Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

A Newton-Raphson algorithm for calculating the effects of changes in weights on an $\mathscr{H}_\infty$ design

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***Abstract*— When weights are adjusted in an $\mathscr{H}_\infty$ design, corresponding modifications occur in the synthesised $\mathscr{H}_\infty$ controller and the resulting closed-loop transfer function matrices of interest. This article proposes an algorithm that allow us to compute the corresponding modifications in the controller and closed-loop transfer functions via calculations that are simpler/faster than solving two brand new Riccati equations from scratch. This is particularly beneficial (in terms of both speed and numerical reliability of the solutions) for high order systems and in iterative adaptive control algorithms that involve weight adjustments at each sampling interval. The algorithm proposed here builds upon our previous results presented in [1] and now allow us to also handle medium-size and large weight adjustments, as opposed to only small weight adjustments.**

***Index Terms*— weight modifications, weight adjustments, $\mathscr{H}_\infty$-control, J-lossless factorisation, chain-scattering.**

I. INTRODUCTION

The problem we consider in this article can be formulated as follows: "Suppose that an $\mathscr{H}_\infty$ control problem for a plant P has been solved for some weights W_i and has delivered a central controller K_c and closed-loop transfer functions T_{zw}. Furthermore, suppose that we now consider the same $\mathscr{H}_\infty$ control problem but with weights $W_{i,\text{new}} := W_i + \Delta W_i$ (where ΔW_i is some adjustment). Via calculations simpler than those used in solving an $\mathscr{H}_\infty$ control problem from scratch, find the new controller $K_{c,\text{new}}$ and the new closed-loop transfer functions $T_{zw,\text{new}}$ as a function of ΔW_i and the variables of the first $\mathscr{H}_\infty$ control problem."

This paper is an extension of [1] in three quite distinct ways. Firstly, this paper includes the innovation of using a Newton-Raphson algorithm, thus giving us a method for handling also medium-sized and large weight changes, as opposed to only small weight changes. Secondly, in this article, we present the material using a more general weighting scheme (which is applicable to a wider class of $\mathscr{H}_\infty$ control problems), since in [1] we restricted attention to only $\mathscr{H}_\infty$ loop-shaping weights. Thirdly, we also answer several questions about domain of applicability of the results and validity of assumptions that have been left open in [1].

To explain the problem setting, consider the following weighted four-block $\mathscr{H}_\infty$ control design problem consisting of finding an internally stabilising controller K such that

$$\left\| \begin{bmatrix} W_2 & 0 \\ 0 & W_4 \end{bmatrix} T_{zw} \begin{bmatrix} W_3 & 0 \\ 0 & W_1 \end{bmatrix} \right\|_\infty < \gamma \tag{1}$$

where γ is some positive number, $W_i \in \mathscr{RH}_\infty$ are square weighting functions, T_{zw} is defined by

$$T_{zw} := \begin{bmatrix} P \\ I \end{bmatrix} (I - KP)^{-1} \begin{bmatrix} -K & I \end{bmatrix}. \tag{2}$$

and $P \in \mathscr{RL}_\infty$ is a MIMO (not necessarily square) plant model. Some algebraic manipulations should convince the reader that the above $\mathscr{H}_\infty$-norm objective can be restated as: "Synthesise an internally stabilising controller K such that

$$\left\| \mathcal{F}_l \left(\left[\begin{array}{cc:c} 0 & \frac{1}{\gamma} W_2 P W_1 & \frac{1}{\gamma} W_2 P \\ 0 & \frac{1}{\gamma} W_4 W_1 & \frac{1}{\gamma} W_4 \\ \hdashline -\tilde{W}_3 & \tilde{P}\tilde{W}_1 & \tilde{P} \end{array} \right], \; K \right) \right\|_\infty < 1. \tag{3}$$

The term in the square brackets of inequality (3) is commonly called "the generalised plant" and we shall denote this by Σ.

As presented in [1, Lemma 1], the work of [2] and [3] tells us that as long as the normalised $\mathscr{H}_\infty$ control problem $\|\mathcal{F}_l(\Sigma, K)\|_\infty < 1$ is solvable, then the set of all admissible controllers is given by $\{K = \text{HM}(\Xi, S) \; : \; S \in \mathscr{RH}_\infty, \|S\|_\infty < 1\}$, where Ξ is a unit in $\mathscr{RH}_\infty$ (with a $(q \times q)$ bi-proper Ξ_{22}) that satisfies

$$\Xi \, J_{pq} \, \Xi^\sim = \begin{bmatrix} I_p & 0 \\ \Sigma_{22} & \Sigma_{21} \end{bmatrix} \begin{bmatrix} \Sigma_{12}^\sim \Sigma_{12} & \Sigma_{12}^\sim \Sigma_{11} \\ \Sigma_{11}^\sim \Sigma_{12} & \Sigma_{11}^\sim \Sigma_{11} - I_r \end{bmatrix}^{-1} \begin{bmatrix} I_p & \Sigma_{22}^\sim \\ 0 & \Sigma_{21}^\sim \end{bmatrix}. \tag{4}$$

Since the solution of equation (4) is not unique, the reader is also referred to [4] for instructions on how to pick out one particular solution Ξ which gives a central member

$$K_c := \text{HM}(\Xi, 0) \tag{5}$$

of the above-mentioned admissible controller set that has the correct properties (e.g. minimum entropy) discussed in the literature [5], [6].

