# On Vinnicombe Metrics and the Stability Robustness of Linear Time-varying Systems[1]

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## Abstract

We present two novel approaches for obtaining stability robustness information about linear time-invariant systems with norm-bounded time-varying output multiplicative perturbations using Vinnicombe metrics. First, a time-varying Vinnicombe metric approximation between a nominal linear time-invariant system and the perturbed linear time-varying system is developed, and we show that a calculable tight upper bound may be placed on the approximation for a subclass of small, periodic output multiplicative errors. Second, we show that a worst-case time-invariant Vinnicombe metric may be used to obtain stability robustness information about the perturbed time-varying system.

## 1 Introduction

Vinnicombe metrics provide a means of quantifying feedback system stability robustness, yet in the linear time-invariant case alone does computation of the metric generally remain straight forward. Of course, linear time-invariance is not a truthful characteristic of 'real-life' physical systems, and it is these 'real-life' systems for which we wish to design feedback controllers such that they are stable and achieve a certain level of performance. It is therefore desirable to construct Vinnicombe metric methods requiring simple calculation for nonlinear and linear time-varying cases if one wishes to utilize the Vinnicombe metric to examine the feedback system's stability robustness. In this paper we present two such methods with respect to the linear time-varying case.

The Vinnicombe, or $\nu$-gap, metric, was developed by [14] for linear time-invariant (LTI) systems. Together with the generalized robust stability margin $b_{P_0,C}$ (for instance, [6]), it was shown to have properties which provided conditions for LTI system feedback control stability [14]. As was done with the gap metric, introduced by [15] and later exploited by [5], it was shown that for a given nominal LTI plant and controller achieving a specified robust stability margin, the same controller is guaranteed to stabilize a perturbed LTI plant provided the distance between the original and perturbed plant measured in the gap or $\nu$-gap metric is sufficiently small. On occasions, the $\nu$-gap metric may be of more advantage than the gap metric in that its calculation is simpler. It also has the additional property that if the distance between a nominal and perturbed LTI plant measured in the $\nu$-gap metric is not sufficiently small, then the perturbed plant will be destabilized by some controller achieving a specified robust stability margin with the nominal.

Nonlinear (for instance, [13] and [1]) and time-varying (such as [1]) Vinnicombe metrics have been developed, just as the gap metric was extended to nonlinear systems [4], though calculation of these metrics has not been analytically possible for classes of general systems. For example, [4] developed an input-output framework for robustness analysis of nonlinear systems utilizing the gap metric, and performed stability calculations for tractable examples. The calculation of nonlinear and time-varying $\nu$-gap metric upper bounds for simple classes of nonlinear and time-varying systems has also been attempted, as has been done in the gap metric case. For instance, [13] places an upper bound on the nonlinear $\nu$-gap metric by describing the nonlinearities in terms of integral quadratic constraints and using convex optimization. [1] placed an upper bound on the time-varying $\nu$-gap metric between two linear time-varying plants related by an output multiplicative

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error (see figure 1 with time-varying $P_0$).

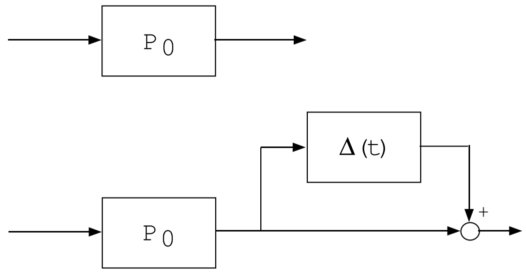


**Figure 1:** System configurations with time-varying output multiplicative error

In this paper, we present two calculable $\nu$-gap metric methods with respect to linear time-varying (LTV) system stability robustness. We consider LTV systems and the time-varying $\nu$-gap metric, even though our ultimate goal is to use the $\nu$-gap metric to study robust stability for nonlinear systems. This is because nonlinear systems can be linearized about trajectories to give LTV systems, and then linear techniques can be utilized for the (much simpler) formulation of the $\nu$-gap metric. Since these trajectories about which linearization is done are generally unknown, it makes sense to consider classes of LTV systems, rather than single systems alone. Also, we are considering only the class of LTI systems in cascade with a small, norm-bounded output multiplicative time-varying error (see figure 1, $P_0$ time-invariant) for simplicity's sake, though the results may be manipulated to the input multiplicative error case. Examples of such classes of systems occur in practice when you have input saturation nonlinearities or output sensor nonlinearities.

The first result is a derivation of a first-order approximation of the time-varying $\nu$-gap metric, defined by [1], between a nominal LTI plant, $P_0$, and a LTV plant, related to the nominal by a norm-bounded output multiplicative error, as in figure 1. We then illustrate a technique which places a computable upper bound on the $\nu$-gap metric approximation and show that for a class of small periodic output multiplicative uncertainties, the upper bound is a tighter result than the general upper bound of [1].

Secondly, we appeal to a result of [11] and [2] which gives necessary and sufficient conditions for the system in figure 2 (that is, the system in figure 1 with closed-loop) to achieve robust stability for all output LTV perturbations, $\Delta(t)$, with $||\Delta|| \leq \epsilon$, and show via the small gain argument that the system in figure 3 shares the same necessary and sufficient conditions for all output stable LTI perturbations, $\epsilon\delta$, with $||\delta||_\infty \leq 1$ for robust stability achievement. It is also known from [14] that the system in figure 3 is stable for all norm-bounded $\delta$ if the nominal closed-loop LTI system is stable and the $\nu$-gap distance between the nominal plant and the

