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Sensor and Network Topologies of Formations with Direction, Bearing and Angle Information between Agents

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***Abstract*— Sensor and network topologies of formations of autonomous agents are considered. The aim of the paper is to suggest an approach for such topologies for formations with direction, bearing and angle information between agents in the plane and in 3-space. A number of results are translated from prior work in this field and in the study of constraints in CAD programming, in rigidity theory, in structural engineering and in discrete mathematics. Some new results are presented both for the plane and for 3-space. A number of unsolved problems are also mentioned.**

I. INTRODUCTION

In a previous paper ([4], see also [5]) we suggested an approach based on rigidity for maintaining formations of autonomous agents with sensor and network topologies that use distance information between agents. In this paper, the approach is extended to the case in which agents use other types of information to maintain rigidity such as directions, bearings and angles. The challenge is that a comprehensive theory of such topologies of formations with communication limitations does not exist. By a formation, we mean a group of mobile autonomous agents moving in real 2 or 3-space. A formation is called rigid if the distance between each pair of agents does not change over time under ideal conditions. Sensing and communication links are used for maintaining fixed distances between agents. It is not necessary to have sensing and communication links between each pair of agents to maintain a rigid formation [4]. Distances between all agent pairs can be held fixed by directly measuring distances between only some agents and keeping them at desired values. Alternatively, the distance between each pair of agents can be held fixed with constraints prescribing directions, bearings and angles between agents along with fewer distances. We refer the reader to [3] for a combination of bearings and distances in formations.

Rigidity of frameworks has studied distance constraints in the plane and in 3-space [12]. Other studies in Computer Aided Design have investigated configurations constrained by mixed types of constraints in the plane [8], and for directions alone in 3-space [9], [10]. The current combinatorial and geometric theory of topologies constrained by distances, directions, bearings and angles is still far from complete in 3-space. A well-developed foundation is eventually needed to provide rigorous techniques to create such formations and to maintain them under operations like agent departure, splitting, merging and reconfiguration.

The aims of this paper are:

1) to suggest an approach for analyzing formations of autonomous agents with a variety of types of information between agents: directions, bearings and angles;
2) to summarize and transfer what is known about direction constraints in the plane, and 3-space, as well as angles, from work in discrete geometry;
3) to develop some useful steps for creating rigid formations in the plane and in 3-space for sensor and network topologies based on directions, bearings and angles.

The paper is organized as follows. We start with an overview of point formations and rigidity in §2. Then we discuss formations with direction and bearing constraints in §3, where there is a complete theory for both the plane and for 3-space. In §4 some initial work on pure angle constraints is described. The paper ends with concluding remarks in §5.

II. POINT FORMATIONS AND RIGIDITY

A point formation $\mathbb{F}_p \triangleq (p, \mathcal{E})$ provides a way of representing a formation of n agents. $p \triangleq \{p_1, p_2, \ldots, p_n\}$ and the points p_i represent the positions of agents in $\mathbb{R}^d$ $\{d = 2 \text{ or } 3\}$ at time t where i is an integer in $\{1, 2, \ldots, n\}$ and denotes the labels of agents. $\mathcal{E}$ is the set of "maintenance links", labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$. The *maintenance links* in $\mathcal{E}$ correspond to constraints between specific agents, such as distances, directions, bearings or angles, which are to be maintained over time by using sensing and communication links between certain pairs of agents. Each point formation $\mathbb{F}_p$ uniquely determines a graph $\mathbb{G}_{\mathbb{F}_p} \triangleq (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} \triangleq \{1, 2, \ldots, n\}$, which is the set of labels of agents, and edge set $\mathcal{E}$. We will denote the set of maintenance links with distance constraints by $\mathcal{L}$, the set of maintenance links with direction constraints by $\mathcal{D}$ and the set of maintenance links with bearing constraints by $\mathcal{B}$. In the sequel, we will explain a way of converting bearings

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to directions, and hence shall be concerned with direction constraints only for studying bearings. Note that an angle constraint is defined on two maintenance links. We will denote an angle constraint by a pair $\{(i,j),(j,k)\}$ where j is the angle's vertex. We will denote the set of angles $\{\{(i,j),(j,k)\}$: $(i,j),(j,k) \in \mathcal{E}\}$ by $\mathcal{A}$. A formation with distance constraints can be represented by $(\mathcal{V},\mathcal{L},f)$ where $f : \mathcal{L} \longmapsto \mathbb{R}$. Each maintenance link $(i,j) \in \mathcal{L}$ is used to maintain the distance $f((i,j))$ between certain pairs of agents fixed. A formation with direction constraints can be represented by $(\mathcal{V},\mathcal{D},g)$ where $g : \mathcal{D} \longmapsto \mathbb{S}^{d-1}$ (the unit sphere of directions). Each maintenance link $(i,j) \in \mathcal{D}$ is used to maintain the direction $g((i,j))$ of the line joining certain pairs of agents fixed with respect to a reference coordinate system. A formation with angle constraints can be represented by $(\mathcal{V},\mathcal{A},h)$ where $h : \mathcal{A} \longmapsto [0,2\pi)$ and $\mathcal{A} \subseteq \mathcal{E} \times \mathcal{E}$. Each angle is used to maintain the angle between two selected links fixed. A single type of constraints [4], [5], [7], [1] or a combination of types of constraints [3] can be used to maintain a rigid formation.

