with error magnitudes comparable to $|x|$. (B) Such 4-cycles are avoided only with nongeneric initializations. Hence we study the ADM with constant X_k, when a forgetting factor is included in $H(z)$, i.e. when $0 < \alpha < 1$. We show that the system parameters can be chosen to make the eventual coding error arbitrarily small. The forgetting factor also allows the requirement $\hat{X}_0 = x_0$ to be relaxed by forcing the initial error to decay.

We study the constant X_k case because of networked control systems, where a remote digital controller controls a plant. Both the actuation signal and the feedback signal are conveyed over bandwidth constrained communication channels, and must consequently be quantized prior to transmission. It has been noted in [4] that variable step quantization of the feedback and actuation signals suffice to achieve closed loop stability. Thus, it behooves one to study ADM behavior in this setting, with transmitters and receivers at both the plant and controller locations respectively, housing fig 1, and fig. 2.

A typical control problem involves forcing the plant output to track a constant signal. This in turn requires that the closed loop input is a constant signal, and to achieve the

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desired performance, at steady state both the output of the digital controller and the sampled plant output should be constant, i.e. both the signals that the ADM's should track should be constants at steady state. Thus at the minimum, desirable performance will necessitate that the signal $\hat{X}_k$ track a constant X_k in figs 1 and 2 with reasonable fidelity.

In section II we present the detailed ADM algorithm of [1], and explain the heuristics that motivate it, and go on to describe the forgetting factor based algorithm. Section III describes some preliminary properties, Section IV provides a stability analysis, and Section V provides design guidelines. Space constraints compel us to compress or omit several proofs.

II. The detailed algorithm

The detailed algorithm of [1] is given in (II.2) - (II.5) below with $\Delta_0 > 0$ and $K > 1$.

$$x_{i+1} = \alpha x_i + \Delta_i e_i \quad \text{(II.2)}$$

$$e_i = \operatorname{sgn}(X_i - x_i) \quad \text{(II.3)}$$

$$\Delta_{i+1} = \Delta_i K^{e_{i+1} e_i} \quad \text{(II.4)}$$

with

$$\operatorname{sgn}(a) = \begin{cases} 1 & \text{if } a \geq 0 \\ -1 & \text{if } a < 0 \end{cases} \quad \text{(II.5)}$$

We first discuss this algorithm in the form proposed in [1], i.e. when $\alpha = 1$. Several features of this algorithm are noteworthy. First observe that as e_i is available at the receiver, so is Δ_i, assuming perfect transmission and an agreed upon value for Δ_0. This is so as Δ_i increases by a factor of K if two successive values of $X_k - x_k$ have the same sign (i.e. $e_{i+1}e_i = 1$), and decreases by the factor K if two successive values of $X_k - x_k$ have opposite signs (i.e. $e_{i+1}e_i = -1$). Thus, the reception of the e_i sequence permits reproduction of Δ_i at the receiver. Consequently if

$$\hat{X}_0 = x_0$$

then the accumulation

$$\hat{X}_{i+1} = \hat{X}_i + \Delta_i e_i \quad \text{(II.6)}$$

ensures that

$$\hat{X}_i = X_i.$$

Second, observe that (II.6) justifies the association of Δ_i with variable step-size as at each sample $\hat{X}_i$ increases or falls by Δ_i, depending on whether x_k and hence $\hat{X}_k$ is below or above X_k. Third, the motivation for updating Δ_i as in (II.3, II.4) can be understood as follows.

Consider in particular a constant Δ, and figure 3, which simultaneously depicts X_k and $\hat{X}_k$. In particular X_k is the signal that ramps up to a constant value while $\hat{X}_k$ is the signal that transitions in steps. In the ramping stage it is desirable to have a large Δ so that the rise in $\hat{X}_k$ tracks X_k at a fast pace. The obverse occurs when X_k has acquired a steady state, as in the second part of figure 3, where a large Δ results in a large granularity in the steady state error between $\hat{X}_k$ and X_k. Contrast this to figure 4 where a smaller Δ is

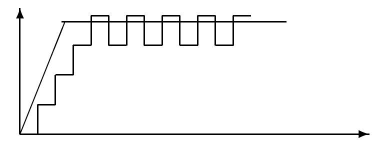

Fig. 3. Large Δ

used. The result is slower tracking when X_k is rising rapidly, but smaller steady state error once X_k has stopped changing. Together these two examples show that when the signal to be tracked changes quickly, a large Δ is desirable. On the other hand when X_k is not changing quickly and $\hat{X}_k$ is close to it, a smaller Δ is desirable. The update laws (II.3, II.4) judge the quality of tracking by whether or not successive values of $X_k - \hat{X}_k$ have the same sign. Their doing so indicates that $\hat{X}_k$ must approach X_k at a more rapid rate requiring a larger Δ. If on the other hand the sign of $X_k - \hat{X}_k$ alternates then $\hat{X}_k$ is likely to be close to X_k and a decrease in Δ is called for.