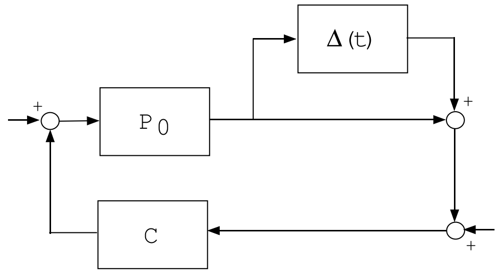


**Figure 2:** Closed-loop system with time-varying perturbation

LTI perturbed plant as shown in figure 4 is sufficiently small (in the generalized robust stability margin sense [14]). Hence, we formulate a calculable worst-case LTI

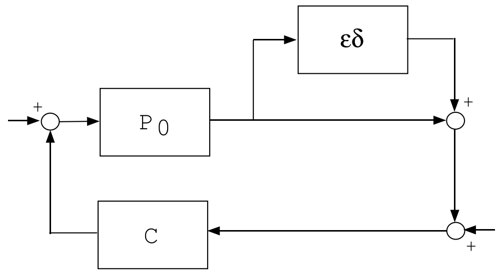


**Figure 3:** Closed-loop system with time-invariant perturbation

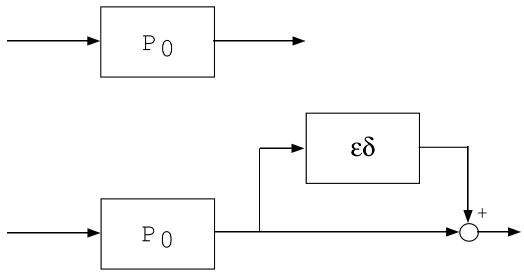


**Figure 4:** System configurations with time-invariant output multiplicative error

$\nu$-gap metric with respect to a norm-bounded $\delta$, which can be used to infer stability robustness information about the original time-varying system. We allude to the scenario where the aim is to design a LTI controller for a nominal LTI plant which has a small nonlinearity on the time-varying gain (after linearization) at the input or output.

The paper is outlined as follows. Section 2 motivates in greater detail the use of classes of LTV systems to study nonlinear system stability robustness. In section 3, the mathematical results required for the time-varying $\nu$-gap metric formulation are given, followed by the formal definition of the time-varying $\nu$-gap metric itself, as developed by [1]. Section 4 states the time-varying $\nu$-gap metric upper bound for LTV systems related by a norm-bounded output multiplicative error, as given by [1]. We then develop our first-order time-varying $\nu$-gap metric approximation for a nominal LTI plant and a LTV plant related to the nominal by a small time-varying output multiplicative perturbation. Following this, the method for placing an upper bound on the approximation is illustrated, and by an example we show that this upper bound is tight when the output multiplicative error is represented by a subclass of periodic functions. Lastly, the worst-case (with respect to norm-bounded $\delta$) time-invariant $\nu$-gap metric, which can be used to infer stability robustness information for the system as shown in figure 2, with output multiplicative norm-bounded time-varying perturbation, $\Delta(t)$, is formulated in section 5.

### 1.1 Notation

Let $A^T$ and $A^*$ denote the transpose and complex conjugate transpose, respectively, of a matrix $A$, and $X^\sim$ denote the adjoint (in the sense of [10]) of a linear operator $X$. $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_\infty$ denote the corresponding Lebesgue spaces, each with norms $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_\infty$, respectively. $\mathcal{L}_2[0,T]$ is the finite horizon Lebesgue space, and $\mathcal{L}_{2e} = \{f \in \mathcal{L}_2[0,T] \text{ for all } T < \infty\}$ describes the extended Lebesgue space. Let $\mathcal{H}_\infty$ denote the Hardy space which is a closed subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane, and with norm notation also $||\cdot||_\infty$, and $\mathcal{RH}_\infty$ denotes the real rational subspace of $\mathcal{H}_\infty$ which consists of all proper and real rational stable transfer matrices. $||\cdot||$ denotes the $\mathcal{L}_2$-induced norm for LTV operators, with the $\mathcal{L}_2$-induced norm equaling the infinity norm of the transfer function for LTI systems.

## 2 Motivation for time-varying system analysis

Nonlinear differential equations describe the dynamics of nonlinear systems. Taking a state vector $x$, an input vector $u$ and an output vector $y$, these nonlinear equations may be written as

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t)), \end{aligned}$$

where $f$, $g$ are continuously differentiable.

Linearizing about some trajectory $(x_0(t), u_0(t))$, the following linear time-varying system is obtained

$$\begin{aligned} \dot{\tilde{x}}(t) &= A(t)\tilde{x}(t) + B(t)\tilde{u}(t) \\ \tilde{y}(t) &= C(t)\tilde{x}(t), \end{aligned}$$

where $A(t) = \frac{\partial f}{\partial x}\Big|_{(x_0(t),u_0(t))}$, $B(t) = \frac{\partial f}{\partial u}\Big|_{(x_0(t),u_0(t))}$ and $C(t) = \frac{\partial g}{\partial x}\Big|_{x_0(t)}$, for which it is now possible to utilize linear techniques for Vinnicombe metric formulation. Since the trajectories that we are linearizing about, however, may be unknown *a priori*, it makes sense to consider classes of LTV systems rather than single systems themselves.

This paper considers only strictly proper LTI systems in cascade with small output multiplicative time-varying errors, though the methods are also applicable if the time-variation occurs at the nominal plant's input. Examples of this occur in practice when you have input saturation nonlinearities or output sensor nonlinearities.

Let the input and output spaces be $\mathcal{U} := \mathcal{L}_{2e}$ and $\mathcal{Y} := \mathcal{L}_{2e}$, respectively. In packed matrix notation, a LTI or LTV system $P$ can be written as

$$P = \left( \begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & D(t) \end{array} \right), \tag{1}$$

where $A$, $B$, $C$ and $D$ of (1) are constant matrices if $P$ is LTI.