Rewriting equation (4) for the $\mathscr{H}_\infty$ control problem posed, we get

$$\Xi \, J_{pq} \, \Xi^\sim = \Gamma(W_{i=\{1,\ldots,4\}}) \tag{6}$$

where

$$\Gamma(W_{i=\{1,\ldots,4\}}) := -\begin{bmatrix} (W_1 W_1^\sim) & 0 \\ 0 & (W_3 W_3^\sim) \end{bmatrix} + \gamma^2 \begin{bmatrix} I \\ P \end{bmatrix} \left[P^\sim (W_2^\sim W_2) P + (W_4^\sim W_4) \right]^{-1} \begin{bmatrix} I & P^\sim \end{bmatrix}. \tag{7}$$

This work was supported by an ARC Discovery-Projects Grant (DP0342683) and National ICT Australia Ltd. National ICT Australia Ltd. is funded through the Australian Government's *Backing Australia's Ability* initiative, in part through the Australian Research Council.

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It is easy to see that once we have the object Ξ (unit in $\mathscr{RH}_\infty$ with bi-proper Ξ_{22}) that solves equation (6), computation of the central controller K_c via equation (5) and computation of the closed-loop transfer function matrices of interest T_{zw} via equation (2) are trivial. Consequently, most of the arguments in this paper will only revolve around the computation of the unimodular object Ξ.

In [1], we analysed the infinitesimal properties of the mapping from weights W_i to the unimodular object Ξ. Put another way, in that article, we examined how an infinitesimally small perturbation in weights W_i maps into an infinitesimally small perturbation in Ξ. It was useful because it gave us a linear approximation to the modification $\Delta\Xi$ in the object Ξ for a given weight adjustments ΔW_i in the weights W_i, with the approximation quality becoming better as the size of the weight adjustments becomes smaller.[1] In fact, in that work, we assumed from the very beginning that all weight adjustments were sufficiently small for the mapping $\Delta W_i \mapsto \Delta\Xi$ to be approximated well by first order terms. In this article, we will refer to such weight adjustments as *"small weight adjustments"*.

All this is summarised in the following theorem (taken out of [1] with minor modifications due to the different weighting scheme considered here) which gives a formula that allows us to compute an approximation of the modification $\Delta\Xi$ in the unimodular object Ξ when "small weight adjustments" ΔW_i are performed.

Theorem 1: Suppose a number $\gamma > 0$, a nominal plant $P \in \mathscr{RL}_\infty$ and some square weights $W_i \in \mathscr{RH}_\infty$ $\forall i \in \{1, \ldots, 4\}$ are given for which the normalised $\mathscr{H}_\infty$ control problem stated in equation (3) *is solvable. Let Ξ (unimodular in $\mathscr{RH}_\infty$) denote the solution of equation* (6) *and force uniqueness on Ξ by pinning down $\Xi(j\infty)$ as described in [4], [11].*

Then consider a "small weight adjustment" ΔW_i for each weight W_i to give corresponding new weights $W_{i,new} := W_i + \Delta W_i$. As a result of these weight changes, the selected Ξ changes to Ξ_{new} and a first order approximation of the change $\Delta\Xi := \Xi_{new} - \Xi$ is given by

$$\Delta\Xi \;\approx\; \Xi\,\Phi J_{pq}, \tag{8}$$

where $\Phi \in \mathscr{RH}_\infty$ satisfies $\Phi(j\infty) = \frac{1}{2}\left[\Xi^{-1}\Gamma(W_{i,new})\Xi^{-\sim} - J_{pq}\right](j\infty)$ (note that $\Phi(j\infty) = 0$ when all ΔW_i are strictly proper) and

$$\Phi + \Phi^{\sim} \;=\; \Xi^{-1}\left(\Gamma(W_{i,new}) - \Gamma(W_i)\right)\Xi^{-\sim} \tag{9}$$

and $\Gamma(\cdot)$ is defined in equation (7)*.*

It is important to clearly understand that in this article we are *not* studying the reverse engineering problem of determining what precise weight change should be made in order to achieve a particular closed-loop transfer function modification. Also, the results in this article cannot be easily applied to that problem because the formula quantifying the controller change is expressed in terms of an object that is only the stable part (in an additive sense) of an $\mathscr{RL}_\infty$ object (see equation (9)). Such additive decompositions (of $\mathscr{RL}_\infty$ objects into $\mathscr{RH}_\infty$ and $\mathscr{RH}_\infty^-$ objects) make it notoriously hard to relate frequency responses and this in turn prohibits use of the results in this paper to solve the reverse-engineering problem. The reader interested in the reverse engineering problem of determining what weight is required in order to achieve a particular closed-loop transfer function is referred to other articles such as [12]–[17] that approach the problem of weight selection in $\mathscr{H}_\infty$ designs from an optimisation perspective.