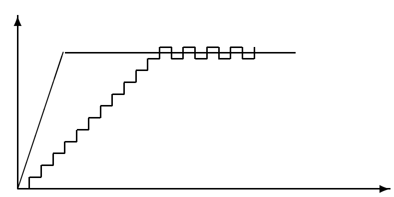

Fig. 4. Small Δ

The results of [5] show that the accumulator based algorithm provides poor performance with constant inputs for generic parameter combinations, and initial conditions. Thus, we study instead the modified algorithm when

$$0 < \alpha < 1. \quad \text{(II.7)}$$

Good design requires that α be close to 1 and

$$\alpha K > 1. \quad \text{(II.8)}$$

Note (II.7,II.8) imply that $K > 1$. As noted earlier our goal is to study this algorithm for constant X_i, i.e. for all i,

$$X_i = x. \quad \text{(II.9)}$$

To foreshadow the major results of this paper, we first note that a major difficulty with the $\alpha = 1$ case is the necessity of identical initialization of x_k and $\hat{X}_k$. As opposed to this, (II.7) ensures that the effect of the difference $x_0 - \hat{X}_0$, diminishes over time.

The second important difference relates to the convergence properties even when exact intialization occurs. In particular when $\alpha = 1$, for generic combinations of Δ_0, K, x_0 and x, for some N and all $k > N$ one has four cycles of the form $x_{k+4} = x_k$, and $\Delta_{k+4} = \Delta_k$, and the largest $|x_k - x|$, during these 4-cycles, can be arbitarily close to the initial error $|x_0 - x|$.

When (II.7) holds on the other hand, the following positive parameter plays a pivotal role:

$$\epsilon = \frac{1-\alpha^3}{1-\alpha^2+\frac{\alpha}{K}} \tag{II.10}$$

Indeed we show that under the right conditions

$$\lim_{i\to\infty} \sup \Delta_i < K\epsilon|x|. \tag{II.11}$$

This in turn will be shown to imply that

$$\lim_{i\to\infty} \sup |x_i - x| \leq \max\{(1-\alpha+K\epsilon)|x|, (\alpha+K\epsilon-1)|x|\} \tag{II.12}$$

Observe that $\alpha + K\epsilon - 1$, can be readily verified as being positive. As will be explained in Section V one can make ϵ arbitrarily small by choosing α arbitrarily close to 1. Consequently, one can achieve an error that is an arbitrarily small fraction of x, the value being encoded. We will explain later why (II.11) and (II.12) cannot generically be achieved when $\alpha = 1$. Finally $\alpha \approx 1$, and $\alpha K \approx 1$ are good design guidelines.

III. Some properties

In this section we present a series of properties of (II.2-II.9), that will allow us to conduct our stability analysis. For simplicity we will assume

$$x > 0, \tag{III.13}$$

as the results translate in an obvious way to the case where $x < 0$. For example under (III.13), the following set of indices that mark the points at which x_i transitions from below to above x, will play an important role.

$$\mathcal{I}_+ = \{i | x_i < x \text{ and } x_{i+1} \geq x\}. \tag{III.14}$$

Henceforth Δ_i for $i \in \mathcal{I}_+$, i.e. at a point of transition of x_i from below to above x, will be referred to as a *transitioning* Δ. For $x < 0$, the transition points are given by the set $\mathcal{I}_- = \{i | x_i \geq x \text{ and } x_i < x\}$. on the other hand, the corresponding indices are those marking transitions in x_i from above x to below. In general the results of this section can be applied to the case of $x < 0$, by replacing inequalities of the form of $x_k \geq x$ by $x_k < x$. The first rather straight forward Lemma shows that e_i must at some point change sign and that the sign changes persist.

Lemma 3.1: Consider the system described in (II.2-II.5) and (II.7-III.13), and $\mathcal{I}_+$ as in (III.14). Then $\mathcal{I}_+$ is an infinite set.

