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STABILITY OF ADAPTIVE DELTA MODULATORS WITH CONSTANT INPUTS

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Abstract

Motivated by applications to feedback control over communication networks where the actuation and feedback signals are transmitted over communication channels, we study the stability of Adaptive Delta Modulators when the coded signal is a constant. The importance of such a setting arises because a common control task is to track a dc input. We show that virtually all combinations of the algorithm parameters result in 4-cycles, that the avoidance of 4-cycles requires a nongeneric initialization, and that steady state oscillations that generically arise can have amplitudes that can be arbitrarily close to the initial error.

Keywords: Adaptive Delta Modulation, Convergence, Stability, Control, Limit Cycles

1 Introduction

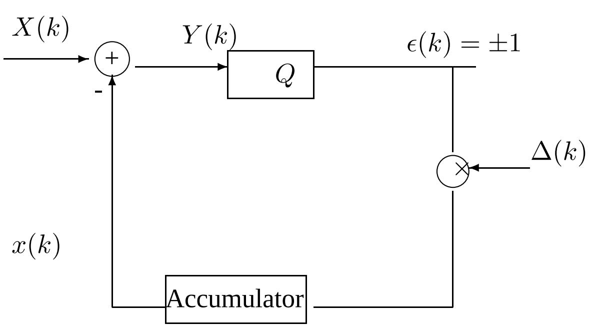

Figure 1. A Delta Modulator at the transmitter

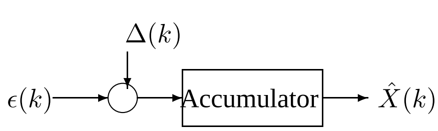

Figure 2. A Delta Modulator at the receiver

Adaptive Delta Modulators (ADM) are used in signal processing and communications for signal quantization with variable step-size. While several variations of this device exist, [1] -[7] the simplest, described in [1] is depicted in figures 1, 2. Essentially, an ADM seeks to increase the dynamic range of the signals that can be accurately tracked while still transmitting only binary values.

Figures 1, 2 depict the ADM algorithm of [1]: fig. 1 is housed in the transmitter, while fig. 2 resides at the receiver. The signal $X(k)$ is coded into the binary sequence $\epsilon(k)$, taking values from $\{-1, 1\}$. It is $\epsilon(k)$ that is actually transmitted. As explained in section 2, the quantity $\Delta(k)$ represents the variable step size. This step size is increased or decreased purely according to the sign pattern in $\epsilon(k)$. Consequently, assuming that the signal at the receiver input is identical to the transmitted value of $\epsilon(k)$, $\Delta(k)$ is known to the receiver and the signal $\hat{X}(k)$ at the receiver is identical to $x(k)$, the accumulator output at the transmitter. Thus should $x(k)$ approach $X(k)$, so also would $\hat{X}(k)$. A heuristic algorithm for updating $\Delta(k)$ with the goal of forcing $\hat{X}(k)$ to approach $X(k)$, is described in [1]. Our principle objective in this paper is to analyze the behavior of $\hat{X}(k)$ and hence $x(k)$ when the signal $X(k)$ is constant.

The motivation for studying the ability of this ADM to track a constant signal stems from networked control systems that are acquiring increasing importance. A typical setting is depicted in figure 3, where the discrete time controller, $C(z)$, is located at a point remote from the continuous time plant, $P(s)$. The actuation signal generated by $C(z)$ and the feedback signal, i.e. the sampled output of the plant $P(s)$, are both conveyed over bandwidth constrained communication channels, and must consequently be quantized prior to transmission. The blocks labeled Tx and Rx are transmitters and receivers, respectively, $w_i(t)$ represent transmission noise, and ZOH represents a zero order hold device, [8]. It has been noted in [9] that variable step quantization of the feedback and actuation signals suffice to achieve closed loop stability. Thus, one must understand the effectiveness of ADM's in this setting, with each Tx and Rx respectively housing figs 1 and 2.