II. A NEWTON-RAPHSON ALGORITHM FOR MEDIUM-SIZED WEIGHT ADJUSTMENTS

There exist weight adjustments ΔW_i which are of a sufficient size that the approximation of the change $\Delta\Xi$ (given in Theorem 1) using the linear analysis may be too crude. Nevertheless, because Ξ and Ξ_{new} are the solutions of a nonlinear equation (i.e. equation (4)) which happens to be susceptible to solutions via Newton-Raphson approximations, when the weight change ΔW_i is modest but not too large, Ξ_{new} can be determined through Newton-Raphson iterations, with the solution Ξ for the initial weights W_i providing the starting point for the iteration. Each iteration involves only linear equations, and the number of iterations to obtain practical convergence can be small, e.g. 3 or 4, essentially because the Newton-Raphson algorithm, if it converges, has quadratic rate of convergence. In this case, the computational advantage derives from the potentially greater attractiveness of solving 3 or 4 linear equations as opposed to one nonlinear equation.

In this section, we will thus propose a Newton-Raphson iterative algorithm, based on the approximation given in Theorem 1, that guarantees quadratic convergence to the exact solution of the problem with changed weights provided an initial quantity lies in a certain region of attraction. We will also explicitly derive an under-bound on the size of this region of attraction and hence this will give us a handle on precisely what are acceptable "medium-sized weight changes". This algorithm will thus allow us to solve the problem statement of Theorem 1 in situations when the first order approximation (8) is too crude to estimate $\Delta\Xi$ since the weight adjustments are not small enough.

Let us begin by proposing this Newton-Raphson iterative algorithm[2] as follows:

1. Let the counter $k = 0$ and $\Xi_0 = \Xi$.
2. Set $\Gamma_k = \Xi_k J_{pq} \Xi_k^{\sim}$.
3. Solve the following equation for $\Phi_k \in \mathscr{RH}_\infty$ with $\Phi_k(j\infty) = \frac{1}{2}\left[\Xi_k^{-1}\Gamma(W_{i,\text{new}})\Xi_k^{-\sim} - J_{pq}\right](j\infty)$:

$$\Phi_k + \Phi_k^{\sim} \;=\; \Xi_k^{-1}\left(\Gamma(W_{i,\text{new}}) - \Gamma_k\right)\Xi_k^{-\sim}. \tag{10}$$

[1] Linear approximations are also computationally attractive, when compared with the usual calculation for solving an $\mathscr{H}_\infty$ problem typically based on Riccati equation solution, since high order problems involving linear matrix equations are solved faster and with solutions that are numerically more reliable than Riccati (i.e. quadratic) equations, and for that matter linear equations involving transfer functions are solved faster than nonlinear equations (such as J-spectral factorisations [2], [7]–[10]).

[2] We shall show in this section that this algorithm possesses properties (e.g. quadratic convergence) of a Newton-Raphson algorithm.

4. Let $\Xi_{k+1} = \Xi_k(I + \Phi_k J_{pq})$.
5. If $\|\Phi_k\|_\infty \ll 1$, then EXIT. Otherwise, increment the counter k by one and go to step 2.

It should be clear that the steps in this algorithm come from the equations and approximations in Theorem 1. The stopping criterion for this algorithm is chosen as $\|\Phi_k\|_\infty \ll 1$ since this guarantees that $\Xi_{k+1} \approx \Xi_k$ (i.e. practically no improvement in the solution) through the equation $\Xi_{k+1} = \Xi_k(I + \Phi_k J_{pq})$.

A. McMillan degrees do not explode!

The algorithm given above has pedagogical value since it is easily related to Theorem 1. However, on preliminary inspection of the algorithm, one may be concerned that the McMillan degrees of Φ_k and Ξ_k increase at every iteration. This is *not* the case as will be shown next. The technique through which we show that the proposed algorithm does not have explosion of degree is also of independent interest because it suggests a computationally simpler way of performing the algorithm.

Let us first study the relation between the McMillan degree of Φ_{k+1} and Φ_k. Towards this end, from equation (10) (used with indices $k+1$ and k) and other quantities in the above algorithm, observe that

$$\begin{aligned}
&\Phi_{k+1} + \Phi_{k+1}^{\sim} + J_{pq} \\
&\quad = \Xi_{k+1}^{-1}\Gamma(W_{i,\text{new}})\Xi_{k+1}^{-\sim} \\
&\quad = (I + \Phi_k J_{pq})^{-1}\Xi_k^{-1}\Gamma(W_{i,\text{new}})\Xi_k^{-\sim}(I + \Phi_k J_{pq})^{-\sim} \\
&\quad = (I + \Phi_k J_{pq})^{-1}(\Phi_k + \Phi_k^{\sim} + J_{pq})(I + \Phi_k J_{pq})^{-\sim} \\
&\quad = J_{pq} - \big[I - (I + \Phi_k J_{pq})^{-1}\big]J_{pq}\big[I - (I + \Phi_k J_{pq})^{-\sim}\big]
\end{aligned}$$

and hence

$$\begin{aligned}
&\Phi_{k+1} + \Phi_{k+1}^{\sim} = \\
&\quad - \big[I - (I + \Phi_k J_{pq})^{-1}\big]J_{pq}\big[I - (I + \Phi_k J_{pq})^{-\sim}\big]. \quad (11)
\end{aligned}$$