In fact $\mathcal{I}_+$ is infinite even when $\alpha = 1$. The next Lemma is also straight forward.

Lemma 3.2: Under (II.2-II.5), (II.7-II.9) and (III.13), if $i \in \mathcal{I}_+$, then

$$\Delta_i > (1-\alpha)x. \tag{III.15}$$

Further if for some j, $x_j < x$ and

$$\Delta_j > (1-\alpha)x_j, \tag{III.16}$$

then $x_{j+1} > x_j$.

In the $\alpha = 1$ case the lower bounds in (III.15) and (III.16) are both zero, and thus trivially hold. We now provide a crucial property of this system. Specifically, after the first sign change in e_i, no more than *two successive* values of x_i may exceed x.

Lemma 3.3: If $i \in \mathcal{I}_+$ and $x_{i+2} \geq x$ then under (II.2-II.5), (II.7-II.9) and (III.13), $x_{i+3} < x$ and

$$x_{i+4} = \alpha^4 x_i + \Delta_i(1-\alpha^2)(1/K - \alpha) < x. \tag{III.17}$$

Further in this case

$$\Delta_{i+4} = \Delta_i. \tag{III.18}$$

Proof: From (III.14), $x_i < x$ and $x_{i+1} \geq x$. Thus, from (II.2-II.5) and $x_{i+2} \geq x$, $e_i\Delta_i = \Delta_i$, $e_{i+1}\Delta_{i+1} = -\Delta_i/K$ and $e_{i+2}\Delta_{i+2} = -\Delta_i$. As $\alpha < 1$ and (III.13),

$$\begin{aligned} x_{i+3} &= \alpha^3 x_i + \alpha^2\Delta_i e_i + \alpha\Delta_{i+1}e_{i+1} + \Delta_{i+1}e_{i+2} \\ &= \alpha^3 x_i + \Delta_i(\alpha^2 - \frac{\alpha}{K} - 1) < \alpha^3 x_i < x. \end{aligned}$$

Also, in this case $e_{i+3}\Delta_{i+3} = \Delta_i/K$. Thus

$$\begin{aligned} x_{i+4} &= \alpha^4 x_i + \alpha^3\Delta_i - \alpha^2\frac{\Delta_i}{K} - \alpha\Delta_i + \frac{\Delta_i}{K} \\ &= \alpha^4 x_i + \Delta_i(1-\alpha^2)(1/K-\alpha) < \alpha^4 x_i < x. \end{aligned}$$

Finally (III.18) follows trivially. ■

Thus if $i \in \mathcal{I}_+$ then either

$$e_{i+1} = e_{i+2} = -1 \text{ and } e_{i+3} = 1, \tag{III.19}$$

or

$$e_{i+1} = -1 \text{ and } e_{i+2} = 1, \tag{III.20}$$

This is also true true when $\alpha = 1$. However, when $\alpha = 1$ from (III.17) $x_{i+4} = x_i$, and $\Delta_{i+4} = \Delta_i$, signaling the onset of 4-cycles. Thus, in the $\alpha = 1$ case *any occurrence of (III.19) will lead to 4-cycles that cannot be arrested.* The next Lemma characterizes conditions under which (III.19) holds and is straight forward.

Lemma 3.4: Consider $i \in \mathcal{I}_+$ and (II.2-II.5), (II.7-II.9) and (III.13). Then $x_{i+2} \geq x$ iff

$$\alpha^2 x_i \geq (\frac{1}{K} - \alpha)\Delta_i + x. \tag{III.21}$$

Because $\alpha < 1$, even if $x_i < x$ and thus $e_i\Delta_i > 0$, x_{i+1} need not exceed x_i. The following Lemma shows, however that if at any point x_i does become less than x, then after at most two samples, its value will increase and will continue do so, as long as it remains below x. It follows by showing that if two successive values of x_i are below x, then subsequent Δ_j obeys (III.16).

Lemma 3.5: Under (II.2-II.5), (II.7-II.9) and (III.13), suppose $i \in \mathcal{I}_+$. Then the following apply.

(A) Suppose (III.19) holds and for some $k \geq 4$ and all $n \in \{3, \cdots, k\}$, $e_{i+n} = 1$. Then for all $n \in \{4, \cdots, k\}$,

$$x_{i+n+1} > x_{i+n}. \tag{III.22}$$

(B) Suppose (III.20) holds and for some $k \geq 4$ and all $n \in \{2, \cdots, k\}$, $e_{i+n} = 1$. Then also (III.22) holds for all $n \in \{4, \cdots, k\}$.