## 3 Time-varying $\nu$-gap metric definition

In this section, we state the time-varying $\nu$-gap metric, as developed by [1]. Firstly, however, the preliminary mathematics necessary for the formulation is presented.

### 3.1 Graph symbol representation

A system, $P$, may be described in terms of its graph. The graph of a system is its set of all possible input-output pairs. Formally, the graph of $P$ is

$$\mathcal{G}_P := \left\{ \begin{pmatrix} Pu \\ u \end{pmatrix} : u \in \mathcal{U}, Pu \in \mathcal{Y} \right\}.$$

Normalized coprime factors (in the sense of [10] and [1]) are a useful way of giving a representation of the graph. $P = NM^{-1}$ (where $M$ and $N$ are linear bounded operators mapping $\mathcal{L}_2$ to $\mathcal{L}_2$) is said to be a right-coprime factorization if $N$ and $M$ are right-coprime (that is, there exists bounded linear operators $X$ and $Y$ mapping $\mathcal{L}_2$ to $\mathcal{L}_2$, such that $XM + YN = I$). If the right-coprime factorization $P = NM^{-1}$ is such that $M^\sim M + N^\sim N = I$, we say that $\{N, M\}$ are normalized right-coprime factors of $P$ [10]. Similarly, $P = \tilde{M}^{-1}\tilde{N}$ is a left-coprime factorization of $P$ if $\tilde{N}$ and $\tilde{M}$ are left-coprime (that is, there exists $\tilde{X}$ and $\tilde{Y}$ such that $\tilde{M}\tilde{X} + \tilde{N}\tilde{Y} = I$), and we say $\{\tilde{N}, \tilde{M}\}$ are normalized left-coprime factors of $P = \tilde{M}^{-1}\tilde{N}$ if $\{\tilde{N}, \tilde{M}\}$ are left-coprime and $\tilde{M}\tilde{M}^\sim + \tilde{N}\tilde{N}^\sim = I$ [10].

The following lemma shows that uniform stabilizability and detectability are sufficient to ensure the existence and uniqueness of stabilizing solutions to standard control and filter differential Riccati equations. This result is used in the formulatioin of normalized right and left graph symbols.

**Lemma 1** *[10] Let $P$ be a finite-dimensional linear time-varying system as in (1) but with $D(t) \equiv 0$ for all $t$ and with uniformly stabilizable and detectable realization. Define the generalized control differential Riccati equation (GCDRE) as*

$$-\dot{X}(t) = A(t)^T X(t) + X(t)A(t) + C(t)^T C(t) - X(t)B(t)B(t)^T X(t), \tag{2}$$

*with $X(t_f) = 0$ as $t_f \to \infty$. Similarly, define the generalized filtering differential Riccati equation (GFDRE) as*

$$\dot{Y}(t) = A(t)Y(t) + Y(t)A(t)^T + B(t)B(t)^T - Y(t)C(t)^T C(t)Y(t), \tag{3}$$

*with $Y(t_s) = 0$ as $t_s \to -\infty$. There exist bounded symmetric solutions $X(t) \geq 0$ and $Y(t) \geq 0$ to the GCDRE and GFDRE respectively. Furthermore, these solutions are stabilizing in the sense that $\dot{x} = (A(t)+B(t)F(t))x$ and $\dot{x} = (A(t) + H(t)C(t))x$ are exponentially stable, where $F(t) := -B(t)^T X(t)$ is the generalized control gain and $H(t) := -Y(t)C(t)^T$ is the generalized filter gain.*

**Proof:** The proof is given in [10]. ∎

If $P$ is a strictly proper LTI system, equations (2) and (3) reduce to the standard generalized control algebraic Riccati equation (GCARE)

$$0 = A^T X + XA - XBB^T X + C^T C \tag{4}$$

and the generalized filter algebraic Riccati equation (GFARE)

$$0 = AY + YA^T - YC^T CY + BB^T, \tag{5}$$

respectively, with stabilizing symmetric solutions $X \geq 0$ and $Y \geq 0$, respectively, constant.

Based on the lemma, it has been shown [10] that the normalized right graph symbol $G = \begin{pmatrix} N \\ M \end{pmatrix}$ for $P$ can be formulated as $\left(\begin{array}{c|c} A(t)+B(t)F(t) & B(t) \\ \hline C(t) & 0 \\ F(t) & I \end{array}\right)$ and the normalized left graph symbol $\tilde{G} = \begin{pmatrix} -\tilde{M} & \tilde{N} \end{pmatrix}$ can be given as $\left(\begin{array}{c|cc} A(t)+H(t)C(t) & -H(t) & B(t) \\ \hline C(t) & -I & 0 \end{array}\right)$. Note that the entries in the realizations are constant if $P$ is LTI.

### 3.2 Time-varying $\nu$-gap metric definition

The $\nu$-gap metric for LTV systems, as developed by [1], is given below. This definition is analogous to the LTI $\nu$-gap metric [14].