Note that equation (11) can be used as a replacement of equation (10) in the computation of Φ_k $\forall k \in \mathbb{Z}_+$ (although Φ_0 still needs to be computed via equation (10)). Given Φ_k, one could solve equation (11) for Φ_{k+1} directly and easily using state-space data, for example, as follows: Letting

$$\Phi_k = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \quad (12)$$

be a minimal state-space realisation, it is easy to see that a state-space realisation for the right side of equation (11) is given by

$$\left[\begin{array}{cc|c} \hat{A} - \hat{B}J_{pq}R^{-1}\hat{C} & (\hat{B}J_{pq}R^{-1})J_{pq}(\hat{B}J_{pq}R^{-1})^T & -(\hat{B}J_{pq}R^{-1})J_{pq}(I - R^{-1})^T \\ 0 & -(\hat{A} - \hat{B}J_{pq}R^{-1}\hat{C})^T & (R^{-1}\hat{C})^T \\ \hline R^{-1}\hat{C} & (I - R^{-1})J_{pq}(\hat{B}J_{pq}R^{-1})^T & -(I - R^{-1})J_{pq}(I - R^{-1})^T \end{array}\right],$$

where $R = (I + \hat{D}J_{pq})$ and $(\hat{A} - \hat{B}J_{pq}R^{-1}\hat{C})$ is Hurwitz since $(I + \Phi_k J_{pq})^{-1} \in \mathscr{RH}_\infty$[3]. Then, there clearly always exists an X that solves

$$\begin{aligned}
X(\hat{A} - \hat{B}J_{pq}R^{-1}\hat{C})^T + (\hat{A} - \hat{B}J_{pq}R^{-1}\hat{C})X \\
+ (\hat{B}J_{pq}R^{-1})J_{pq}(\hat{B}J_{pq}R^{-1})^T = 0 \quad (13)
\end{aligned}$$

so that a similarity transform $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ on the above state-space realisation yields

$$\Phi_{k+1} = \left[\begin{array}{c|c} \hat{A} - \hat{B}J_{pq}R^{-1}\hat{C} & -X(R^{-1}\hat{C})^T - (\hat{B}J_{pq}R^{-1})J_{pq}(I - R^{-1})^T \\ \hline R^{-1}\hat{C} & -\tfrac{1}{2}(I - R^{-1})J_{pq}(I - R^{-1})^T \end{array}\right], \quad (14)$$

via equation (11). It is unclear whether this last state-space realisation is minimal or not. However, letting $\deg(\cdot)$ denote the McMillan degree of $(\cdot)$, we have shown above that

$$\deg(\Phi_{k+1}) \le \deg(\Phi_k) \qquad \forall k \in \{0\} \cup \mathbb{Z}_+. \quad (15)$$

It is worth pointing out at this stage that one could use equations (12), (13) and (14) to calculate Φ_k at each $k \in \mathbb{Z}_+$ (except $k = 0$) instead of Step 3 of the algorithm.[4] This clearly is computationally more efficient and ensures that the McMillan degree of Φ_k does not increase at each iteration.

Now let us check that the McMillan degree of Ξ_k also does not increase at each iteration. Towards this end, note that

$$\begin{aligned}
& \Phi_k + \Phi_k^{\sim} = \Xi_k^{-1}\Gamma(W_{i,\text{new}})\Xi_k^{-\sim} - J_{pq} \\
\Leftrightarrow\ & \Xi_k\Phi_k J_{pq} + \Xi_k\Phi_k^{\sim}J_{pq} = \Gamma(W_{i,\text{new}})\Xi_k^{-\sim}J_{pq} - \Xi_k \\
\Leftrightarrow\ & \Xi_{k+1} = \Gamma(W_{i,\text{new}})\Xi_k^{-\sim}J_{pq} - \Xi_k\Phi_k^{\sim}J_{pq} \\
\Leftrightarrow\ & \Xi_{k+1} = \begin{bmatrix}\Xi_k & \breve{\Gamma}_{\text{new}}^s\end{bmatrix}\begin{bmatrix} -\Phi_k^{\sim}J_{pq} \\ \breve{\Gamma}_{\text{new}}^a\Xi_k^{-\sim}J_{pq}\end{bmatrix} \quad (16)
\end{aligned}$$

