GENERATION OF PRESCRIBED NONSTATIONARY COVARIANCES

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This paper is concerned with establishing connections between the matrices describing a system excited by white noise, and the covariance of the output of the system. More precisely, systems of the form

$$\dot{x} = F(t)x + G(t)u \tag{1a}$$

$$y = [H(t)]'x + J(t)u \tag{1b}$$

will be considered, with input covariance matrix

$$E[u(t)u'(\tau)] = I\delta(t-\tau) \tag{2}$$

The covariance of the output of (1), denoted by $R_y(t,\tau)$, is of the general form

$$R_y(t,\tau) = C(t)\delta(t-\tau) + A'(t)B(\tau)1(t-\tau) + B'(t)A(\tau)1(\tau-t) \tag{3}$$

Two basic problems are associated with eqs. (1) - (3). The first is: given $F(\cdot)$, $G(\cdot)$, $H(\cdot)$ and $J(\cdot)$ in (1), determine $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ in (3). This problem, essentially one of analysis, is discussed in [1] for the case where $J(t) = 0$ for all t, and in [2] for the more general case of nonzero J. In both these papers it was found helpful to define the symmetric covariance matrix of $x(t)$, viz.

$$P(t) = E[x(t)x'(t)] \tag{4}$$

Then, assuming that $P(t_0)$ is known (possibly with $t_0 = -\infty$), $P(t)$ can readily be found from F and G; $R_y(t,\tau)$ can then readily be specified in terms of P, F, G and H.

The second problem associated with eqs. (1) - (3) is: given $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ in (3), determine $F(\cdot)$, $G(\cdot)$, $H(\cdot)$ and $J(\cdot)$ in (1), and if necessary, an initial-state covariance matrix $P(t_0)$. The significance of $P(t_0)$ is as follows: $R_y(t,\tau)$ may be defined only for $t, \tau > t_0$, or alternatively, we may only be interested in simulating $R_y(t,\tau)$ for $t, \tau > t_0$ using a system (1); then it is necessary to specify not merely the matrices $F(\cdot)$ through $J(\cdot)$ of (1), but also an initial condition on $x(t_0)$ for (1). This initial condition is not necessarily a deterministic one, but may be a stochastic one, of a form requiring $E[x(t_0)x'(t_0)] = P(t_0)$, where $P(t_0)$ is some prescribed symmetric nonnegative definite matrix.

This second problem has been termed the spectral factorization problem;

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though solutions are comparatively well-known for a stationary $R_y(t,\tau)$ [i.e. $R_y(t,\tau)$ is of the form $R_y(t-\tau)$], little has been achieved in the nonstationary case. In fact the best results appear to be those of Stear, [3], who requires for the determination of a suitable system the solution of simultaneous nonlinear integral equations. In this paper we summarise the results of [4, 5] giving a new solution to the spectral factorization problem.

The solution proceeds by finding first from $R_y(t,\tau)$ the matrix $P(t)$, the covariance of $x(t)$. Then $P(t)$ is used in defining the system (1). Thus the factorization procedure in some sense reverses the analysis procedure discussed above as the "first problem", where system $1 \rightarrow P(t) \rightarrow$ covariance 3 was the route taken.

The first step in the solution requires the observation that $F(\cdot)$ in (1) is undetermined by $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$. The reason is that $R_y(t,\tau)$ reflects only input-output properties of the system, and in no way reflects the choice of coordinate basis used for the state-space. It is known that given one coordinate basis with a certain 'F' matrix, there always exists another basis where the new 'F' matrix is entirely arbitrary, see [6]. Moreover, change of basis cannot affect input-output properties of the system. This means that we are free in solving the spectral factorization problem to choose any F matrix.

There are some obviously good choices and obviously bad choices of the F matrix. However, if the initial time t_0 is desired to be $-\infty$, the F matrix should be asymptotically stable; otherwise any conclusions will be physically meaningless. In any case, the F matrix should be taken as bounded. Also, in the case where $R_y(t,\tau)$ is stationary and it is desired to simulate $R_y(t,\tau)$ with a constant system [i.e. F, G, H and J in (1) must be constant matrices], there are very definite restrictions on F, see [7].

It also turns out that $R_y(t,\tau)$ does not determine $P(t_0)$, and that this matrix can be chosen to be almost an arbitrary nonnegative definite matrix. A useful choice, and one that is always possible, is $P(t_0) = 0$.