**Definition 1** *[1] Suppose that $P_1$ and $P_2$ are two finite-dimensional plants with the same number of inputs and outputs and with state-space realizations $P_i = \left[\begin{array}{c|c} A_i(t) & B_i(t) \\ \hline C_i(t) & D_i(t) \end{array}\right]$, where $A$ is continuous and $A$, $B$, $C$ and $D$ are bounded. Suppose further that each $\{A_i(t), B_i(t), C_i(t), D_i(t)\}$ is uniformly stabilizable and detectable and that normalized coprime factorization descriptions $P_i = N_i M_i^{-1} = \tilde{M}_i^{-1}\tilde{N}_i$ are fixed with $G_i = \begin{bmatrix} N_i \\ M_i \end{bmatrix}$ and $\tilde{G}_i = \begin{bmatrix} -\tilde{M}_i & \tilde{N}_i \end{bmatrix}$. Define*

$$\delta_\nu(P_1, P_2) = \begin{cases} \|\tilde{G}_1 G_2\| & \text{if } (i) \text{ and } (ii) \text{ hold,} \\ 1 & \text{otherwise,} \end{cases}$$

*where $(i)$ is the condition that the number of positive Lyapunov exponents (see [8] for definition) of (6)[2] equals the number of positive Lyapunov exponents of (7)[3], and $(ii)$ is the requirement that the Lyapunov exponents of (6) are all nonzero.*

Lyapunov exponents [8] generalize the concept of poles to LTV systems, so $(i)$ and $(ii)$ are equivalent to requiring $\det(G_1^* G_2)$ to have zero-winding number and to being not equal to zero on the $j\omega$ axis, respectively, in the time-invariant case.

It is evident that difficulties will arise regarding the calculation of the time-varying $\nu$-gap metric, such as the requirement to solve generalized differential Riccati equations for graph symbol formulation which is in general not analytically possible. For instance, consider a simple LTV plant $P = \left(\begin{array}{c|c} -1 & 1 \\ \hline 1+\epsilon \sin at & 0 \end{array}\right)$ with $\epsilon \in [0, 1]$. From (2), the GCDRE is $-\dot{X}(t) = -2X(t) - X(t)^2 + (1 + \epsilon \sin at)^2$, which, to the best of our knowledge, cannot be solved analytically. We thus turn our attention to formulating upper bounds or an approximation to the $\nu$-gap metric for classes of time-varying systems as an alternative to calculating the metric directly.

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$$\begin{pmatrix} -(A_1 - B_1 S D_2^T C_1)^T & C_1^T (I + D_2 D_1^T)^{-1} C_2 \\ B_2 S^T B_1^T & A_2 - B_2 D_1^T (I + D_2 D_1^T)^{-1} C_2 \end{pmatrix}, \tag{6}$$

where $S := (I + D_2^T D_1)^{-1}$.

3

$$\begin{pmatrix} -(A_1 + B_1 F_1)^T & (C_1 + D_1 F_1)^T (C_2 + D_2 F_2) + F_1^T F_2 \\ 0 & A_1 + B_2 F_2 \end{pmatrix} \tag{7}$$

## 4 Upper bound and approximation to the time-varying $\nu$-gap metric

In this section, we derive a first-order approximation to the time-varying $\nu$-gap metric between a LTI plant and a plant that is related to the nominal by a norm-bounded time-varying perturbation. We then formulate an upper bound on the approximation, and show by example that this bound is tight for a small periodic time-varying perturbation with respect to the upper bound given by [1].

### 4.1 Time-varying $\nu$-gap metric upper bound

The following corollary of [1] places an upper bound on the time-varying $\nu$-gap metric between two LTV plants related by a norm-bounded output multiplicative error, as shown in figure 1 with time-varying $P_0$.

**Corollary 1** *[1] Let $P_0 = NM^{-1}$ be a normalized coprime factorization description of a finite-dimensional linear time-varying plant and let $\Delta$ be a stable linear operator with bound $||\Delta|| < 1$ so that an output multiplicative perturbation of $P_0$ is given by $(I+\Delta)P_0 = (I+\Delta)NM^{-1}$. Then*

$$\delta_\nu(P_0, (I+\Delta)P_0) \leq ||\Delta\ N||. \tag{8}$$

**Proof:** The proof is given in [1]. ∎

For example, consider $P_0 = \left(\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right)$ and the perturbed LTV plant $(I+\Delta)P_0 = \left(\begin{array}{c|c} -1 & 1 \\ \hline 1+\epsilon \sin at & 0 \end{array}\right)$. Solving (4) gives a stabilizing solution of $X = -1+\sqrt{2}$, from which one obtains $N = \frac{1}{s+\sqrt{2}}$. Then, from (8), an upper bound for the time-varying $\nu$-gap metric between $P_0$ and $(I+\Delta)P_0$ is $\delta_\nu(P_0,(I+\Delta)P_0) \leq ||\epsilon \sin at.N|| \leq |\epsilon|||\sin at||||N||_\infty = \frac{\epsilon}{\sqrt{2}}$.

### 4.2 Time-varying $\nu$-gap metric approximation

Now consider a strictly proper LTI system, $P_0$, and a small time-varying error $\Delta(t)$ with $||\Delta|| \ll 1$, such that an output multiplicative perturbation of $P_0$ is given by $(I+\Delta)P_0$ (as shown in figure 1 with $P_0$ time-invariant). Assume that the corresponding GCDRE as given by (2) for the perturbed plant is well-conditioned in the sense of [9], and the perturbed solution to the GCDRE is well-defined. A first-order time-varying $\nu$-gap metric approximation is given as follows.

**Theorem 1** *Suppose $P_0$ is a finite-dimensional LTI system with state-space realization $P_0 = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)$. Suppose further that $(A, B)$ and $(C, A)$ are stabilizable and detectable, respectively, and that $P_0 = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ are normalized coprime factorizations with $G = \begin{pmatrix} N \\ M \end{pmatrix}$ and $\tilde{G} = \begin{pmatrix} -\tilde{M} & \tilde{N} \end{pmatrix}$. Let $\Delta(t)$ be a sufficiently small time-varying perturbation such that (i) and (ii) are satisfied and second/higher order terms are negligible. Then a first-order approximation of the time-varying $\nu$-gap metric between a LTI plant $P_0$ and a LTV plant of the form $(I+\Delta)P_0$ is given by*

$$\delta_\nu(P_0, (I+\Delta)P_0) \approx ||\tilde{M}\ \Delta\ N||. \tag{9}$$

**Proof:** To be published elsewhere. ∎

The approximation is useful since it gives an approximate value of a time-varying $\nu$-gap metric rather than simply an upper bound. Furthermore, note that an upper bound on the time-varying $\nu$-gap metric approximation (9) is $||\tilde{G}_1 G_2|| \approx ||\tilde{M}\ \Delta\ N|| \leq ||\tilde{M}||_\infty ||\Delta\ N|| = ||\Delta\ N||$, since $||\tilde{M}||_\infty = 1$ always for a normalized coprime factor of a strictly proper plant $P_0$. This is the same as (8) for small $\Delta(t)$ and LTI $P_0$.