where $\breve{\Gamma}_{\text{new}}^s \in \mathscr{RH}_\infty$ and $\breve{\Gamma}_{\text{new}}^a \in \mathscr{RH}_\infty^-$ satisfy $\breve{\Gamma}_{\text{new}}^s\breve{\Gamma}_{\text{new}}^a = \Gamma(W_{i,\text{new}})$. Note that such a decomposition is easily computed, in state-space data for example, as we do not require $\breve{\Gamma}_{\text{new}}^s$ and $\breve{\Gamma}_{\text{new}}^a$ to be square or units! It follows that

$$\begin{bmatrix}\Xi_{k+1} & \breve{\Gamma}_{\text{new}}^s\end{bmatrix} = \begin{bmatrix}\Xi_k & \breve{\Gamma}_{\text{new}}^s\end{bmatrix}\begin{bmatrix} -\Phi_k^{\sim}J_{pq} & 0 \\ \breve{\Gamma}_{\text{new}}^a\Xi_k^{-\sim}J_{pq} & I\end{bmatrix}. \quad (17)$$

Now, before we proceed, we will need the following lemma whose proof is trivial.

Lemma 2: Given $F, G \in \mathscr{RH}_\infty$ and $H \in \mathscr{RH}_\infty^-$ satisfying $F = GH$, then

$$\deg(F) \le \deg(G).$$

[3]It will be shown in subsection II-B via inequalities (29) and (32) that $\|\Phi_k\|_\infty < 1$ $\forall k \in \{0\} \cup \mathbb{Z}_+$ (provided the initial quantities lie in a region of attraction) which automatically guarantees that $(I + \Phi_k J_{pq})^{-1} \in \mathscr{RH}_\infty$ $\forall k \in \{0\} \cup \mathbb{Z}_+$.

[4]Note that the calculations giving the state-space formula for Φ_{k+1} in equation (14) work also when the given realisation for Φ_k in equation (12) is not necessarily minimal but $\hat{A}$ is Hurwitz.

Furthermore, let $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *and* $H = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ *be state-space realisations with* A *and* $(-\bar{A})$ *Hurwitz. Then*

$$F = \left[\begin{array}{c|c} A & B\bar{D} - X\bar{B} \\ \hline C & D\bar{D} \end{array}\right] \tag{18}$$

where the matrix X *satisfies* $AX - X\bar{A} + B\bar{C} = 0$.

Using Lemma 2 on equation (16), we get

$$\deg(\Xi_{k+1}) \le \deg\left(\begin{bmatrix}\Xi_k & \breve{\Gamma}^s_{\text{new}}\end{bmatrix}\right) \tag{19}$$

and doing the same on equation (17), we get

$$\deg\left(\begin{bmatrix}\Xi_{k+1} & \breve{\Gamma}^s_{\text{new}}\end{bmatrix}\right) \le \deg\left(\begin{bmatrix}\Xi_k & \breve{\Gamma}^s_{\text{new}}\end{bmatrix}\right). \tag{20}$$

Hence, letting $\alpha := \deg\left(\begin{bmatrix}\Xi_0 & \breve{\Gamma}^s_{\text{new}}\end{bmatrix}\right)$, it follows from inequality (20) that

$$\deg\left(\begin{bmatrix}\Xi_k & \breve{\Gamma}^s_{\text{new}}\end{bmatrix}\right) \le \alpha \quad \forall k \in \{0\} \cup \mathbb{Z}_+$$

and thus through inequality (19) that

$$\deg(\Xi_k) \le \alpha \quad \forall k \in \mathbb{Z}_+. \tag{21}$$

Consequently, the McMillan degree of Ξ_k does not increase beyond that of $\begin{bmatrix}\Xi_0 & \breve{\Gamma}^s_{\text{new}}\end{bmatrix}$.

It is worth noting also at this stage that the state-space construction in equation (18) given in Lemma 2 can be used to construct a state-space realisation for Ξ_k at each $k \in \mathbb{Z}_+$ that is of non-increasing order. This is clearly more advantageous than the direct computation in Step 4 of the above algorithm. In order to do this, let

$$\begin{bmatrix}\Xi_k & \breve{\Gamma}^s_{\text{new}}\end{bmatrix} = \left[\begin{array}{c|c} \grave{A} & \grave{B} \\ \hline \grave{C} & \grave{D} \end{array}\right] \tag{22}$$

$$\text{and} \quad \begin{bmatrix}-\Phi_k^{\sim} J_{pq} \\ \breve{\Gamma}^a_{\text{new}} \Xi_k^{-\sim} J_{pq}\end{bmatrix} = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] \tag{23}$$

be realisations with $\grave{A}, (-\bar{A})$ Hurwitz. Then, applying the state-space construction of equation (18) in Lemma 2 on equations (16) and (17), there exists a Y satisfying

$$\grave{A}Y - Y\bar{A} + \grave{B}\bar{C} = 0 \tag{24}$$

and consequently state-space realisations for Ξ_{k+1} and $\begin{bmatrix}\Xi_{k+1} & \breve{\Gamma}^s_{\text{new}}\end{bmatrix}$ are given by

$$\Xi_{k+1} = \left[\begin{array}{c|c} \grave{A} & \grave{B}\bar{D} - Y\bar{B} \\ \hline \grave{C} & \grave{D}\bar{D} \end{array}\right] \tag{25}$$

$$\begin{bmatrix}\Xi_{k+1} & \breve{\Gamma}^s_{\text{new}}\end{bmatrix} = \left[\begin{array}{c|cc} \grave{A} & \grave{B}\bar{D} - Y\bar{B} & \grave{B}\begin{bmatrix}0 \\ I\end{bmatrix} \\ \hline \grave{C} & \grave{D}\bar{D} & \grave{D}\begin{bmatrix}0 \\ I\end{bmatrix} \end{array}\right]. \tag{26}$$