A typical control problem involves forcing the plant output to track a constant signal. This in turn requires that the closed loop input is a constant signal, and to achieve

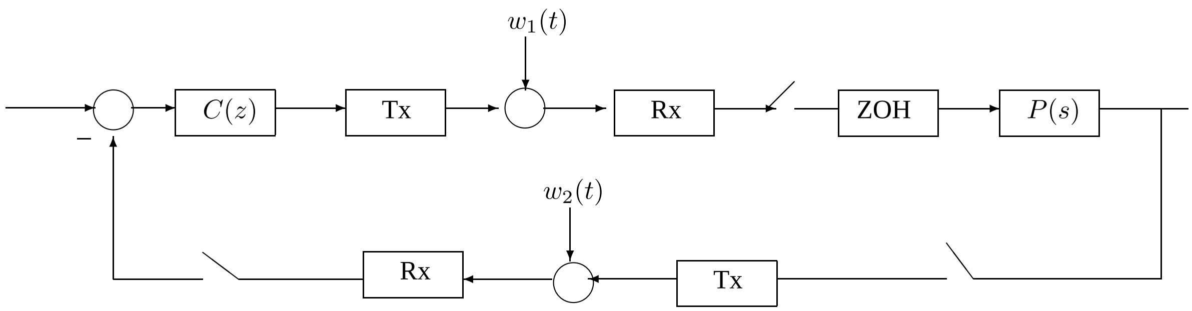

Figure 3. A Remote Control Setting

the desired performance, at steady state both the output of $C(z)$ and the sampled plant output should be constant, i.e. both the signals that the ADM should track should be constants at steady state. Thus at the minimum, desirable performance will necessitate that the signal $\hat{X}(k)$ in fig. 2 track a constant $X(k)$ in fig. 1 with reasonable fidelity.

In section 2 we present the detailed ADM algorithm of [1], and explain the heuristics that motivate it. In section 3 we demonstrate through an example that not only does $\hat{X}(k)$ not always converge to a constant $X(k)$, but rather may enter into 4-cycles of amplitudes comparable to the initial error $X(k) - x(0)$. Section 4 provides a detailed analysis under the assumption of a constant $X(k) = x$ and demonstrates that:

(A) Either $\hat{X}(k)$ converges to x or enters into a 4-cycle.

(B) 4-cycles are avoided only with nongeneric initializations.

Section 5 provides further discussion of these results and section 6 is the conclusion.

2 The detailed algorithm

The detailed algorithm of [1] is given in (2.1) - (2.4) below with $\Delta(0) > 0$ and $K > 1$.

$$x(i+1) = x(i) + \Delta(i)\epsilon(i) \tag{2.1}$$

$$\epsilon(i) = \text{sgn}(X(i) - x(i)) \tag{2.2}$$

$$\Delta(i+1) = \Delta(i)K^{\epsilon(i+1)\epsilon(i)} \tag{2.3}$$

with

$$\text{sgn}(a) = \begin{cases} 1 & \text{if } a \geq 0 \\ -1 & \text{if } a < 0 \end{cases} \tag{2.4}$$

Several features of this algorithm are noteworthy. First observe that as $\epsilon(i)$ is available at the receiver, so is $\Delta(i)$, assuming perfect transmission and an agreed upon value for $\Delta(0)$. This is so as $\Delta(i)$, increases by a factor of K if two successive values of $X(k) - x(k)$ have the same sign (i.e. $\epsilon(i+1)\epsilon(i) = 1$), and decreases by the factor K if two successive values of $X(k) - x(k)$ have opposite signs (i.e. $\epsilon(i+1)\epsilon(i) = -1$). Thus, the reception of the $\epsilon(i)$ sequence permits reproduction of $\Delta(i)$ at the receiver. Consequently if

$$\hat{X}(0) = x(0)$$

then the accumulation

$$\hat{X}(i+1) = \hat{X}(i) + \Delta(i)\epsilon(i) \tag{2.5}$$

ensures that

$$\hat{X}(i) = x(i).$$

Second, observe that (2.5) justifies the association of $\Delta(i)$ with variable step-size as at each sample $\hat{X}(i)$ increases or falls by $\Delta(i)$, depending on whether $x(k)$ and hence $\hat{X}(k)$ is below or above $X(k)$. Third, the motivation for updating $\Delta(i)$ as in (2.2, 2.3) can be understood as follows.

