LINEAR OPTIMAL CONTROL SYSTEM DESIGN WITH CONSTRAINED CLOSED-LOOP POLE POSITIONS

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In this paper, we are concerned with the problem of optimizing a linear system, subject to a quadratic performance *criterion*, and, *simultaneously*, *ensuring* that the closed-loop system poles lie to the left of a line $Re[s] = -\alpha$ for some prescribed $\alpha > 0$. We state an optimization procedure, and note additional properties of the closed-loop system.

Optimization of systems described by equations of the form

$$\dot{x} = Fx + Gu; \qquad x(t_0) = x_0 \qquad (1)$$

(x an n-vector, u a p-vector, F and G constant matrices), in the sense of minimizing a performance index

$$V = \int_{t_0}^{\infty} (u'Ru + x'Qx)dt \qquad (2)$$

where R is constant positive definite, Q is constant non-negative definite is a well-known problem in optimal control, see [1-3].

The optimal control law, given by constant linear feedback of the form $u = -K'x$ (at least for a completely controllable system) yields a closed-loop system with several pleasing properties.

(a) if $[F, Q^{\frac{1}{2}}]$ is completely observable, the closed-loop system is asymptotically stable, i.e. its poles all have negative real part, [2, 3].

(b) sensitivity of the system trajectories to plant parameter variations is smaller for the closed-loop system than for the equivalent open-loop system, [3, 4].

(c) a computable amount of time delay may be tolerated in the loop; equivalently, the phase margin of the closed-loop system is at least $60°$. The gain margin is theoretically infinite. [This result is a simple consequence of an identity used to establish (b).]

(d) a substantial amount of nonlinearity may be tolerated at the input without altering the system stability [5].

Another well-known control problem is to define a control law $u = -K'x$ for the system (1), such that the closed-loop system $\dot{x} = (F-GK')x$ has arbitrary poles. Solutions to or remarks concerning this problem may be found in e.g. [6 - 11].

It is clearly important to be able to combine features of the optimal control and pole positioning problems. Other than for a certain asymptotic problem [3], little is however known about the positions of the closed-loop poles of an optimally-designed system. Plainly one cannot hope to completely specify the closed-loop poles and then generate these from an optimal control problem; but it seems reasonable to expect that certain classes of optimal control problems could put general constraints on pole positions.

To obtain one such constraint, we consider here

$$V = \int_{t_0}^{\infty} (u'Ru + x'Qx)e^{2\alpha t}dt \qquad (3)$$

subject to (1), where R and Q are as before, and $\alpha > 0$. We assume $[F, Q^{\frac{1}{2}}]$ is completely observable $[F, G]$ completely controllable. In order that V be finite, it is plainly necessary for the optimal u and x to decay faster than $e^{-\alpha t}$; we thus anticipate a closed-loop system with poles to the left of $Re[s] = -\alpha$.

We now outline the main results.

With the identification $\hat{x} = xe^{\alpha t}$, $\hat{u} = ue^{\alpha t}$, the minimization of (3) subject to (1) becomes equivalent to the minimization of

$$V = \int_{t_0}^{\infty} (\hat{u}'R\hat{u} + \hat{x}'Q\hat{x})dt \qquad (4)$$

subject to

$$\dot{\hat{x}} = (F + \alpha I)\hat{x} + G\hat{u} \qquad \hat{x}(t_0) = e^{\alpha t}x(t_0) \qquad (5)$$

Therefore, taking into account the results of [1 - 3] concerning the optimization problem defined by (1) and (2), and observing that $\hat{u} = -K_\alpha'\hat{x}$ implies $u = -K_\alpha'x$, we have the following result.

<u>The optimal control for the system (1) with performance index (3) is $u = -K_\alpha'x$, where $K_\alpha = P_\alpha GR^{-1}$ and P_α is the unique positive definite solution of</u>

$$P_\alpha(F + \alpha I) + (F' + \alpha I)P_\alpha - P_\alpha GR^{-1}G'P_\alpha + Q = 0 \qquad (6)$$

Now $\hat{u} = -K_\alpha'\hat{x}$, the control law for (5), is such $\dot{\hat{x}} = (F + \alpha I - GK_\alpha')\hat{x}$, the associated closed-loop system has all poles in $Re[s] < 0$. <u>The closed-loop system associated with (1) is $\dot{x} = (F - GK_\alpha')x$, and it follows simply that all poles are in $Re[s] < -\alpha$.</u>

In terms of the quantities defined above, it is possible to show, for the single-input case, that

$$|1 + K_\alpha'(j\omega I-F)^{-1}G|^2 = 1 + G'(-j\omega I-F')^{-1}(2\alpha P_\alpha + Q)(j\omega I-F)^{-1}G$$

Also, with α set equal to zero,

$$|1 + K_0'(j\omega I-F)^{-1}G| = 1 + G'(-j\omega I-F')^{-1}Q(j\omega I-F)^{-1}G$$

Evidently (with inequality holding for almost all ω):

$$|1 + K_\alpha'(j\omega I-F)^{-1}G| \geq |1 + K_0'(j\omega I-F)^{-1}G| \geq 1 \quad (7)$$

(There is also a straightforward generalization of this result to multiple input systems.)

An immediate conclusion follows from (7) in respect of sensitivity, using technical arguments as in [3, 4]: the reduction of sensitivity of trajectories to plant parameter variation in changing from an open- to a closed-loop configuration is greater for $\alpha > 0$ than for $\alpha = 0$.

Equation (7) can also be used to conclude that at a fixed frequency, the closed-loop system corresponding to $\alpha > 0$ can tolerate more phase shift or equivalently more time delay than the closed-loop system corresponding to $\alpha = 0$.

Where $\alpha = 0$ the type of nonlinearity which may be tolerated is of the sort where, instead of having $u = -K'x$, a feedback law of the type $u = -\psi(t)K'x$ exists with $\frac{1}{2} < \psi(t) < \infty$ for all t, and where the time-variation in ψ really reflects nonlinear behaviour. For $\alpha > 0$, it can be shown that the same sort of feedback law is permissible, except that a smaller lower bound on $\psi(t)$ applies. Then for all $\psi(t)$ for which,

$$\frac{1-\beta}{2} < \psi(t) < \infty \qquad \text{for all } t \qquad (8)$$

where $\beta = \lambda_{min}(\alpha P_\alpha + Q) / \lambda_{max}(P_\alpha GG'P_\alpha)$, the matrix P_α being the same as that referred to earlier, the control law $u = -\psi(t)K'x$ yields an asymptotically stable system. For $\alpha > 0$, β is always positive, in general extending the bound over the $\alpha = 0$ case.

Evidently, if a transducer with a known nonlinearity has to be incorporated in a design, it might be possible to experiment with different values of α to secure the desired lower limit in (8).

In retrospect, much appears to be gained by using the performance index (3) rather than (2). One might well ask if anything is lost. The form of (3) suggests that more of the controlling would be done near t_0 then would be the case for (2). Accordingly, one would expect that larger and larger α would put higher and higher control requirements, i.e. energy per unit time, on the closed-loop system (and particularly the input transducers). This appears to be the only obvious drawback.

The paper generates a number of open questions. On the one hand, one could ask what functions $f(t)$ might reasonably replace $e^{\alpha t}$ in (3) to obtain interesting results, and on the other hand, what would ensure other bounds on closed-loop poles of optimally designed systems. It would for example be advantageous to have a scheme ensuring that all closed-loop poles have a damping ratio greater than a prescribed minimum.

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