We now illustrate the method for placing an alternative upper bound on the approximation which, for a class of periodic output multiplicative errors, is tighter at the expense of requiring more difficult calculations.

### 4.3 $\nu$-gap metric approximation upper bound

Note that $\tilde{G}_1 G_2 \approx \tilde{M}\ \Delta\ N = -\tilde{M}\ \Delta\ N =$

$$\left(\begin{array}{cc|c} A - YC^TC & YC^T\Delta(t)C & 0 \\ 0 & A - BB^TX & B \\ \hline C & -\Delta(t)C & 0 \end{array}\right). \tag{10}$$

(10) can be expressed as a function of an output signal, $z(t)$, as follows. (10) is described by the equations

$$\begin{aligned} \dot{x}(t) &= \hat{A}(t)x(t) + \hat{B}v(t) \\ z(t) &= \hat{C}(t)x(t), \end{aligned}$$

where $\hat{A}(t) = \begin{pmatrix} A - YC^TC & YC^T\Delta(t)C \\ 0 & A - BB^TX \end{pmatrix}$, $\hat{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}$ and $\hat{C}(t) = \begin{pmatrix} C & -\Delta(t)C \end{pmatrix}$.

Let $x(-\infty) = 0$. Associated with these equations is a state transition matrix $\phi(t,\tau)$, which is the solution to the array of first-order differential equations $\frac{d}{dt}\phi(t,\tau) = \hat{A}(t)\phi(t,\tau)$, where $\phi(\tau,\tau) = I$. Note that $\phi(t,\tau)$ can easily be obtained analytically due to the special form of the upper triangular matrix with LTI terms on the main diagonal. Then the output signal of the system (10) is

$$z(t) = \int_{-\infty}^{\infty} h(t,\tau)v(\tau)d\tau, \tag{11}$$

where $h(t,\tau) = \mu(t-\tau)\begin{pmatrix} C & -\Delta(t)C \end{pmatrix}\phi(t,\tau)\begin{pmatrix} 0 \\ B \end{pmatrix}$ [7], with $\mu(\cdot)$ denoting a unit step function (ie: $\mu(\alpha) = 1$ when $\alpha \geq 0$ and $\mu(\alpha) = 0$ otherwise).

The following two lemmas define the $\mathcal{L}_1$- and $\mathcal{L}_\infty$-induced norms, $c_1$ and $c_\infty$, respectively, and give conditions for $\mathcal{L}_1$ and $\mathcal{L}_\infty$ stability of the system given by

(10). These results are utilized to formulate the upper bound on the $\mathcal{L}_2$-induced norm of system given by (10) in theorem 2.

**Lemma 2** *[3] Let $z(\cdot)$ and $v(\cdot)$ be related by (11), where $v(\cdot)$, $z(\cdot) : \mathbb{R} \to \mathbb{R}$. Suppose $v \in \mathcal{L}_1$ and that $h(t, \cdot)$ is bounded $\forall t \in \mathbb{R}$. Then $v \in \mathcal{L}_1 \Rightarrow z \in \mathcal{L}_1$, and moreover, there exists a constant $c < \infty$ such that $||z||_1 \leq c||v||_1$ if and only if $\sup_{\tau\in\mathbb{R}} \int_{-\infty}^{\infty} |h(t,\tau)|dt = c_1 < \infty$.*

**Proof:** The proof is given in [3]. ■

**Lemma 3** *[3] Let $z(\cdot)$ and $v(\cdot)$ be related by (11), where $v(\cdot)$, $z(\cdot) : \mathbb{R} \to \mathbb{R}$. Suppose $v \in \mathcal{L}_\infty$ and that $h(t, \cdot)$ is bounded $\forall t \in \mathbb{R}$. Then $v \in \mathcal{L}_\infty \Rightarrow z \in \mathcal{L}_\infty$, and moreover, there exists a constant $c < \infty$ such that $||z||_\infty \leq c||v||_\infty$ if and only if $\sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} |h(t,\tau)|d\tau = c_\infty < \infty$.*

**Proof:** The proof is given in [3]. ■

**Theorem 2** *[3] Consider (11). If (a) some constant $c_1$ satisfies $\int_{-\infty}^{\infty} |h(t,\tau)|dt \leq c_1 < \infty$ for all $\tau \in \mathbb{R}$, and (b) some constant $c_\infty$ satisfies $\int_{-\infty}^{\infty} |h(t,\tau)|d\tau \leq c_\infty < \infty$ for all $t \in \mathbb{R}$, then $v \in \mathcal{L}_2 \Rightarrow z \in \mathcal{L}_2$, and moreover, $||z||_2 \leq \sqrt{c_1 c_\infty}||v||_2$.*