Consider in particular a constant Δ, and figure 4, which simultaneously depicts $X(k)$ and $\hat{X}(k)$. In particular $X(k)$ is the signal that ramps up to a constant value while $\hat{X}(k)$ is the signal that transitions in steps. In the ramping stage it is desirable to have a large Δ so that the rise in $\hat{X}(k)$ tracks $X(k)$ at a fast pace. The obverse occurs when $X(k)$ has acquired a steady state, as in the second part of figure 4, where a large Δ results in a large granularity in the steady state error between $\hat{X}(k)$ and $X(k)$. Contrast this to figure 5 where a smaller Δ is used. The result is slower tracking when $X(k)$ is rising rapidly, but smaller steady state error once $X(k)$ has stopped changing. Together these two examples show that when the signal to be tracked changes quickly, a large Δ is desirable. On the other hand when $X(k)$ is not changing quickly and $\hat{X}(k)$ is close to it, a smaller Δ is desirable. The update laws (2.2, 2.3) judge the quality of tracking by whether or not successive values of $X(k) - \hat{X}(k)$ have the same sign. Their doing so indicates that $\hat{X}(k)$ must approach $X(k)$ at a more rapid rate requiring a larger Δ. If on the other hand the sign of $X(k) - \hat{X}(k)$ alternates then $\hat{X}(k)$ is likely to be close to $X(k)$ and a decrease in Δ is called for.

3 An example

Henceforth we make the following standing assumption.

Assumption 3.1 *The signal $X(k)$ in fig. 1 and (2.2) obeys, for some constant x,*

$$X(k) = x, \quad \forall\, k.$$

■

Now consider the situation where $x(0) = 0$, and $x > 0$. Define L to be an integer such that

$$\Delta(0)\frac{K^L - 1}{K-1} < x \tag{3.6}$$

but

$$\Delta(0)\frac{K^{L+1} - 1}{K-1} \geq x. \tag{3.7}$$

Thus, because of (2.1-2.4),

$$x(i) < x \quad \forall\, i \leq L,$$

$$x(L+1) \geq x$$

and

$$\Delta(L) = \Delta(0)K^L. \tag{3.8}$$

Thus as $\epsilon(L+1) = -1$ and $\epsilon(L) = 1$,

$$\Delta(L+1) = \Delta(L)/K.$$

Further now

$$x(L+2) = x(L) + \Delta(L) - \Delta(L)/K > x(L).$$

Thus a combination of $\Delta(0)$ and x can always be found such that $x(L+2) \geq x$, while the previous equations continue to hold. Then

$$\Delta(L+2) = \Delta(L)$$

and

$$x(L+3) = x(L) + \Delta(L) - \Delta(L)/K - \Delta(L) < x(L) < x.$$

Thus,

$$\Delta(L+3) = \Delta(L)/K$$

and

$$\begin{aligned} x(L+4) &= x(L) + \Delta(L) - \Delta(L)/K - \Delta(L) + \Delta(L)/K \\ &= x(L) < x. \end{aligned}$$

Further, one also has

$$\Delta(L+4) = \Delta(L)$$

connoting the onset of 4-cycles. Further the swing between the maximum and minimum values of $x(i)$ in this cycle is

$$\Delta(L)(K+1)/K = \Delta(0)(K^L + K^{L-1}),$$

which in view of (3.7) and (3.6) has a comparable magnitude to the initial error between x and $x(0)$, being $O(K^L)$. Thus the fidelity of reconstruction is almost as poor as the initial error, in fact approaching it arbitrarily closely for large K.

In the next section we examine just how common the occurence of 4-cycles are.

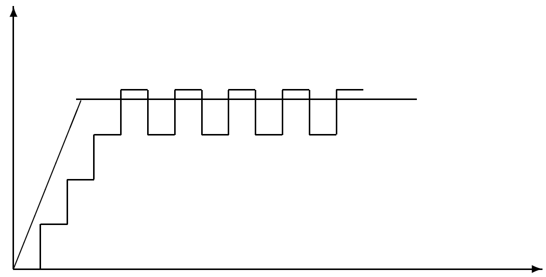

Figure 4. Large Δ

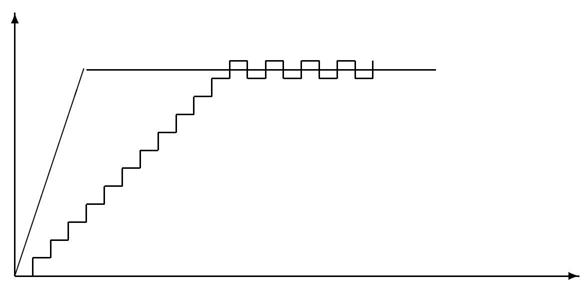

Figure 5. Small Δ

4 The main result

Throughout this section we assume that the standing Assumption 3.1 is in force. In analyzing the stability under Assumption 3.1, we first observe that

$$\Delta(i) > 0 \qquad \text{for all } i. \tag{4.9}$$

The first lemma shows that regardless of the choice of parameters and initializations at some N, $x(N)$ crosses x. This first crossing plays a pivotal role in subsequent analysis.

