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Information Structures to Secure Control of Rigid Formations with Leader-Follower Architecture

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Abstract— **This paper is concerned with rigid formations of mobile autonomous agents that have leader-follower architecture. In a previous paper, Baillieul and Suri gave a proposition as a necessary condition for stable rigidity. They also gave a separate theorem as a sufficient condition for stable rigidity. This paper suggests an approach to analyze rigid formations that have leader-follower architecture. It proves that the third condition in the proposition given by Baillieul and Suri is redundant, and it proves that this proposition is a necessary and sufficient condition for stable rigidity. Simulation results are also presented to illustrate rigidity.**

I. Introduction

Formations of autonomous agents have lately received considerable attention due to recent advances in computation and communication technologies (see for example [1], [2], [8], [9], [10], [14], [15], [18]). A *formation* is a group of agents moving in real 2- or 3-dimensional space. A formation is called *rigid* if the distance between each pair of agents does not change over time under ideal conditions. Sensing/communication links are used for maintaining fixed distances between agents. The interconnection structure of sensing/communication links is called *sensor/network topology*. In practice, actual agent groups cannot be expected to move exactly as a rigid formation because of sensing errors, vehicle modelling errors, etc. The ideal benchmark point formation against which the performance of an actual agent formation is to be measured is called a *reference formation*. In reality, agents are entities with physical dimensions. For modeling purposes, agents are represented by points called *point agents*. Distances between all agent pairs can be held fixed by directly measuring distances between only some agents and keeping them at desired values.

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Two agents connected by a sensing/communication link are called *neighbors*. There are two types of neighbor relations in rigid formations. In the first type, the neighbor relation is symmetric, i.e., if agent i senses/communicates with agent j and uses the received information (such as distance or bearing information) for motion planning, so does agent j with agent i. A link with a symmetric neighbor relation is represented graphically by a straight line. In the second type, the neighbor relation is asymmetric, i.e., if agent i senses/communicates with agent j and uses the received information (such as distance or bearing) for motion planning, then agent j does not make use of any information received from agent i although it may sense/communicate with agent i. For example, rigid formations with a leader-follower architecture have the asymmetric neighbor relation. A link with an asymmetric neighbor relation between a leader and a follower is represented by a directed edge pointing from the follower to the leader. The terms 'undirected formation' and 'directed formation' are also used to describe formations with symmetric neighbor relations and formations with leader-follower architecture [14]. Eren et al. [5], [6] and Olfati-Saber and Murray [9] suggested an approach based on rigidity for maintaining formations of autonomous agents with sensor/network topologies that use distance information between agents, where the neighbor relation is symmetric. For formations that have a leader-follower architecture, Baillieul and Suri gave two separate conditions for stable rigidity for formations that use distance information between agents, one of which is a necessary condition and the other is a sufficient condition [1]. Desai et al. made use of both distance and bearing information to maintain formations that have a leader-follower architecture [4]. This paper suggests an approach to analyze rigid formations with a leader-follower architecture and proves that the necessary condition given by Baillieul and Suri is a necessary and sufficient condition for stable rigidity in formations that have a leader-follower architecture.

In this paper, we will restrict our attention to formations in 2-dimensional space. The paper is organized as follows. In §II, we start with definitions of point formations and rigidity. We then review rigid formations with symmetric neighbor relations. We investigate stably rigid formations that have a leader-follower architecture in §III. Simulation results are given in §IV Finally, the paper ends with concluding remarks in §V.

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II. Rigidity and Point Formations

A point formation $\mathbb{F}_p \triangleq (p, \mathcal{L})$ provides a way of representing a formation of n agents. $p \triangleq \{p_1, p_2, \ldots, p_n\}$ and the points p_i represent the positions of agents in $\mathbb{R}^d$ $\{d = 2$ or $3\}$ where i is an integer in $\{1, 2, \ldots, n\}$ and denotes the labels of agents. $\mathcal{L}$ is the set of "maintenance links," labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$. The *maintenance links* in $\mathcal{L}$ correspond to constraints between specific agents, such as distances, which are to be maintained over time by using sensing/communication links between certain pairs of agents. Each point formation $\mathbb{F}_p$ uniquely determines a graph $\mathbb{G}_{\mathbb{F}_p} \triangleq (\mathcal{V}, \mathcal{L})$ with vertex set $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$, which is the set of labels of agents, and edge set $\mathcal{L}$. A formation with distance constraints can be represented by $(\mathcal{V}, \mathcal{L}, f)$ where $f : \mathcal{L} \longmapsto \mathbb{R}$. Each maintenance link $(i, j) \in \mathcal{L}$ is used to maintain the distance $f((i, j))$ between certain pairs of agents fixed.