Having chosen $F(\cdot)$ and $P(t_0)$, the spectral factorization problem becomes one of passing from $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $F(\cdot)$ and $P(t_0)$ to $G(\cdot)$, $H(\cdot)$ and $J(\cdot)$ (possibly with $t_0 = -\infty$).

Define $\Phi(t,\tau)$ to be the transition matrix associated with the homogeneous equation $\dot{x} = Fx$. Also define the matrix $\Gamma(t)$ as any matrix satisfying

$$\Phi(t,\tau) = \Gamma(t)\Gamma^{-1}(\tau) \tag{5}$$

The following theorem can be stated, (for proofs see [4]).

Theorem Let $R_y(t,\tau)$ be a covariance of the form of (3), where it is assumed that $C(t)$ is a positive definite matrix such that for all t, $\alpha_1 I - C(t)$ and $C(t) - \alpha_2 I$ are also positive definite for some positive constants α_1 and α_2. Further let $F(\cdot)$ be specified, together with a symmetric nonnegative definite matrix $P(t_0)$. Then, a system of the form (1), when driven by white noise with covariance matrix (2), has $R_y(t,\tau)$ as its output covariance, and is defined by $F(t)$,

$$H(t) = [\Gamma^{-1}(t)]'A(t), \tag{6}$$

$$J(t) = C^{\frac{1}{2}}(t), \tag{7}$$

where $C^{\frac{1}{2}}(t)$ is the unique positive definite symmetric square root of $C(t)$, and

$$G(t) = [\Gamma(t)B(t) - P(t)H(t)]J^{-1}(t) \tag{8}$$

Here $P(t)$ is the solution for $t > t_0$ of the matrix Riccati differential equation

$$\dot{P} = P(F'-HC^{-1}B'\Gamma') + (F-\Gamma BC^{-1}H')P$$
$$+ PHC^{-1}H'P + \Gamma BC^{-1}B'\Gamma' \tag{9}$$

with initial condition $P(t_0)$ at t_0. The matrix $P(t)$ is the covariance of $x(t)$, the state of the system defined by F, G, H and J.

Some comments on this theorem will now be made. Evidently F, H and J in (1) are more or less straightforward in their determination from R_y in (3). The difficult matrix to determine is G. As (8) shows, $G(t)$ depends on another matrix $P(t)$, which is actually the covariance of $x(t)$; moreover (9) gives a procedure for determining P, essentially from $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $F(\cdot)$ and $P(t_0)$.

The above theorem says nothing about the existence of solutions to (9), and thus fails to indicate whether (1) is always well-defined. One instance where (9) immediately breaks down is when C is singular; it is in fact possible to refine the above theorem to consider this case, see [4], but arguments in this reference and others do point out the less practical nature of the problem under this condition. As an example of the difficulties associated with singular $C(t)$, it turns out that any whitening filter for $R_y(t,\tau)$ has to contain at least one differentiator.

Reference 5 is especially concerned with the fundamental problem of existence. It turns out that the existence of solutions to (9) is guaranteed by a fundamental result of optimal control, relating existence problems of Riccati equations to covariance requirements on certain matrices [8]. Of course, a few more conditions on $R_y(t,\tau)$ are required; thus, see [5],

<u>Theorem 2</u> With the same hypothesis as theorem 1, suppose F, H, ΓB and C are bounded, $[F, H']$ is completely observable and $P(t_0)$ is zero. Then $P(t)$, the solution of (9) exists and is well-defined for all $t > t_0$.

To ensure that P(t) (and thus G(t)), is bounded for all $t \geq t_0$, we have

Corollary A With the same hypothesis as theorem 2 and the assumption $[F, H']$ is uniformly completely observable, P(t) is bounded

and

Corollary B With the same hypothesis as Corollary A, and the assumption F is asymptotically stable, $R_y(t,\tau) - \eta\delta(t-\tau)$ is a covariance for some positive η, then a bounded solution of (9) exists with the initial condition $\lim_{t_0 \to -\infty} P(t_0) = 0$

Together with Theorem 1, Theorem 2 solves the spectral factorization theorem over an interval $[t_0, T]$ for arbitrary T, corollary A over $[t_0, \infty)$ and corollary B over $(-\infty, \infty)$.

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