Lemma 4.1 *Consider (2.1-2.4) under assumption 3.1. Then, for all $x(0), x, \Delta(0) > 0, K > 1$ there exists i such that*

$$sgn(x - x(i))sgn(x - x(i+1)) < 0 \tag{4.10}$$

Proof: Suppose (4.10) is false. Then

$$\epsilon(n) = \epsilon(n+1) \quad \forall n \tag{4.11}$$

and

$$\begin{aligned}\Delta(n) &= \Delta(n-1)K \\ &= K^n\Delta(0)\forall n.\end{aligned}$$

Thus

$$x(n) = x(0) + \epsilon(0)\Delta(0)\sum_{i=0}^{n-1} K^i$$

which leads to

$$x - x(n) = x - x(0) - \text{sgn}(x - x(0))\Delta(0)\sum_{i=0}^{n-1} K^i$$

as $K > 1$

$$\Delta(0)\sum_{i=0}^{n-1} K^i$$

must for some n have magnitude greater than $|x(0) - x|$. Thus at this n, sgn$(x - x(0))$ is different from sgn$(x - x(n))$. ■

Choose N to be the smallest integer for which

$$\text{sgn}(x - x(0))\text{sgn}(x - x(N)) = -1.$$

Observe that for a given K, x, $\Delta(0)$ and $x(0)$, N is uniquely determined. Specifically,

$$N = \begin{cases} 1 + \left\lfloor \log_K \left[1 + \frac{(K-1)|x-x(0)|}{\Delta(0)}\right]\right\rfloor & \text{if } x \geq x(0) \\ \left\lceil \log_K \left[1 + \frac{(K-1)|x-x(0)|}{\Delta(0)}\right]\right\rceil & \text{if } x < x(0) \end{cases}. \tag{4.12}$$

We now have a lemma that shows that for all $i \geq N$, three consecutive values of sgn$(x - x(i))$ cannot be same, and that further if at any point two consecutive values of sgn$(x - x(i))$ are the same then 4-cycles immediately eventuate.

Lemma 4.2 *Consider the algorithm in (2.1-2.4) under Assumption 3.1, and N as defined in (4.12). Then the following hold:*

(a) If

$$\textit{sgn}(x - x(i))\textit{sgn}(x - x(i+1)) < 0 \tag{4.13}$$

then either (i) or (ii) below hold.

(i) sgn$(x - x(i+2))$sgn$(x - x(i)) > 0$, or

(ii) sgn$(x - x(i+3))$sgn$(x - x(i)) > 0$.

(b) If for any $i \geq N$,

$$\textit{sgn}(x - x(i))\textit{sgn}(x - x(i+1)) > 0, \tag{4.14}$$

then

$$\textit{sgn}(x-x(i-1)) = -\textit{sgn}(x-x(i)) = \textit{sgn}(x-x(i+2)). \tag{4.15}$$

(c) If for any $i \geq N$,

$$\textit{sgn}(x - x(i))\textit{sgn}(x - x(i+1)) = 1 \tag{4.16}$$

then for all $n \geq i-1$, $x(n+4) = x(n)$ and $\Delta(n+4) = \Delta(n)$.

Proof: Suppose (4.13) holds but (i) fails. Then

$$\Delta(i+1) = \Delta(i)/K \tag{4.17}$$

and

$$\Delta(i+2) = \Delta(i). \tag{4.18}$$

Thus

$$\begin{aligned} x - x(i+3) &= x - x(i) - \text{sgn}(x - x(i))\Delta(i)[1 - 1/K - 1] \\ &= x - x(i) + \text{sgn}(x - x(i))\Delta(i)/K, \end{aligned}$$

and hence (ii) must hold because of (4.9) and the definition of sgn.

To prove (b) one simply notes that in view of (a) an inductive argument proves that after N, two consecutive crossings by $x(i)$ of x cannot be separated by more that 2-samples.