A *trajectory* of a formation is a continuously parameterized one-parameter family of curves $(q_1(t), q_2(t), ..., q_n(t))$ in $\mathbb{R}^{nd}$ which contain p and on which for each t, $\mathbb{F}_{q(t)}$ is a formation with the same measured values under f, g. A *rigid motion* is a trajectory along which point formations contained in this trajectory are congruent to each other. We will say that two point formations $\mathbb{F}_p$ and $\mathbb{F}_r$, where $p, r \in q(t)$, are congruent if they have the same graph and if p and r are congruent. p is *congruent* to r in the sense that there is a distance-preserving map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T(r_i) = p_i, i \in \{1, 2, \ldots, n\}$. If rigid motions are the only possible trajectories then the formation is called *rigid*; otherwise it is called *flexible* [6].

A. Rigidity in Point Formations with Symmetric Neighbor Relations

Whether a given point formation is rigid or not can be studied by examining what happens to the given point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ with m maintenance links, along the trajectory $q([0, \infty)) \triangleq \{\{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ on which the Euclidean distances $d_{ij} \triangleq ||p_i - p_j||$ between pairs of points (p_i, p_j) for which (i, j) is a link are constant. Along such a trajectory

$$(q_i - q_j) \cdot (q_i - q_j) = d_{ij}^2, \quad (i, j) \in \mathcal{L}, \quad t \geq 0 \quad (1)$$

We note that the existence of a trajectory is equivalent to the existence of a piecewise analytic path, with all derivatives at the initial point [12]. Assuming a smooth (piecewise analytic) trajectory, we can differentiate to get

$$(q_i - q_j) \cdot (\dot{q}_i - \dot{q}_j) = 0, \quad (i, j) \in \mathcal{L}, \quad t \geq 0 \quad (2)$$

Here, $\dot{q}_i$ is the velocity of point i. The m equations can be collected into a single matrix equation

$$R_{\mathcal{L}}(q)\dot{q} = 0 \quad (3)$$

where $\dot{q} = [\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n]^T$ (T stands for transpose) and $R_{\mathcal{L}}(q)$ is a specially structured $m \times 2n$ matrix called the *rigidity matrix* [11], [16], [17].

Example 2.1: Consider a planar point formation $\mathbb{F}_p$ shown in Figure 1. This has a rigidity matrix as shown in Table I.

Let $\mathcal{M}_p$ be the manifold of points congruent to p. Because any trajectory of $\mathbb{F}_p$ which lies within $\mathcal{M}_p$, is one along which $\mathbb{F}_p$ undergoes rigid motion, (2) automatically holds along any trajectory which lies within $\mathcal{M}_p$. From this, it follows that the tangent space to $\mathcal{M}_p$ at p, written $\mathcal{T}_p$, must be contained in the kernel of $R_{\mathcal{L}}(p)$. If the points $p_1, p_2, \ldots, p_n$ are in general position (which means that the points $p_1, p_2, \ldots, p_n$ do not lie on any hyperplane in $\mathbb{R}^n$), then $\mathcal{M}_p$ is $n(n+1)/2$ dimensional since it arises from the $n(n-1)/2$-dimensional manifold of orthogonal transformations of $\mathbb{R}^n$ and the n-dimensional manifold of translations of $\mathbb{R}^n$ [11]. Thus $\mathcal{M}_p$ is 3-dimensional for $\mathbb{F}_p$ in $\mathbb{R}^2$. We have rank $R_{\mathcal{L}}(p) = 2n -$ dimension$\{$kernel$(R_{\mathcal{L}}(p))\}$ $\leq 2n - n(n+1)/2$. The following theorem holds [11], [16]:

Theorem 2.2: Assume $\mathbb{F}_p$ is an n-point formation with at least 2 points in 2-dimensional space where rank $R_{\mathcal{L}}(p) = \max\{\text{rank } R_{\mathcal{L}}(x) : x \in \mathbb{R}^2\}$. $\mathbb{F}_p$ is rigid in $\mathbb{R}^2$ if and only if

$$\text{rank } R_{\mathcal{L}}(p) = 2n - 3.$$

This theorem leads to the notion of the "generic" behavior of rigidity. When the rank is less than the maximum, the formation may still be rigid. However this type of rigidity lacks the generic behavior and thus is not addressed in this paper.