IV. Stability

In this section we provide conditions under which (II.11) and (II.12) hold. The first few Lemmas focus on the values of transitioning Δ's. Specifically, the first states that if a given transitioning Δ exceeds ϵx, then the next transitioning Δ can be no greater.

Lemma 4.1: Suppose (II.2-II.5), (II.7-II.9) and (III.13) hold. Consider i, j two consecutive members of $\mathcal{I}_+$, with $j > i$. Suppose $\Delta_i > \epsilon x$, then $\Delta_j \leq \Delta_i$ and $j \leq i + 4$.

Proof: From Lemma 3.3 one of two cases apply.

Case I: $x_{i+2} < x$**.** We will argue now that if $\Delta_j > \Delta_i$, then $\Delta_i < \epsilon x$. Now in this the case, $\Delta_{i+1} e_{i+1} = -\Delta_i/K$, $\Delta_{i+2} e_{i+2} = \Delta_i/K^2$, and for all $k \in \{i+2, \cdots, j\}$, $\Delta_k e_k = \Delta_i K^{k-i-4}$. Thus, $\Delta_j > \Delta_i$ requires $j > i+4$. So $x_{i+4} < x$ and $x_{i+5} < x$. Also,

$$x_{i+5} \;=\; \alpha^4 x_{i+1} - \alpha^3 \frac{\Delta_i}{K} + \alpha^2 \frac{\Delta_i}{K^2} + \alpha \frac{\Delta_i}{K} + \Delta_i.$$

Thus $x_{i+5} < x$ and $x_{i+1} \geq x$ imply

$$\Delta_i < \frac{x(1-\alpha^4)}{1 + \frac{\alpha}{K} + \frac{\alpha^2}{K^2} - \frac{\alpha^3}{K}}. \tag{IV.23}$$

Thus, to prove that $\Delta_j \leq \epsilon x$, it is enough to show that the upper bound in (IV.23) is smaller than ϵx, or equivalently by (II.10) that $(1+\alpha+\alpha^2)(1+\frac{\alpha}{K}(1-\alpha^2+\frac{\alpha}{K}))$ is greater than $(1+\alpha+\alpha^2+\alpha^3)(1-\alpha^2+\frac{\alpha}{K})$. This can be shown through routine calculations using $K > 1$, and (II.7) and (II.8). Hence $\Delta_i > \epsilon x$ implies $\Delta_j \leq \Delta_i$. Further the second conclusion follows as $\Delta_j \leq \Delta_i$ iff $j \leq (i+4)$.

Case II: $x_{i+2} \geq x$**.** In this case $\Delta_{i+2} e_{i+2} = -\Delta_i$, and as from Lemma 3.3, x_{i+3} and x_{i+4} are less than x, $\Delta_{i+3} e_{i+3} = \Delta_i/K$ and $\Delta_{i+4} e_{i+4} = \Delta_i$. As $x_{i+2} \geq x$, and $\Delta_i > \epsilon x$, we can show that $x_{i+5} > x$. Hence $j = i+4$ and from Lemma 3.3 $\Delta_j = \Delta_i$. ∎

When $\alpha = 1$, the four-cycles referred to earlier occur, when $j = i+4$ and $x_{i+2} \geq x$. The Lemma below shows that if in fact a transitioning Δ exceeds $K\epsilon x$, *and* x_i stays above x only once, then the next transitioning Δ will be smaller.

Lemma 4.2: Suppose (II.2-II.5), (II.7-II.9) and (III.13) hold. Consider i, j, two consecutive members of $\mathcal{I}_+$, with $j > i$. Suppose $\Delta_i > K\epsilon x$ and $x_{i+2} < x$. Then $\Delta_j < \Delta_i$.

Proof: Because, $x_{i+2} < x$, from the definition of $\mathcal{I}_+$, $\Delta_{i+1} e_{i+1} = -\Delta_i/K$, $\Delta_{i+2} e_{i+2} = \Delta_i/K^2$, and for all $k \in \{i+2, \cdots, j\}$, $\Delta_k e_k = \Delta_i K^{k-i-4}$. Thus, if $j = i+2$, then $\Delta_j < \Delta_i$. Then the Lemma is proved because $j > i+2$, and $x_{i+1} \geq x$,one can show that $x_{i+4} \geq x$. ∎

The third step is to show in Lemma 4.3 that if a transitioning Δ is less than or equal to ϵx, then the next transitioning Δ cannot exceed $K\epsilon x$.