Now suppose (4.16) holds. Then as $i \geq N$, we have from (a,b) that (4.15) holds. Thus

$$\Delta(i) = \Delta(i-1)/K, \tag{4.19}$$

$$\Delta(i+1) = \Delta(i-1) \tag{4.20}$$

and

$$\Delta(i+2) = \Delta(i-1)/K \tag{4.21}$$

$$\begin{aligned} x(i+3) &= x(i-1) \\ &+ \text{sgn}(x - x(i-1)) \\ &\quad \Delta(i-1)[1 - 1/K - 1 + 1/K] \\ &= x(i-1). \end{aligned}$$

Thus

$$\text{sgn}(x-x(i+2)) = \text{sgn}(x-x(i+3)) = \text{sgn}(x-x(i-1))$$

and hence from (4.21),

$$\Delta(i+3) = \Delta(i-1).$$

Thus the result holds. ■

Remark 4.1 *Thus the only way that 4-cycles can be avoided is if once $x(i)$ crosses x, sgn$(x - x(i))$ alternates between ± 1.*

This leads to our main result:

Theorem 4.1 *Consider the algorithm in (2.1-2.4) under Assumption 3.1 with $\Delta(0) > 0$, $K > 1$. Then with N as defined either $x(i)$ enters a 4-cycle or*

$$\lim_{k\to\infty} x(k) = x. \tag{4.22}$$

Further (4.22) holds, iff

$$|x - x(N)| = \Delta(N)\frac{K}{K+1}. \tag{4.23}$$

Remark 4.2 *Note that $x(N)$, N, $\Delta(N)$ are completely determined by $x(0)$, x, $\Delta(0)$ and K through (4.12). Thus, the values of $x(0)$, x, $\Delta(0)$ and K that result in the satisfaction of (4.23) form a set of zero measure. In fact with N as in (4.12), (4.23) becomes*

$$|x - x(0) - \mathit{sgn}(x - x(0))\Delta(0)\frac{K^N - 1}{K-1}| = \Delta(0)\frac{K^{N+1}}{K+1}. \tag{4.24}$$

Proof: Suppose 4-cycles do not occur. Then from Remark 4.1 for all $i \geq N$, $\text{sgn}(x - x(i))\text{sgn}(x - x(i+1)) = -1$. Thus

$$\text{sgn}(x - x(i)) = \begin{cases} \text{sgn}(x - x(N)) & i - N \text{ even} \\ -\text{sgn}(x - x(N)) & i - N \text{ odd} \end{cases} \tag{4.25}$$

and

$$\epsilon(i)\Delta(i) = \text{sgn}(x - x(N))\Delta(N)\left(\frac{-1}{K}\right)^{i-N}. \tag{4.26}$$

Further, under (4.26, 4.25), and all $i \geq N$

$$\begin{aligned} x(i) &= x(N) + \sum_{n=N}^{i-1}\Delta(n)\epsilon(n) \\ &= x(N) + \Delta(N)\text{sgn}(x - x(N))\sum_{n=N}^{i-1}\left(\frac{-1}{K}\right)^{n-N} \\ &= x(N) + \Delta(N)\text{sgn}(x - x(N))\frac{1 - \left(\frac{-1}{K}\right)^{i-N}}{\frac{1}{K}+1} \\ &= x(N) + \text{sgn}(x - x(N))\frac{K}{K+1}\Delta(N) \\ &\quad \left[1 - \left(\frac{-1}{K}\right)^{i-N}\right]. \end{aligned}$$

Hence

$$\begin{aligned} x - x(i) &= \text{sgn}(x - x(N))\left[|x - x(N)| - \frac{K}{K+1}\Delta(N)\right. \\ &+ \left.\frac{K}{K+1}\Delta(N)\left(\frac{-1}{K}\right)^{i-N}\right]. \end{aligned} \tag{4.27}$$

If (4.23) holds then for all $i \geq N$

$$x - x(i) = \text{sgn}(x - x(N))\frac{K}{K+1}(\frac{-1}{K})^{i-N}\Delta(N)$$

which goes exponentially to zero and obeys (4.25). If (4.23) is violated, then for some $\delta \neq 0$,

$$|x - x(N)| = \Delta(N)\frac{K}{K+1} + \delta. \tag{4.28}$$

Then (4.27) becomes

$$x - x(i) = \text{sgn}(x - x(N))\left[\delta + \frac{K}{K+1}\left(\frac{-1}{K}\right)^{i-N}\Delta(N)\right]. \tag{4.29}$$

Hence there exists M such that for all $i > M$

$$\text{sgn}(x - x(i)) = \text{sgn}(\delta(x - x(N)))$$

which violates (4.25), necessary for the absence of 4-cycles.