1) Generic Rigidity: We define a type of rigidity, called "generic rigidity," that is more useful for our purposes. A set $\mathcal{A} = (\alpha_1, \ldots, \alpha_m)$ of distinct real numbers is said to be *algebraically dependent* if there is a non-zero polynomial $h(x_1, \ldots, x_m)$ with integer coefficients such that $h(\alpha_1, \ldots, \alpha_m) = 0$. If $\mathcal{A}$ is not algebraically dependent, it is called *generic* [3]. We say that $p = (p_1, \ldots, p_n)$ is generic in 2-dimensional space, if its $2n$ coordinates are generic. It can be shown that the set of generic p's form an open connected dense subset of $\mathbb{R}^{2n}$ [13]. A graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is called *generically rigid*, if $\mathbb{F}_p = (p, \mathcal{L})$ is rigid for a generic p.

The concept of generic rigidity does not depend on the precise distances between the points of $\mathbb{F}_p$ but examines how well the rigidity of formations can be judged by

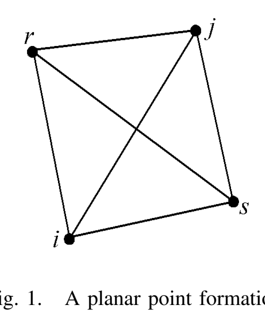

Fig. 1. A planar point formation.

$R_{\mathcal{L}}(p)$	i		j		r		s	
(i,j)	$x_i - x_j$	$y_i - y_j$	$x_j - x_i$	$y_j - y_i$	0	0	0	0
(i,r)	$x_i - x_r$	$y_i - y_r$	0	0	$x_r - x_i$	$y_r - y_i$	0	0
(i,s)	$x_i - x_s$	$y_i - y_s$	0	0	0	0	$x_s - x_i$	$y_s - y_i$
(j,r)	0	0	$x_j - x_r$	$y_j - y_r$	$x_r - x_j$	$y_r - y_j$	0	0
(j,s)	0	0	$x_j - x_s$	$y_j - y_s$	0	0	$x_s - x_j$	$y_s - y_j$
(r,s)	0	0	0	0	$x_r - x_s$	$y_r - y_s$	$x_s - x_r$	$y_s - y_r$

TABLE I

RIGIDITY MATRIX EXAMPLE FOR DISTANCES

knowing the vertices and their incidences, in other words, by knowing the underlying graph. For this reason, it is a desirable specialization of the concept of a "rigid formation" for our purposes. The following theorem holds for a generically rigid graph [16]:

Theorem 2.3: The following are equivalent:

1) a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is generically rigid in 2-dimensional space;
2) for some p, the formation $\mathbb{F}_p$ with the underlying graph $\mathbb{G}$ has rank$\{R_{\mathcal{L}}(p)\}= 2|\mathcal{V}| - 3$ where $|\mathcal{V}|$ denotes the cardinal number of $\mathcal{V}$;
3) for almost all p, the formation $\mathbb{F}_p$ with the underlying graph $\mathbb{G}$ is rigid.

For 2-dimensional space, we have a complete combinatorial characterization of generically rigid graphs, which was first proved by Laman in 1970 [7].

Theorem 2.4 (Laman [7]): A graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is generically rigid in 2-dimensional space if and only if there is a subset $\mathcal{L}' \subseteq \mathcal{L}$ satisfying the following two conditions: (1) $|\mathcal{L}'| = 2|\mathcal{V}| - 3$, (2) For all $\mathcal{L}'' \subseteq \mathcal{L}', \mathcal{L}'' \neq \emptyset, |\mathcal{L}''| \leq 2|\mathcal{V}(\mathcal{L}'')| - 3$, where $|\mathcal{V}(\mathcal{L}'')|$ is the number of vertices that are end-vertices of the edges in $\mathcal{L}''$.