Lemma 4.3: Suppose (II.2-II.5), (II.7-II.9) and (III.13) hold. Consider i, j, two consecutive members of $\mathcal{I}_+$, with $j > i$. Suppose $\Delta_i \leq \epsilon x$. Then $\Delta_j \leq K\epsilon x$.

Proof: We need to consider the two cases $x_{i+2} < x$ and $x_{i+2} \geq x$.

Case I: $x_{i+2} < x$**.** In this case $\Delta_{i+1} e_{i+1} = -\Delta_i/K$, $\Delta_{i+2} e_{i+2} = \Delta_i/K^2$, and for all $k \in \{i+2, \cdots, j\}$, $\Delta_k e_k = \Delta_i K^{k-i-4}$. Suppose $j = i+n$. If $n \leq 5$, then $\Delta_j \leq K\Delta_i \leq K\epsilon x$ proving the result. If $n \geq 6$, and $\Delta_j > K\epsilon x$, because of $x_j = x_{i+n} < x$

$$x_{i+n} = \alpha^2 x_{i+n-2} + \alpha \frac{\Delta_j}{K^2} + \frac{\Delta_j}{K} < x.$$

Thus,

$$x_{i+n-2} \;<\; \frac{x}{\alpha^2}(1 - (\frac{\alpha}{K} + 1)\epsilon) \tag{IV.24}$$

On the other hand because of Lemma 3.2, $\Delta_i > (1-\alpha)x$. Thus,

$$x_{i+4} \;>\; \alpha^3 x + x\frac{1-\alpha}{K}(1 - \alpha^2 + \frac{\alpha}{K}) \tag{IV.25}$$

Further, as $n \geq 6$, from Lemma 3.5, $x_{i+n-2} \geq x_{i+4}$. Then a contradiction ys established by showing that the upper bound in (IV.24) is less than the lower bound in (IV.25) using, $K > 1$, (II.10), (II.7) and (II.8).

Case II: $x_{i+2} \geq x$**.** In this case $\Delta_{i+2} e_{i+2} = -\Delta_i$, and from Lemma 3.3, x_{i+3} and x_{i+4} are less than x, and $\Delta_{i+4} = \Delta_i$.

With n defined as in the proof of Case I, suppose, that $\Delta_{i+n} > K\epsilon x$. Then in this case $n \geq 6$, as $\Delta_{i+5} = K\Delta_i \leq K\epsilon x$. From (II.10),

$$x_{i+n-1} = \frac{x_{i+n} - \frac{\Delta_{i+n}}{K}}{\alpha} < x\left(\frac{1-\epsilon}{\alpha}\right). \tag{IV.26}$$

Further, because of Lemma 3.5 and the fact that $x_{i+2} \geq x$, $x_{i+5} > x_{i+4}$ and $\Delta_i > (1-\alpha)x$,

$$x_{i+5} \;>\; x[\alpha^3 + (1-\alpha)(1-\alpha^2 + \frac{\alpha}{K})] \tag{IV.27}$$

As from Lemma 3.4, $\Delta_{i+n-1} > \Delta_{i+5}$, for all $n \geq 6$, a contradiction is established because the upper bound in (IV.26) is smaller than the lower bound in (IV.27), from $K > 1$, (II.7) and (II.8). ∎

We now establish conditions that ensure (II.11). Our strategy will be to show that there exists an N such that for all $i \geq N$ and $i \in \mathcal{I}_+$, there holds:

$$\Delta_i \leq K\epsilon x. \tag{IV.28}$$

We argue that this ensures (II.11). Assume such an N exists and consider any consecutive elements i, j of $\mathcal{I}_+$, obeying $j > i \geq N$. Define $i < k < j$ as the unique point where $x_k < x$ and $x_{k-1} \geq x$. Then we know that for all $i \leq l \leq k-1$, $\Delta_i \geq \Delta_l$. Likewise for all $k \leq l \leq j$, $\Delta_l \leq \Delta_j$. This proves that if (IV.28) holds for all $i \in \mathcal{I}_+$, and $i \geq N$, then it also holds for all $i \geq N$.