These formulae, from equation (22) to equation (26), can be used repeatedly to generate Ξ_k at each $k \in \mathbb{Z}_+$ on noting that equation (22) is updated by equation (26) and an update of equation (23) can be easily constructed from its individual components Φ_k, Ξ_k and $\breve{\Gamma}^a_{\text{new}}$ which are all available.

B. Convergence of the algorithm

Now let us study how $\Gamma_k = \Xi_k J_{pq} \Xi_k^{\sim}$ changes at each iteration and whether it approaches $\Gamma(W_{i,\text{new}})$ as k increases. If this is the case, then it would also implicitly imply that Ξ_k approaches Ξ_{new} as k increases, which is the desired outcome from this algorithm. To this end, note that from Step 3 of the algorithm we have

$$\left\| \Xi_k^{-1} \Gamma(W_{i,\text{new}}) \Xi_k^{-\sim} - J_{pq} \right\|_\infty \le 2 \left\| \Phi_k \right\|_\infty. \tag{27}$$

Therefore $\Xi_k^{-1}\Gamma(W_{i,\text{new}})\Xi_k^{-\sim} \to J_{pq}$, or equivalently $\Gamma_k = \Xi_k J_{pq} \Xi_k^{\sim} \to \Gamma(W_{i,\text{new}})$, as $\Phi_k \to 0$. Consequently, we only need to show that $\Phi_k \to 0$ as $k \to \infty$, as this will then guarantee that the algorithm yields a solution Ξ_k that converges to Ξ_{new} as $k \to \infty$.

Hence, how do we now prove that the above algorithm does have the property that $\Phi_k \to 0$ as $k \to \infty$? Letting $\sigma_j(\cdot)$ (where $\sigma_j(\cdot) > \sigma_{j+1}(\cdot)$) denote distinct Hankel singular values of $(\cdot)$ each having possibly some multiplicity, $\overline{\sigma}(\cdot)$ denote the maximum singular value of a real or complex matrix and $\beta = \left[\deg\left(\Xi_0^{-1}\Gamma(W_{i,\text{new}})\Xi_0^{-\sim}\right) + \frac{1}{2}\right]$, we get

$$\begin{aligned}
&\|\Phi_{k+1}\|_\infty \\
&\quad \le \overline{\sigma}[\Phi_{k+1}(j\infty)] + 2\sum_{j=1}^{N_{k+1}} \sigma_j(\Phi_{k+1}) \\
&\quad \le \overline{\sigma}[\Phi_{k+1}(j\infty)] + 2N_{k+1}\,\sigma_1(\Phi_{k+1}) \\
&\quad = \frac{1}{2}\overline{\sigma}[\Phi_{k+1}(j\infty) + \Phi_{k+1}(j\infty)^*] \\
&\qquad\qquad + 2N_{k+1} \inf_{\eta \in \mathscr{H}_\infty^-} \|\Phi_{k+1} + \eta\|_\infty \\
&\quad \le \left(2N_{k+1} + \frac{1}{2}\right) \left\|\Phi_{k+1} + \Phi_{k+1}^{\sim}\right\|_\infty \\
&\quad \le \left(2N_{k+1} + \frac{1}{2}\right) \left\|I - (I + \Phi_k J_{pq})^{-1}\right\|_\infty^2 \text{ via (11)} \\
&\quad = \left(2N_{k+1} + \frac{1}{2}\right) \left\|(I + \Phi_k J_{pq})^{-1}\Phi_k\right\|_\infty^2 \\
&\quad \le \beta \frac{\|\Phi_k\|_\infty^2}{(1 - \|\Phi_k\|_\infty)^2}
\end{aligned} \tag{28}$$

where N_{k+1} is the number of distinct Hankel singular values of Φ_{k+1}. Note that $\beta \ge (2N_k + \frac{1}{2}) \quad \forall k \in \mathbb{Z}_+$ since the McMillan degree of Φ_k is non-increasing at each k as shown in inequality (15).