III. RIGIDITY IN POINT FORMATIONS WITH LEADER-FOLLOWER ARCHITECTURE

First, we give some definitions from graph theory, which are relevant to point formations with leader-follower architecture. A graph in which each edge is replaced by a directed edge is called a *directed graph*, also called a *digraph*. When there is a danger of confusion, we will call a graph, which is not a directed graph, an *undirected graph*. A directed graph having no multiple edges or loops (corresponding to a binary adjacency matrix with 0's on the diagonal) is called a *simple directed graph*.

An *arc*, or *directed edge*, is an ordered pair of end-vertices. It can be thought of as an edge associated with a direction. Each directed edge is denoted with a line directed from the first element to the second element of the pair. For example, for a given directed edge (i, j), the direction is from i to j. Symmetric pairs of directed edges are called *bidirected edges*. In the context of formations, a birected edge is equivalent to an undirected edge in the underlying graph of a formation. We will use only directed graphs with no bidirected edges in formations that have a leader-follower architecture. The number of inward directed graph edges to a given graph vertex i in a directed graph $\mathbb{G}$ is called the *in-degree* of the vertex and is denoted by $d^-_{\mathbb{G}}(i)$. The number of outward directed graph edges from a given graph vertex i in a directed graph $\mathbb{G}$ is called the *out-degree* of the vertex and is denoted by $d^+_{\mathbb{G}}(i)$. The set of neighbors of i such that the directed edge is pointed from i to the other vertex, denoted by $\mathcal{N}_{\mathbb{G}}(i)$, is called a (open) neighborhood. When i is also included, it is called a closed neighborhood and is denoted by $\mathcal{N}_{\mathbb{G}}[i]$. The out-neighborhood $\mathcal{N}^+_{\mathbb{G}}(i)$ of a vertex i is $\{j \in \mathcal{V} : (i, j) \in \mathcal{L}\}$, and the in-neighborhood $\mathcal{N}^-_{\mathbb{G}}(i)$ of a vertex i is $\{j \in \mathcal{V} : (j, i) \in \mathcal{L}\}$. A *path* is a sequence $\{i, j, k, \ldots, r, s\}$ such that (i, j), (j, k), ..., (r, s) are edges of the graph. A *cycle* of a graph $\mathbb{G}$ is a subset of the edge set of $\mathbb{G}$ that forms a path such that the first vertex of the path corresponds to the last. A *directed cycle* is an oriented cycle such that all arcs go the same direction. A directed graph is *acyclic* if it does not contain any directed cycle.

In a formation with leader-follower architecture, each link is denoted with a line directed from follower to leader. There is one global leader and one first-follower of the global leader. The global leader does not follow any other agent, and the first-follower only follows the global leader. They are connected with one link pointed from the first-follower to the global leader. The rest of the agents are followers of at least two other agents. They can also be leaders of other agents.

Recall that the first follower has a link of out-degree 1. Since each agent in rigid formation, except the global-leader and the first follower, has at least two links with an out-degree of 2, we expect at least $2(n-2)+1 = 2n-3$ links in total.

For point formations with leader-follower architecture, Baillieul and Suri define stably rigid formations [1]. They first introduce a general model for distributed relative distance control of a point formation:

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \end{pmatrix} = \sum_{j \in \mathcal{N}^+_{\mathbb{G}}(i)} u_{ij}(d_{ij}, \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}) \begin{pmatrix} x_i - x_j \\ y_i - y_j \end{pmatrix} \quad (4)$$

for $i \neq 1, 2$ where d_{ij} is the set-point distance between agents i and j, and u_{ij} is a function of both the set-point and the measured distance. The definition of stable rigidity is as follows: a formation is *stably rigid* under a distributed

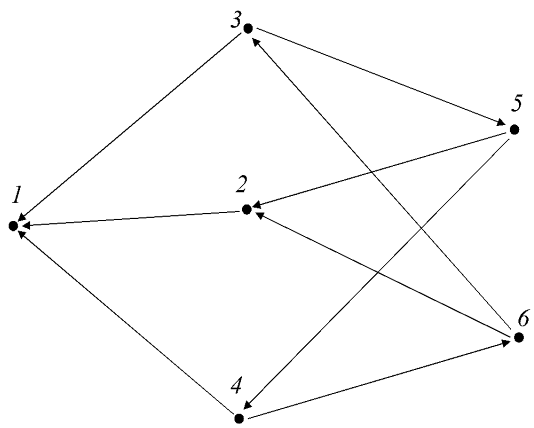

Fig. 2. The figure given by Baillieul and Suri in [1] as a counterexample.

relative distance control law as given in (4), if for any sufficiently small perturbation in the relative positions of the agents, the control law steers them asymptotically back into the prescribed formation in which the relative distance constraints are satisfied. The following theorem is given in [1] as a sufficiency condition for stably rigid formations:

Theorem 3.1 (Baillieul and Suri - Theorem 1 [1]): If a formation is constructed from a single directed edge by a sequence of vertex addition operation, then it is stably rigid.