Because of Lemmas 4.1 and 4.3, if any transitioning Δ becomes less than or equal to $K\epsilon x$, all future transitioning Δ's must be bounded by $K\epsilon x$. Thus to prove (II.11), it suffices to have the following condition: That for every $i \in \mathcal{I}_+$, at which $\Delta_i > K\epsilon$, there exists a $j > i$ and $j \in \mathcal{I}_+$, such that $\Delta_j < \Delta_i$. Then as all changes in Δ are factors that are powers of K, (IV.28) must hold for all suitably large $i \in \mathcal{I}_+$. Now suppose, a given $i \in \mathcal{I}_+$, with $\Delta_i > K\epsilon$, has the property that for all $j > i$ and $j \in \mathcal{I}_+$, $\Delta_j \geq \Delta_i$. By Lemma 4.1, at all such j, in fact $\Delta_j = \Delta_i$. By Lemma 4.2 this implies that for all $j \geq i$ and $j \in \mathcal{I}_+$, $x_{j+2} \geq x$. Since, in this case Lemma 3.3 asserts that $x_{j+3} < x$, $x_{j+4} < x$, and $\Delta_{j+4} = \Delta_j$, this also means that $j + 4 \in \mathcal{I}_+$, as failure to transition at this point will result in a large transitioning Δ, thereby violating Lemma 4.1. This argument thus shows the following: if for all N there exists $i \geq N$, such that (IV.28) fails, then there must exist an $i \in \mathcal{I}_+$, such that for all nonnegative integer n,

$$i + 4n \in \mathcal{I}_+, \; x_{i+4n+2} \geq x \text{ and } \Delta_{i+4n} = \Delta_i > K\epsilon x. \tag{IV.29}$$

The result is in fact a 2-cycle in Δ (in this case for all $\Delta_{i+4n+2} = \Delta_i$ and $\Delta_{i+4n+1} = \Delta_i/K$) with potentially large amplitudes.

It is possible for such cycles to occur. Consider for example the situation where for some i

$$\Delta_i \geq \Delta^* = \frac{(1+\alpha^2)|x|}{\alpha - \frac{1}{K}}. \tag{IV.30}$$

Now select, $x_i = x^*$ defined below.

$$x^* = -\text{sgn}(x)\frac{\left(\alpha - \frac{1}{K}\right)\Delta_i}{1+\alpha^2}. \tag{IV.31}$$

Suppose $x > 0$. In this case since $x_i = x^* < 0 < x$,

$$x_{i+1} = \alpha x_i + \Delta_i = \frac{1+\frac{\alpha}{K}}{1+\alpha^2}\Delta_i \geq \frac{\alpha + K}{\alpha K - 1}x > x, \tag{IV.32}$$

where the last inequality is obtained by using the fact that $K > 1$ and $\alpha < 1$. Thus, $i \in \mathcal{I}_+$, and $\Delta_{i+1}e_{i+1} = -\Delta_i/K$. Consequently, from (IV.30)

$$x_{i+2} = \alpha^2 x_i + (\alpha - \frac{1}{K})\Delta_i = \frac{\alpha - \frac{1}{K}}{1+\alpha^2}\Delta_i > x. \tag{IV.33}$$

Thus, from Lemma 3.3,

$$\begin{aligned} x_{i+4} &= \alpha^4 x_i - (\alpha - \frac{1}{K})(1-\alpha^2)\Delta_i \\ &= \alpha^4 x_i + (1+\alpha^2)(1-\alpha^2)x_i = x_i. \end{aligned}$$

Thus, in this case 2-cycles result in Δ and 4-cycles in x_i. Further the resulting Δ_i oscillate with bounds of Δ^* and Δ^*/K. As $\alpha - 1/K$ must be small, Δ^* is much larger than x. Also, the x_i sequence changes sign during this cycle. We show that these features are necessary for such large oscillations in Δ to occur.

Theorem 4.1: Suppose $x > 0$, (respectively, $x < 0$) and at least one of the following two conditions hold: (i) For some $i \in \mathcal{I}_+$, (respectively, $i \in \mathcal{I}_-$), (IV.30) is violated. (ii) For all $i \in \mathcal{I}_+$, (respectively, $i \in \mathcal{I}_-$), $x_i \geq 0$, (respectively, $x_i \leq 0$). Then under (II.2-II.5) and (II.7-II.9) there exists a finite N, such that for all $i \geq N$, (IV.28) holds.