∎

5 Discussion

Thus to avoid 4-cycles one must have the satisfaction of (4.23). The example in section 3 also demonstrates that these cycles could occur in a way that there is only a modest improvement in the resulting tracking error over the initial value of $|x - x(0)|$, especially if K is large, in turn required for fast tracking. In a control setting stabilization of the closed loop dynamics would generally require that the value of K be large enough to keep pace with the most unstable mode of the plant in the transient phase, posing a competing challenge with the need of accurately converging to the steady state values of the ADM input. This example also shows that performance is potentially sensitive to the selection of $x(0)$.

Consider now (4.23) further. By the definition of N,

$$\text{sgn}(x - x(N) = -\text{sgn}(x - x(0)).$$

Then (4.24) reduces to

$$|x - x(0)| = \Delta(0)\left[\frac{K^N - 1}{K-1} - \frac{K^{N+1}}{K+1}\right]. \tag{5.30}$$

Thus convergence necessitates that there exist a positive integer N, for which (5.30) holds. For a specified choice of $\Delta(0)$ and K, the right hand side of (5.30) can only take a countable set of values. Therefore it follows that for a given combination of x, $\Delta(0)$ and K, only a countable set of $x(0)$ will lead to convergence, making convergence a highly non-robust prospect, as x of course is not a design parameter available to base this initialization on.

We next examine the relation between the overall swing in the 4-cycles that occurs should (4.23) be violated, and the extent of this violation. Specifically, suppose for some $\delta \neq 0$, (4.28) holds. Suppose j is the smallest index greater than or equal to N for which

$$\text{sgn}(x - x(j)) = \text{sgn}(x - x(j+1)). \tag{5.31}$$

Of course by definition

$$\text{sgn}(x - x(j)) = -\text{sgn}(x - x(j-1)). \tag{5.32}$$

Observe from Lemma 4.2 that this signals the onset of 4-cycles. Analyis in the proof of Lemma 4.2 also shows that the swing between the maximum and minimum values of this 4-cycle must equal

$$\frac{\Delta(j-1)(K+1)}{K}. \tag{5.33}$$

Now for all $j \geq i \geq N$, (4.25) must hold. Then in view of (4.26) and (4.29), one has for all $j \geq i \geq N$,

$$x - x(i) = \text{sgn}(x - x(N)) \left[\delta + \frac{K}{K+1} \epsilon(i) \Delta(i) \right]. \quad (5.34)$$

We now argue that

$$\Delta(j-1) \geq |\delta|(K+1)/K. \quad (5.35)$$

Indeed if

$$\Delta(j-1) < |\delta|(K+1)/K$$

then because of (5.32),

$$\Delta(j) < |\delta|(K+1)/K^2.$$

Then from (5.34) that

$$\text{sgn}(x - x(j)) = \text{sgn}(x - x(j-1)) = \text{sgn}(x - x(N))\text{sgn}(\delta),$$

violating (5.32). Thus (5.35) holds. Consequently, the resulting swing cannot be smaller than

$$|\delta|(K+1)^2/K^2.$$

It is evident from (5.30) and the definition of N that both N and hence the range of values that δ can take increase with the intial error $|x - x(0)|$. Consequently, barring fotuitous initializations, large oscillations in the likely 4-cycles is quite possible.

To complete the picture let us examine how large $|\delta|$ can potentially be. Observe

$$|\delta| = \left| |x - x(0)| - \Delta(0) \left[\frac{K^N - 1}{K-1} - \frac{K^{N+1}}{K+1} \right] \right|. \quad (5.36)$$

From the definition of N it is possible that

$$\Delta(0) \frac{K^N - 1}{K-1} \approx |x - x(0)|.$$

Then from (5.36)

$$|\delta| \approx \frac{K(K-1)}{K+1} |x - x(0)| + \frac{K\Delta(0)}{K+1}. \quad (5.37)$$

Thus for large K, the swing in the oscillations can be of comparable magnitude to the initial error.

6 Conclusion

Motivated by networked control applications we have studied the behavior of the ADM algorithm of [1] when the input to be coded is a constant. We have shown that for generic initializations, convergence is not possible, and 4-cycles must arise. We have also shown by example that these 4-cycles could result in large coding errors. Though motivated by control considerations, these results also bring into question the efficacy of this algorithm in communications and signal processing settings.

Areas of further work could involve studying this ADM with non-constant signals. Possible approaches may involve looking at external signals whose frequency content is concentrated well below the sampling rate. Then it might be that a singular perturbation argument could allow one to draw similar conclusions to those obtained for a constant signal. It is also useful to look directly at stabilizability issues concerning the setting of fig. 3 when the ADM blocks are used in transception.

7 Acknowledgements

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