The following proposition is given in [1] as a necessary condition for stable rigidity:

Proposition 3.2 (Baillieul and Suri - Proposition 1 [1]): If a formation with directed links is stably rigid then the following three conditions hold for the underlying graph: i) the undirected underlying graph is generically minimally rigid; ii) the directed graph is acyclic; iii) the directed graph has no vertex with an out-degree greater than 2.

It is stated in Proposition 3.2 that the conditions in Proposition are not sufficient because there is a counterexample graph shown in Figure 2. It is stated that this graph satisfies the conditions of Proposition 3.2 but it is not stably rigid. However, we note that this graph actually does not satisfy the conditions of Proposition 3.2, because there is a cycle $(3, 5, 4, 6, 3)$ in the graph; hence it violates the condition ii) of Proposition 3.2.

It can be proved that the conditions given in Proposition 3.2 are also sufficient conditions; hence these conditions are necessary and sufficient conditions for stable rigidity. Minimal rigidity together with acyclicity in a directed graph implies all vertices have out-degree at most 2. Therefore, the third condition in Proposition 3.2 is redundant. We have the following proposition:

Proposition 3.3: A point formation in 2-dimensional space with directed links is stably rigid if and only if the following conditions hold for the underlying directed graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$: i) the undirected graph is generically minimally rigid; ii) the directed graph is acyclic.

Proof: The necessity part of the proof is proved in [1]. Here we prove the sufficiency part only. Let us assume that the directed graph is acyclic. Then we can take the directed edges to define a partial order on the vertices: $a \geq b$ if the directed edge is pointed from a to b. We can extend this by transitivity. Since there are no cycles, this is a partial order with all vertices distinct. Since the graph is minimally generically rigid, all vertices have degree at least 2. Any maximal elements in this partial order have only outgoing edges - and therefore has two such edges. This can be removed (by the reversed vertex addition operation) to give a smaller, minimally rigid graph satisfying all of the conditions. We continue this down to one directed edge. The end points of this directed edge become the global leader and the first follower. Since this reduction sequence can be reversed, the graph is constructed using only the vertex addition operation. By Theorem 3.1, such graphs are stably rigid. □

Corollary 3.4: Equivalently a point formation in 2-dimensional space that has a leader-follower architecture is stably rigid if and only if the point formation can be constructed from the initial edge by the vertex addition operation.

The edge split operation is not used in [1] because this operation, as described in [1], results in vertices of out-degree 3. However, the edge split operation can be defined in such a way that the out-degrees of vertices remain less than 3. The definition given below for the edge split operation on directed minimally rigid graphs results vertices of out-degree 2.

2) Sequential Techniques: As with undirected graphs, one operation for graph expansion is *vertex addition*: given a minimally rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$, we add a new vertex i of out-degree 2 with two edges directed from i to two other vertices in $\mathcal{V}^*$. The second operation is *edge splitting*: given a minimally rigid graph $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$, we remove a directed edge (j, k) (directed from j to k) in $\mathcal{L}^*$ and then we add a new vertex i of out-degree 2 and in-degree 1 with three edges by inserting two edges (j, i), (i, k), and one edge between i and one other vertex (other than j, k) in $\mathcal{V}^*$ such that the edge (j, i) is directed from j to i and the other two edges are directed form i to the other vertices.

Now we are ready to present the following theorems (We omit the proofs here, and leave it to the full paper version.):

Theorem 3.5 (vertex addition - directed case): Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a directed graph with a vertex i of out-degree 2 in 2-dimensional space; let $\mathbb{G}^* = (\mathcal{V}^*, \mathcal{L}^*)$ denote the subgraph obtained by removing i and the edges incident with it. Then $\mathbb{G}$ is stably rigid if and only if $\mathbb{G}^*$ is stably rigid.

Example 3.6: The vertex addition operation for a directed graph is shown in Figure 3.