Proof: We will prove the result when $x > 0$. Suppose for every N, there exists $i \geq N$, such that (IV.28) is violated. In view of the argument given after Lemma 4.3, this implies that there exists i such that for all $n \geq 0$, (IV.29) holds. Thus, for all $n \geq 0$, one has (see Lemma 3.3),

$$x_{i+4(n+1)} = \alpha^4 x_{i+4n} - \left(\alpha - \frac{1}{K}\right)(1-\alpha^2)\Delta_{i+4n}.$$

With x^* defined in (IV.31) there follows:

$$x_{i+4(n+1)} - x^* \;\; = \;\; \alpha^4(x_{i+4n} - x^*).$$

Thus

$$\lim_{n\to\infty} x_{i+4n} = x^*, \tag{IV.34}$$

and as $x^* < 0$, (ii) must be violated. Further, observe that as $i \in \mathcal{I}_+$, the second equation in (IV.29) ensures that for all $n \geq 0$

$$x_{i+4n+2} = \alpha^2 x_{i+4n} + (\alpha - \frac{1}{K})\Delta_i \geq x.$$

Because of (IV.34) this requires that

$$x \leq \alpha^2 x^* + (\alpha - \frac{1}{K})\Delta_i = \frac{\alpha - \frac{1}{K}}{1+\alpha^2}\Delta_i$$

where the last equality follows from (IV.31). Thus (IV.30) holds for this $i \in \mathcal{I}_+$, and in fact all subsequent transitioning Δ must be no smaller than this Δ_i. As this Δ_i also obeys the last inequality in (IV.29), because of Lemma 4.1 no previous transitioning Δ can be less than this Δ_i either, i.e. (i) must be violated. ■

The example given before Theorem 4.1 also shows that (i) and (ii) in Theorem 4.1 together constitute sufficient conditions for these potentially large amplitude 4-cycles to be possible, e.g. when $x_0 = x^*$. In the $\alpha = 1$ case, (ii) in Theorem 4.1 is not necessary for such undesirable cycles to occur. In particular, (ii) stems from the requirement of (IV.34). Because of the equation before (IV.34), (IV.34) need not hold if $\alpha = 1$. Further, as noted after Lemma 3.3, when $\alpha = 1$, such cycles in Δ_k, and indeed 4-cycles in x_k are guaranteed for $\alpha = 1$, if even once $x_{i+2} \geq x$ for $i \in \mathcal{I}_+$. This in general is not true when (II.7) holds. We will now prove (II.12).

Theorem 4.2: Suppose under (II.2-II.5) and (II.7-II.9) there exists a finite N, such that for all $i \geq N$, (IV.28) holds. Then (II.12) also holds.

Proof: We will prove the result when $x > 0$. Choose succesive members l and j of $\mathcal{I}_+$, $l < j$ and both greater than N. Clearly, from Lemma 3.3 at most x_{l+1} and x_{l+2} can be greater than or equal to x. Further as $x > 0$, and (II.7) holds, $x_{l+2} < x_{l+1}$. Thus the maximum value of x_k for all $k \in \{l, l+1, \cdots, j\}$ is x_{l+1}. Because of (IV.28),

$$x_{l+1} = \alpha x_l + \Delta_l < (\alpha + K\epsilon)x. \tag{IV.35}$$

Consider now the unique i for which $l < i < j$, $x_{i-1} \geq x$ and $x_i < x$. Then from Lemma 3.3 either $i = l + 2$ or

$i = l+3$, and so from Lemma 3.5 the only candidates for minimum x_k with $k \in \{l, l+1, \cdots, j\}$ are x_i, x_{i+1} and x_{i+2}. Call $\Delta_{i-1} = \Delta \leq K\epsilon x$. Clearly,

$$x_i = \alpha x_{i-1} - \Delta \geq (\alpha - K\epsilon)x. \tag{IV.36}$$

We now show that neither x_{i+1}, nor x_{i+2} can be less than $(\alpha - K\epsilon)x$. If $x_{i+1} \geq x_i$, then of course $x_{i+1} \geq (\alpha - K\epsilon)x$. Suppose, $x_{i+1} < x_i$. As $\Delta_i = \Delta/K$, the second part of Lemma 3.2, leads to the conclusion that $\Delta/K < x(1-\alpha)$. Then,

$$x_{i+1} = \alpha^2 x_{i-1} + \Delta(\frac{1}{K} - \alpha) > x[(\alpha^2 + (1-\alpha K)(1-\alpha)]. \tag{IV.37}$$

Using $K > 1$ and (II.7) one can show that $x_{i+1} > (\alpha - K\epsilon)x$ by comparing the coefficient of x in (IV.37) with $(\alpha - K\epsilon)$. Now consider x_{i+2}. We have

$$x_{i+2} = \alpha^3 x_{i-1} + \Delta\left[1 - \alpha^2 + \frac{\alpha}{K}\right] > \alpha^3 x.$$

Again because $K > 1$, $\alpha^3 > (\alpha - K\epsilon)$. ∎

This Theorem only requires that Δ_i become smaller than $K\epsilon x$ and nothing else.