**Proof:** The proof is given in [3]. ■

The method is illustrated with the following example where the output multiplicative error is given by a small periodic function. Let $P_1 = \left(\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right)$ be the nominal LTI plant and $P_2 = \left(\begin{array}{c|c} -1 & 1 \\ \hline 1+\epsilon\sin at & 0 \end{array}\right)$ be the perturbed LTV plant, with $\epsilon$ small. Stabilizing solutions to (4) and (5) are $X = -1 + \sqrt{2}$ and $Y = -1 + \sqrt{2}$, respectively, such that, from (10),

$$\tilde{G}_1 G_2 \approx \left(\begin{array}{cc|c} -\sqrt{2} & (-1+\sqrt{2})\epsilon\sin at & 0 \\ 0 & -\sqrt{2} & 1 \\ \hline 1 & -\epsilon\sin at & 0 \end{array}\right). \qquad (12)$$

With a state transition matrix, $\phi(t,\tau) =$

$$\begin{pmatrix} e^{-\sqrt{2}(t-\tau)} & \frac{(-1+\sqrt{2})\epsilon e^{-\sqrt{2}(t-\tau)}(\cos a\tau - \cos at)}{a} \\ 0 & e^{-\sqrt{2}(t-\tau)} \end{pmatrix},$$

system (12) expressed as a function of system output is $z(t) = \int_{-\infty}^{\infty} \mu(t-\tau)\epsilon e^{-\sqrt{2}(t-\tau)} f(t,\tau)v(\tau)d\tau$, where $f(t,\tau) = \frac{-1+\sqrt{2}}{a}(\cos a\tau - \cos at) - \sin at$, and hence

$$c_1 = \sup_{\tau\in\mathbb{R}} \epsilon \left\{ \int_{\tau}^{\infty} e^{-\sqrt{2}t}|f(t,\tau)|dt\ e^{\sqrt{2}\tau} \right\} \qquad (13)$$

$$c_\infty = \sup_{t\in\mathbb{R}} \epsilon \left\{ \int_{-\infty}^{t} e^{\sqrt{2}\tau}|f(t,\tau)|d\tau\ e^{-\sqrt{2}t} \right\}. \qquad (14)$$

A minor difficulty in calculating $c_1$ and $c_\infty$ arises from the modulus signs appearing in the integrals of (13) and (14). To calculate these integrals, observe that $f(t,\tau)$ is periodic in $t$ and $\tau$, respectively. Finding the zeros of $f(t,\tau)$ allows for each integral to be split into the sum of two geometric series, and the modulus signs are dealt with explicitly by manipulating the sign of $f(t,\tau)$ to make it always equal to $|f(t,\tau)|$.

Once the integrals are calculated, the resulting formulae in the parentheses of (13) and (14) can be plotted in figures, such as figures 5 and 6 with frequency $a = 1$ for example, to find the suprema and hence the induced norms. For example, from figures 5 and

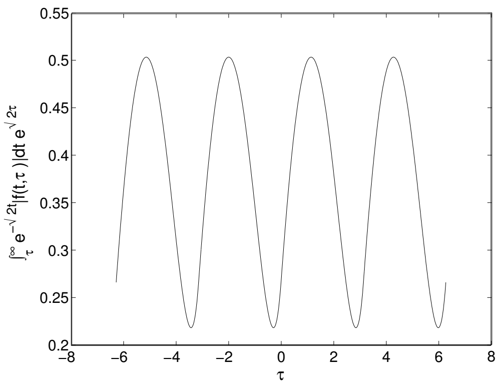


**Figure 5:** $c_1$

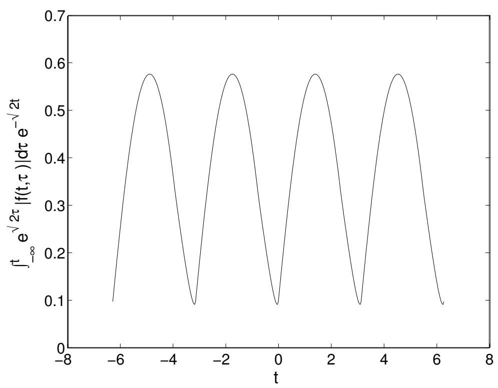


**Figure 6:** $c_\infty$

6 we obtain $c_1$ and $c_\infty$ to be $\epsilon 0.5043$ and $\epsilon 0.5773$, respectively. Therefore, from theorem 2, an upper bound on the approximation to the time-varying $\nu$-gap metric between systems $P_1$ and $P_2$ for $a = 1$ is $||\tilde{M}\ \Delta\ N|| \leq \sqrt{c_1 c_\infty} = \sqrt{\epsilon 0.5043 \epsilon 0.5773} = \epsilon 0.5396$.

Taking the calculation one step further, a worst-case upper bound may be found by calculating each upper bound for all frequencies, $a$, over the range as shown in figure 7. So for the class of sine function output

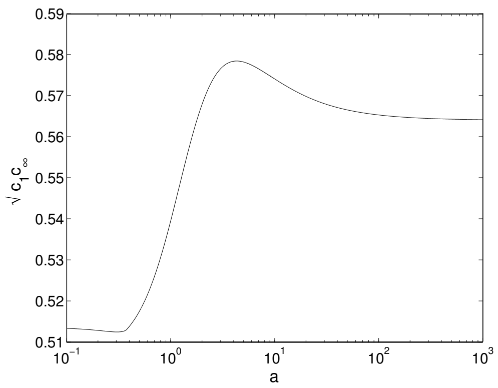


**Figure 7:** Worst-case upper bound

multiplicative errors, $\epsilon \sin at$, with $\epsilon$ small, a worst-case upper bound to the time-varying Vinnicombe metric approximation is $\epsilon 0.5786$, and this is a tighter result than that of [1], which in this case is $\epsilon 0.7071$.