Theorem 3.7 (edge splitting - directed case): Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph with a vertex i of out-degree 2 and in-degree 1 (where this edge is between i and j), and let $\mathbb{G}' = (\mathcal{V}', \mathcal{L}')$ be the subgraph obtained by deleting i and its three incident edges. Then $\mathbb{G}$ is stably rigid if and only

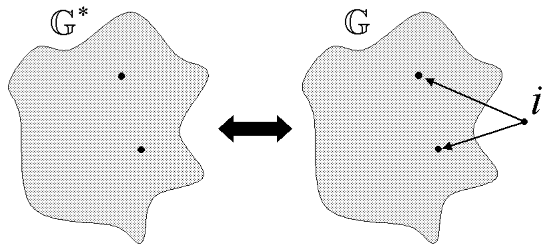

Fig. 3. Vertex Addition - directed case.

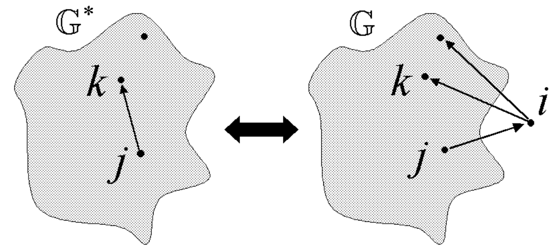

Fig. 5. Edge Splitting - directed case.

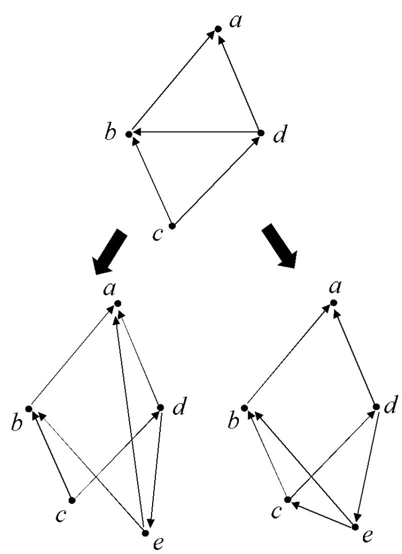

Fig. 4. Two examples of the edge splitting operation on a minimally rigid directed graph. The split edge is (d, b). We note that the resulting directed graph on the left has no cycles. On the other hand, the resulting directed graph on the right has a cycle (c, d, e, c). We note that the acyclic directed graph on the left can also be obtained by a series of vertex additions starting from a single edge.

if there is a directed edge of a pair j, k (directed from j to k) of the neighborhood $\mathcal{N}_{\mathbb{G}}(i)$ such that the directed edge (j, k) is not in $\mathcal{L}$ and the graph $\mathbb{G}^* = (\mathcal{V}', \mathcal{L}' \bigcup (j, k))$ is stably rigid.

Example 3.8: The edge splitting operation for a directed graph is shown in Figure 5.

When edge splitting does not lead to a cycle, the resulting graph can always be created by using only vertex addition (from Proposition 3.3 and Corollary 3.4). Hence all stably rigid formations can be created by using the vertex addition operation.

IV. Simulation Results

Figure 6 shows a 5-point formation created by vertex addition only. This formation satisfies the criteria given in Proposition 3.3. The agent with out-degree 0 (global leader) is indexed with 1 and the agent with out-degree 1 is indexed with 2. As the global leader moves, the rigidity is preserved. This can be seen in Figure 7. The distances between all agent pairs remain constant over time as the formation moves.

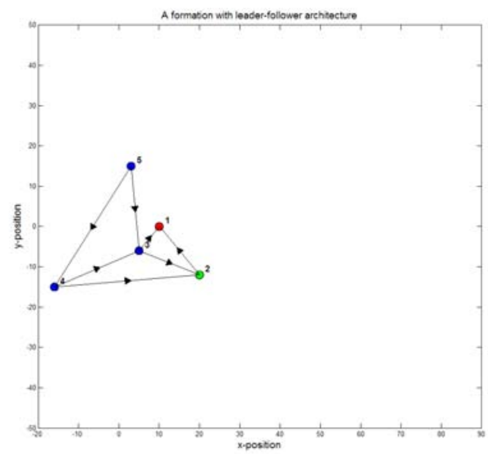

Fig. 6. A rigid formation. The agent with out-degree 0 (global leader) is depicted with color red and index 1. The agent with out-degree 1 is depicted with green. The agents with out-degree 2 are depicted with color blue.