Now we shall use inequality (28) to show that provided the initial Φ_0 lies in certain a region of attraction, then $\|\Phi_k\|_\infty$ decreases at each k down to zero and in fact converges to zero quadratically. Thus, let an under-bound on the region of attraction be defined as follows:

$$\|\Phi_0\|_\infty < \frac{1}{\beta\epsilon} \qquad (\le 1), \tag{29}$$

where

$$\epsilon := \frac{1}{\beta} + \frac{1}{2} + \sqrt{\frac{1}{\beta} + \frac{1}{4}}. \tag{30}$$

If Φ_0 lies in the region specified by inequality (29), then it follows that

$$\frac{1}{(1-\|\Phi_0\|_\infty)^2} < \epsilon \tag{31}$$

as $\left(\frac{\epsilon}{\epsilon-1/\beta}\right)^2 = \epsilon$ from equation (30). Then using inequalities (28), (31) and (29) and noting that the function $\frac{1}{(1-x)^2}$ is monotonically increasing as x increases in the interval $x \in [0, 1)$, we get

$$\|\Phi_{k+1}\|_\infty < \|\Phi_k\|_\infty \qquad \forall k \in \{0\} \cup \mathbb{Z}_+ \tag{32}$$

$$\frac{1}{(1-\|\Phi_k\|_\infty)^2} < \epsilon \qquad \forall k \in \{0\} \cup \mathbb{Z}_+. \tag{33}$$

Hence $\|\Phi_k\|_\infty$ decreases at each k and $\|\Phi_k\|_\infty < 1$ for all $k \in \{0\} \cup \mathbb{Z}_+$ (implying that $(I + \Phi_k J_{pq})^{-1} \in \mathscr{RH}_\infty$ is automatically guaranteed by $\Phi_k \in \mathscr{RH}_\infty$). Also, using inequalities (28) and (33), it is easy to see that

$$\|\Phi_{k+1}\|_\infty < \beta\epsilon \|\Phi_k\|_\infty^2, \tag{34}$$

which in turn yields

$$\|\Phi_k\|_\infty < \frac{1}{\beta\epsilon} (\beta\epsilon \|\Phi_0\|_\infty)^{2^k} \quad \forall k \in \mathbb{Z}_+. \tag{35}$$

This shows that $\|\Phi_k\|_\infty \to 0$ quadratically as $k \to \infty$.

Note that the bound on the region of attraction given in inequality (29) is very conservative due to the series of inequalities above used to obtain it. One would expect that the region of attraction is much larger, as occurs in the later numerical example.

If one insists on using the proposed algorithm for an initial Φ_0 outside the guaranteed region of attraction (29) and notices that the algorithm still converges to a fixed solution, then this solution must be the correct solution Ξ_{new} even though the algorithm started outside the guaranteed region of attraction. This is because from Steps 4 and 3 of the algorithm we can see that as $\Xi_{k+1} \to \Xi_k$ (i.e. the algorithm is converging) then $\Phi_k \to 0$ and consequently $\Gamma_k \to \Gamma(W_{i,\text{new}})$ yielding the required conclusion. It is important to note that this is independent of whether the weight adjustment satisfied inequality (29) or not!

We are now ready to state what we mean by a medium-sized weight change. The term "medium-sized weight change" should be taken to mean: (a) the algorithm given at the beginning of this section converges[5], and (b) the new $\mathscr{H}_\infty$ control problem with changed weight remains solvable.

III. NUMERICAL EXAMPLE

In this section, we will illustrate the results presented in this paper. We will consider the following system $P(s)$:

$$P(s) = \frac{10}{(s-1)(0.2s+1)}. \tag{36}$$

A first four-block $\mathscr{H}_\infty$ controller is designed for the system $P(s)$ with the following weights:

$$W_1(s) = \frac{0.1s+1}{0.003(100s+1)} \qquad W_2 = \frac{1}{20} \tag{37}$$

$$W_3 = \frac{1}{30} \qquad W_4 = 1 \tag{38}$$

As we can see, in this first design, only the sensitivity function T_{22} is effectively constrained via the weights. Using the weights (37)-(38) and the Riccati-based $\mathscr{H}_\infty$ control design method, we obtain the following central controller:

$$K_c = \frac{164.1246(s+4.993)(s+0.8347)}{(s+0.01)(s^2+46.53s+1053)} \tag{39}$$

This first controller delivers three acceptable closed-loop transfer functions on four. The only problem is the closed-loop transfer function $T_{21} = K_c/(1+K_cP)$ whose resonance peak is too high as can be seen in Figure 1. This resonance

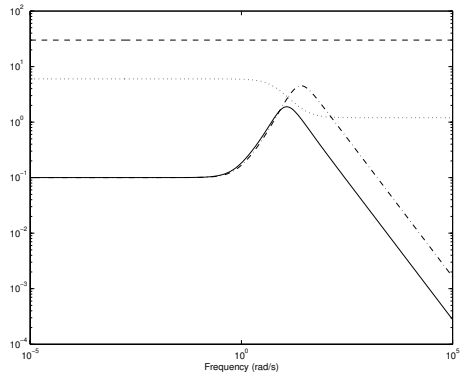

Fig. 1. $|T_{21}(j\omega)|$ (dash-dot) and $|T_{21,new}(j\omega)|$ (solid), $|W_3(j\omega)^{-1}|$ (dashed), $|W_{3,new}(j\omega)^{-1}|$ (dotted)

peak must be decreased. So, we choose the following new weight $W_{3,new}$:

$$W_{3,new} = \frac{\frac{1}{6}s+1}{6(\frac{1}{30}s+1)} \tag{40}$$

The 4-block $\mathscr{H}_\infty$ problem is solved with this new weight keeping unmodified the three others and we obtain the following new central controller:

$$K_{c,new} = \frac{27.8746(s+4.996)(s+30)(s+0.7208)}{(s+25.53)(s+0.01)(s^2+25.15s+307.7)} \tag{41}$$