V. Design guidelines

The parameter ϵ in (II.10) obeys $K\epsilon < K^3(1-\alpha^3)/\alpha$. Thus, for a given K one can make $K\epsilon$ as small as one pleases by making $\alpha \approx 1$. Thus (II.12) indicates that the error in $x_i - x$ can be made arbitrarily small by choosing a sufficiently small $K\epsilon$. Of course, a practical limit on how close α can be made to 1 is imposed by the competing role of α as an instrument to diminish the effect of $x_0 - \hat{X}_0$. Observe, by choosing K to be modest in magnitude, one can achieve the objective of keeping $\alpha K \approx 1$ while still satisfying (II.7), $K > 1$ and $\epsilon \approx 0$.

We now turn to satisfying the requirement to ensure (II.11). The first such design strategy assumes that lower and upper bounds on $|x|$ and its sign are available. Frequently, it is desirable to keep this lower bound greater than zero to permit x to rise above a noise floor. A strategy assuming such a bound, justified in Lemma 5.1, requires that

$$\Delta_0 < \frac{1+\alpha^2}{K(\alpha K - 1)}|x| = \frac{\Delta^*}{K^2}. \tag{V.38}$$

The Lemma assumes that $|x_0| < |x|$ and $x_0 x \geq 0$. We will comment later on how to modify (V.38) when $|x_0| > |x|$ and $x_0 x > 0$.

Lemma 5.1: Consider (II.2-II.5), (II.7-II.9) with (V.38) in force. Suppose $|x_0| < |x|$ and $x_0 x \geq 0$. Then (II.11) and hence (II.12) holds if (V.38) holds.

Proof: Assume $x > 0$. Suppose i is the first element in $\mathcal{I}_+$. From Theorem 4.1 it suffices to show that $\Delta_i < \Delta^*$. If $i \leq 2$ then $\Delta_i \leq K^2 \Delta_0 < \Delta^*$. If $i \geq 3$. As by definition $x_k < x$ for all $k \leq i$ and $i \geq 3$, and

$$x \quad > \quad x_i \geq \frac{\Delta_i}{K^2}(K^2 + \alpha K + \alpha^2).$$

Thus, $\Delta_i < x$. As $\Delta^* > x$, one has $\Delta_i < \Delta^*$, proving the result. ∎

If $|x_0| \geq |x|$ and $x_0 x > 0$ then replace $|x|$ in (V.38) by $|x - x_0|$.

What if the bounds on $|x|$ are not available? Because of Theorem 4.1 undesirably large cycles cycles are indicated by two events occuring together: (a) The x_i undergo sign changes over a period of 4 samples. (b) The Δ_i enter 2-cycles. Occurrence of both are detectable at the transmitter and receiver. Then the following modification of the ADM forces (II.11) to hold. Reduce Δ by a factor of K, whenever (a) and (b) both occur. In the worst case then Δ_i will be artificially reduced below $\min[\Delta^*, K\epsilon x]$, and will never again exceed that value, *allowing (II.12) to hold.* This reduction in Δ_i is also consistent with the motivation of the step-size adaptation rule as (a,b) further indicate that Δ overshoots x. Even when X_k is non-constant, and changes sign, if as is the case with ADM, the sampling rate is well in excess of the bandwidth of X_k, changes in the sign of X_k are unlikely to occur over a period of mere 4 samples. In such a case (a,b) certainly indicate that Δ_i is large enough to overshoot the changes in X_k. Thus, the motivation for adaptation of Δ calls for reducing Δ when (a,b) occur. As Lemmas 3.1 to 4.3, and Theorem 4.2 can be verified to hold even when $x = 0$, this scheme also leads to (II.12) when $x = 0$.

VI. Conclusion

Motivated by networked control applications we have studied the behavior of an ADM algorithm with a forgetting factor, when the coded signal is a constant. We have shown that unlike its counterpart without a forgetting factor, arbitrarily small coding errors can be acheived through suitable design selections. Areas of further work include studying this ADM with non-constant signals with essential bandwidth well below the sampling rate, by using a singular perturbation method. It is also useful to look directly at stabilizability issues in a remote control setting.

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