Figure 8 shows a 6-point formation. This formation does not satisfy the criteria given in Proposition 3.3. The agent of out-degree 0 (global leader) is colored in red and the agents of out-degree 1 are colored with green. The agents of out-degree 2 are colored in blue. As the global leader moves, the rigidity is lost. This can be seen in Figure 9. The distances between the agent pairs, where there is no link, change over time as the formation moves.

V. Concluding Remarks

In this paper, we suggested a way of analyzing rigid formations that have a leader-follower architecture in 2-dimensional space. The necessity condition for stable rigidity is given in [1]. We proved that the third condition in this proposition is redundant. We also proved that this proposition is a necessary and sufficient condition for stable rigidity. Equivalently, we proved that a point formation in 2-dimensional space that has a leader-follower architecture is stably rigid if and only if the point formation can be constructed from the initial edge by the vertex addition operation.

The sequel will offer the following:

- an extension of Proposition 3.3 and the associated sequential techniques in $\mathbb{R}^3$;

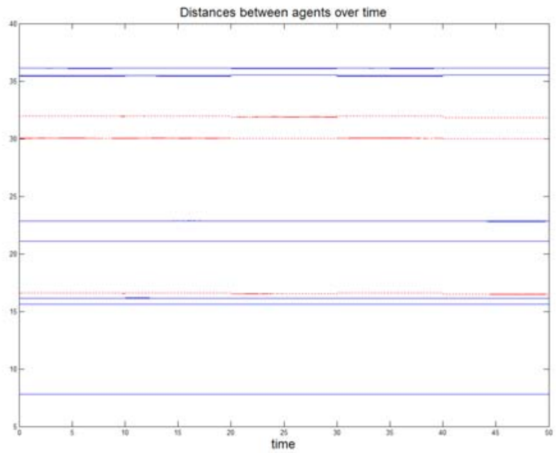

Fig. 7. Distances between agent pairs are shown over time as the global leader moves. The blue solid lines show the distance between agent pairs and the existence of links between those agent pairs. The dotted red lines show the distances between agent pairs and the absence of links between those agents pairs.

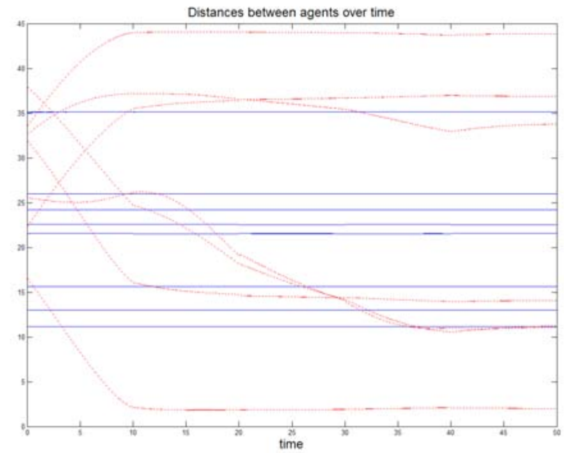

Fig. 9. Distances between agent pairs are shown over time as the global leader moves. The blue solid lines show the distance between agent pairs and the existence of links between those agent pairs. The dotted red lines show the distances between agent pairs and the absence of links between those agents pairs.

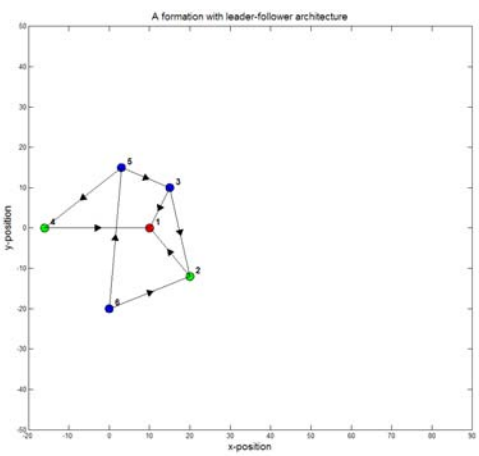

Fig. 8. A flexible formation. The agent with out-degree 0 (global leader) is depicted with color red and index 1. The agents with out-degree 1 are depicted with green. The agents with out-degree 2 are depicted with color blue.

- an analysis of creating leader-follower architectures from a given undirected rigid formation;
- an analysis of the effects of cycles;
- an analysis of formations that have both directed and undirected links.

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