## 5 A worst-case linear time-invariant $\nu$-gap metric

In this section, we formulate a worst-case time-invariant $\nu$-gap metric (with respect to a norm-bounded time-invariant perturbation, $\delta$), which can be used to postulate stability robustness of a LTV system. Again, we consider the nominal plant to be LTI, and the LTV plant to consist of a norm-bounded, time-varying, output multiplicative perturbation to the nominal. Our formulation, given in section 5.2, applies to single-input single-output (SISO) systems, though the necessary and sufficient condition for robust stability of a LTI plant with a structured LTV perturbation given in theorem 3 holds for the multi-input multi-output (MIMO) case. This theorem is stated in section 5.1, where it is also shown by use of a small gain argument that, in the SISO case, a perturbed LTI system with structure as shown in figure 3 shares the same necessary and sufficient condition for robust stability. Finally, in section 5.3, we state a stability robustness result for the output multiplicatively perturbed LTV plant which utilizes the worst-case LTI $\nu$-gap metric.

### 5.1 The time-varying system stability robustness problem

Suppose $P_0$ is a strictly proper LTI plant. Let $\tilde{\mathbf{\Delta}}$ denote the set of all block diagonal, LTV, causal perturbations, $\tilde{\Delta}$, with $||\tilde{\Delta}|| \leq 1$. For simplicity, assume that each block is a square LTV matrix of dimension $p_i \times p_i$.

The stability robustness problem for the time-varying closed-loop system in figure 2 can be cast as shown in figure 8, where $G = \begin{pmatrix} 0 & \epsilon P_0 \\ I & P_0 \end{pmatrix}$ is the transfer function matrix mapping input signals to output signals. Denoting $Z := \mathcal{F}_l(G, C)$, the system of figure 8 is further reduced to the system shown in figure 9.

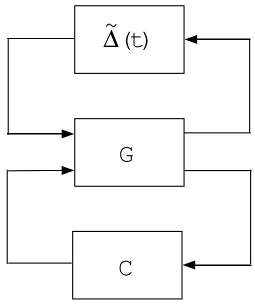


**Figure 8:** Recasting of the closed-loop time-varying system

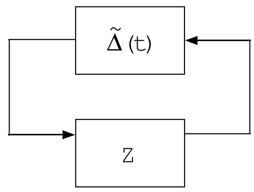


**Figure 9:** Stability robustness problem

Associate with $\tilde{\mathbf{\Delta}}$ a set of scalings that commute with the set of perturbations. For instance, choose $\mathbf{D} = \{\text{diag }(d_1 I_{p_1}, d_2 I_{p_2}, \ldots, d_n I_{p_n}) \mid d_i \in \mathbb{R}, d_i > 0\}$ such that $D^{-1}\tilde{\Delta}(t)D = \tilde{\Delta}(t)$ for all $\tilde{\Delta}(t) \in \tilde{\mathbf{\Delta}}$. Consider the closed-loop system in figure 10. It follows that the stability robustness condition for $Z$ and $D^{-1}ZD$ is the same for any $D \in \mathbf{D}$.

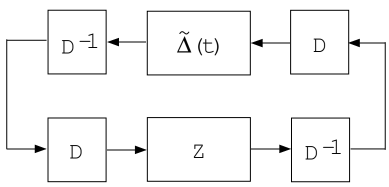


**Figure 10:** Re-scaled system

We now quote a theorem from [11] and [2] that gives the stability robustness condition for the time-varying system.

**Theorem 3** *The system in figure 9 achieves robust stability for all* $\tilde{\Delta}(t) \in \tilde{\mathbf{\Delta}}$ *if and only if* $\inf_{D \in \mathbf{D}} ||D^{-1}ZD||_\infty < 1$.

**Proof:** The proof is given in [11] and [2]. ■

We now proceed considering only SISO systems. In this case, the $D$-scales drop from the formulation and hence the necessary and sufficient condition for robust stability in theorem 3 becomes simply $||Z||_\infty < 1$.

If the LTI system in figure 3 is similarly recast as a robust stability problem as in figure 9 but with $\delta \in \mathcal{RH}_\infty$ and $||\delta||_\infty \leq 1$ replacing $\tilde{\Delta}(t) \in \tilde{\mathbf{\Delta}}$, then by the small gain theorem (for instance, see [16]), the LTI system shares the same condition for robust stability, that is, the closed-loop LTI system is also robustly stable for all $\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1$ if and only if $||Z||_\infty < 1$. We thus have lemma 4.

**Lemma 4** *Robust stability for all $\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1$ of the LTI system in figure 3 is equivalent to robust stability for all $\Delta(t) : ||\Delta|| \leq \epsilon$ of the LTV system in figure 2.*

**Proof:** Since both $\delta$ and $\Delta(t)$ are SISO perturbations, $||Z||_\infty < 1$ is equivalent to both the first and second parts of the lemma statement. ■

### 5.2 Formulation of a worst-case LTI $\nu$-gap metric

We now recast the stability problem for the LTI system in figure 3 in terms of the $\nu$-gap metric. From [14], we know that if the distance between $P_0$ and the perturbed plant $(1+\epsilon\delta)P_0$, measured in the $\nu$-gap metric, is sufficiently small in the generalized robust stability margin sense for all $\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1$, and the nominal closed-loop system is stable, then the time-invariant perturbed system as shown in figure 3 is stable for all $\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1$. That is, if $[P_0, C]$ is stable and

$$\delta_\nu(P_0, (1+\epsilon\delta)P_0) < b_{P_0,C}$$

$\forall \delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1$, then $[(1+\epsilon\delta)P_0, C]$ is stable $\forall \delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1$. Clearly, this statement can be rewritten as follows: if $[P_0, C]$ is stable and

$$\sup_{\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1} \delta_\nu(P_0, (1+\epsilon\delta)P_0) < b_{P_0,C}, \qquad (15)$$

then $[(1+\epsilon\delta)P_0, C]$ is stable for all $\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1$.