The controller $K_{c,new}$ is represented in Figure 2 and the new closed-loop transfer functions $T_{21,new}$ in Figure 1. In these two last figures, the modified transfer functions are compared to the corresponding transfer function of the initial $\mathscr{H}_\infty$ problem (i.e. the one with the weights (37) and (38)).

The controller $K_{c,new}$ in (41) has been computed using the classical design method based on Riccati equations. Now, we will show that the results presented in this paper provide us an alternative way to accurately compute the controller $K_{c,new}$. Note nevertheless that, as opposed to the Riccati-based method, our method requires as starting point the

[5] A sufficient condition for convergence is that inequality (29) is satisfied.

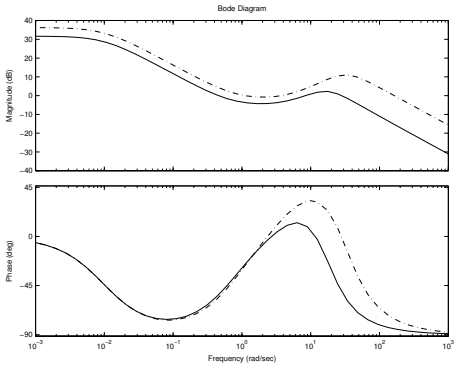

Fig. 2. $|K_c(j\omega)|$ (dash-dot) and $|K_{c,new}(j\omega)|$ (solid)

knowledge of the matrix $\Xi(s)$ corresponding to the initial $\mathscr{H}_\infty$ control design (i.e. the one with the weights (37) and (38)) to compute $K_{c,new}$ (see the algorithm given at the beginning of Section II).

In order to compute the new controller, consider Figure 1 and note that the change of weight $W_{3,new} - W_3$ is quite significant. The significance of the weight change can also be evidenced from the large $\mathscr{H}_\infty$ norm of the quantity $\Phi(s)$ computed via (9): $\|\Phi(s)\|_\infty = 3003.8 >>> 1$. Consequently, $W_{3,new} - W_3$ is a *large weight change*. In this case, in order to compute $K_{c,new}$, we have to divide the *large* weight adjustment $W_{3,new} - W_3$ into a number of *medium-sized* weight changes.

In order to deduce those medium-sized steps, we proceed as follows. As a first guess, we choose to modify $W_3 = 1/30$ to $W_3 = 1/6$ (which is the static gain of the final weight $W_{3,new}$). We then verify whether the Newton-Raphson iterative procedure of Section II can be used to compute the change $\Delta\Xi$ in the matrix Ξ corresponding to this weight change. This is not the case: the algorithm does not converge. Consequently, we have to reduce the weight change. In order to do that, we divide the previous change in two. This leads to a weight change from $W_3 = 1/30$ to $W_3 = 1/10$. For that weight change, the Newton-Raphson algorithm converges and three iterations are sufficient to get an accurate expression of the matrix Ξ corresponding to a weight W_3 equal to $1/10$. We then proceed further by modifying $W_3 = 1/10$ into $W_3 = 1/6$ which now leads to a Newton-Raphson procedure delivering, after three iterations, an accurate expression of the matrix Ξ corresponding to the weight $W_3 = 1/6$. By continuing this procedure, we finally observe that the large weight change W_3 to $W_{3,new}$ can be divided into four *medium-sized* weight changes:

$$W_{3,0} = W_3 \;\rightarrow\; W_{3,1} = \frac{1}{10} \;\rightarrow\; W_{3,2} = \frac{1}{6}$$
$$\rightarrow\; W_{3,3} = \frac{\frac{1}{10}s + 1}{6(\frac{1}{30}s + 1)} \;\rightarrow\; W_{3,4} = W_{3,new}$$

For each of these four steps, only three Newton-Raphson iterations have been necessary to obtain an accurate expression of the corresponding matrix Ξ. Of course, such a division of a large weight change into medium-sized weight changes can be easily automated.

After these four steps, we obtain an expression for the transfer function Ξ_{new} corresponding to the $\mathscr{H}_\infty$ control design problem with weight $W_{3,new}$ and, consequently, using (5), an expression for the central controller $K_{c,new}$ can be deduced:

$$K_{c,new} = \frac{27.8673(s + 0.7208)(s + 4.996)(s + 30)}{(s + 25.53)(s + 0.01)(s^2 + 25.15s + 307.6)}. \quad (42)$$

By comparing (42) and (41), we see that our new methodology and the classical methodology give the same controller.

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