Using the SISO chordal distance formula for the LTI $\nu$-gap metric given by [12], we can now reformulate $\sup_{\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1} \delta_\nu(P_0, (1+\epsilon\delta)P_0)$ into an easily calculable form for which the winding number condition is always satisfied by finding upper and lower bounds on the worst-case $\nu$-gap metric, which both equate to $\delta_\nu(P_0, (1-\epsilon)P_0)$.

**Theorem 4** *Given $\epsilon \in [0, 1)$,*

$$\sup_{\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1} \delta_\nu(P_0, (1+\epsilon\delta)P_0) = \delta_\nu(P_0, (1-\epsilon)P_0).$$

**Proof:** To be published elsewhere. ■

From theorem 4 and (15), we have the following lemma.

**Lemma 5** *If $[P_0, C]$ is stable and $\delta_\nu(P_0, (1-\epsilon)P_0) < b_{P_0,C}$, then the LTI closed-loop system $[(1+\epsilon\delta)P_0, C]$ as shown in figure 3 is stable for all $\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1$.*

**Proof:** Follows directly from theorem 4 and theorem 3.8 from [12]. ■

### 5.3 Stability robustness result for time-varying systems

We now have the following stability robustness result for output multiplicatively perturbed SISO time-varying systems.

**Theorem 5** *Suppose $P_0$ is a nominal LTI plant and the closed-loop system $[P_0, C]$ is stable. If $\delta_\nu(P_0, (1-\epsilon)P_0) < b_{P_0,C}$, then the LTV closed-loop system $[(1+\Delta)P_0, C]$ as shown in figure 2 is stable for all causal LTV perturbations $\Delta(t)$ satisfying $||\Delta|| \leq \epsilon$.*

**Proof:** By lemma 5, we know that if the LTI $\nu$-gap metric between $P_0$ and $(1-\epsilon)P_0$ is sufficiently small, then the LTI closed-loop system $[(1+\epsilon\delta)P_0, C]$ is stable for all $\delta \in \mathcal{RH}_\infty : ||\delta||_\infty \leq 1$. Then, by lemma 4, the LTV closed-loop system$[(1+\Delta)P_0, C]$ must be stable for all $\Delta(t)$ such that $||\Delta|| \leq \epsilon$. ■

The following example illustrates the power of theorem 5. Again consider $P_0 = \frac{1}{s+1}$. The worst-case time-invariant $\nu$-gap metric calculation gives $\delta_\nu(P_0, (1-\epsilon)P_0) = \frac{\epsilon}{\sqrt{2}\sqrt{\epsilon^2-2\epsilon+2}}$ for some $\epsilon \in [0, 1)$. This function is shown in the following figure (solid line), along with the time-varying $\nu$-gap metric upper bound, $\frac{\epsilon}{\sqrt{2}}$, from [1] (dashed line), and the time-varying $\nu$-gap metric approximation upper bound, $\epsilon 0.5786$, obtained in section 4.3 (dotted line), for output multiplicative error $\Delta(t) = \epsilon \sin at$. Note that the upper bound on the

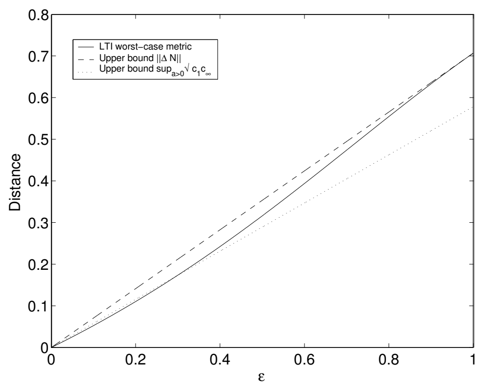


**Figure 11:** Comparison of examples

approximation is only valid for small $\epsilon$ since it is an approximation.

From the figure, we can deduce that if the time-varying Vinnicombe metric upper bound of $\frac{\epsilon}{\sqrt{2}}$ is to be used to provide stability robustness information about the time-varying system with form $(1 + \epsilon \sin at)P_0$, one would require a larger generalized robust stability margin, $b_{P_0,C}$. This is the same as saying that the upper bound from [1] gives a more conservative test for robust stability of LTV systems than the results derived in this paper. For known subclasses of output multiplicative errors and small $\epsilon$, the upper bound on the $\nu$-gap metric approximation may be able to be used, thus requiring a smaller $b_{P_0,C}$, though at the expense of more difficult calculations. However, for both ease of calculation and for the requirement of a smaller generalized robust stability margin, the stablity test of theorem 5 is shown to be a powerful alternative for determining LTV system stability.

## 6 Conclusion

One aim of this paper was to present a first-order approximation to the time-varying Vinnicombe metric between a LTI nominal system and a LTV system, related to the nominal by an output multiplicative error. An approach for placing a relatively tight upper bound on the approximation was shown for a subclass of small periodic output multiplicative errors. This calculable upper bound on the approximation may then be used to postulate stability robustness of the time-varying system. For larger or for more general classes of output multiplicative perturbations, the upper bound from [1] may be used at the expense of more conservative results. It was also shown that a worst-case LTI $\nu$-gap metric which is easily calculated can instead be used to postulate stability robustness of a SISO LTV output multiplicatively perturbed plant. This worst-case LTI $\nu$-gap metric calculation is simple to compute and gives a result that is sufficiently tight in the sense that it requires a smaller generalized robust stability margin to guarantee robust stability of the LTV system. We have made an initial attempt to generalize the results of this paper to general MIMO systems and our calculations seem very promising. These MIMO generalizations will be published elsewhere.

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