A *trajectory* of a formation is a continuously parameterized one parameter family of curves $(q_1(t),q_2(t),...,q_n(t))$ in $\mathbb{R}^{nd}$ which contain p and on which for each t, $p(t)$ is a formation with the same measured values under f,g,h. A *rigid motion* is a trajectory along which point formations contained in this trajectory are congruent to each other. We will say that two point formations $\mathbb{F}_p$ and $\mathbb{F}_q$ are congruent if they have the same graph and if p and q are congruent. p is *congruent* to q in the sense that there is a distance preserving map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T(q_i) = p_i, i \in \{1,2,\dots,n\}$. If rigid motions are the only possible trajectories then the formation is called *rigid*, otherwise *flexible*. We refer the reader to [4], [5] for such formations.

A *parallel rigid motion* is a trajectory along which point formations contained in this trajectory are translations or dilations of each other. Two point formations $\mathbb{F}_p$ and $\mathbb{F}_q$ are *parallel* if they have the same graph and their corresponding maintenance links are parallel to each other. If parallel rigid motions are the only possible trajectories then the formation is called *parallel rigid*, otherwise *parallel flexible*.

A *homothetic rigid motion* is a trajectory along which point formations contained in this trajectory are similar to each other. Two point formations $\mathbb{F}_p$ and $\mathbb{F}_q$ are *similar* if they have the same graph and one can be obtained from the other by congruence and scaling transformations. If homothetic rigid motions are the only possible trajectories then the formation is called *homothetic rigid*, otherwise *homothetic flexible*.

§III shall be concerned with parallel rigid formations, and §IV shall be concerned with homothetic rigid formations. In the context of each section we will use the word rigid and flexible throughout the section knowing that the adjectives (parallel, homothetic) are implicit unless there is a danger of confusion.

III. POINT FORMATIONS BASED ON DIRECTIONS AND BEARINGS

A direction constraint is determined by the direction of the line joining two agents. We will denote a point formation based on directions by $\mathbb{F}_p = (\{p_1,p_2,\dots,p_n\},\mathcal{D})$, where $\mathcal{D}$ is the set of direction constraints. Central to the development of direction constraints will be the use of parallel drawings of configurations ([9], [10] [8]).

A. Direction Constraints in the Plane

Note that bearing constraints in the plane can easily convert into direction constraints. Bearing constraints between agents are determined by the angle between a maintenance link and the reference coordinate system of an agent. For example, the bearing constraints between two agents i and j are the angles θ_{ij} and θ_{ji} between the link (i,j) and the reference coordinate systems of agents i and j respectively. The conversion of bearing constraints to direction constraints is as follows: We pick an arbitrary point p_i in p and establish its location and reference coordinate system as the base by freezing out rotations and translations. Then for each attached link, we have the direction of that link. Now, at new vertices linked in, we can propagate through to that reference coordinate system, and then give directions to all attached links. Then we go out from those links to any new vertices, and repeat the process of extracting directions for new links. In the end, every link has directions (up to initial choice) and we can apply the usual theory of directions and parallel drawings. From now on, we will be concerned with only directions in the section.

Given a point formation in the plane with direction constraints $\mathbb{F}_p$, we are interested in parallel point formations $\mathbb{F}_q$ in which $q_i - q_j$ is parallel to $p_i - p_j$ for all $(i,j) \in \mathcal{D}$. Using the $\perp$ operator, for turning a vector by $90°$ counterclockwise, these constraints can be written

$$(p_i - p_j)^{\perp} \cdot (q_i(t) - q_j(t)) = 0, \quad (i,j) \in \mathcal{D}, \quad t \geq 0 \quad (1)$$

Such constraints are also called *normal constraints*. This gives a system of $|\mathcal{D}|$ homogenous linear equations. A solution of this system is called a *parallel point formation*. Trivially parallel point formations are translations and dilations of the original point formation, including the parallel point formation in which all points are coincident. All others are non-trivial. A point formation with direction constraints is called *rigid* if all parallel point formations are trivially parallel. Otherwise it is called *flexible*. Taking the derivative of (1), we obtain

$$(p_i - p_j)^{\perp} \cdot (\dot{q}_i(t) - \dot{q}_j(t)) = 0, \quad (i,j) \in \mathcal{D}, \quad t \geq 0 \quad (2)$$

These equations can be rewritten in matrix form as

$$T(\mathbb{F}_p)\dot{q} = 0 \quad (3)$$

where $\dot{q}$ = column $\{\dot{q}_1,\dot{q}_2,\dots,\dot{q}_n\}$. If the formation contains at least 3 points which are not contained in any proper hyperplane within $\mathbb{R}^2$, then there are always three independent solutions of (2) corresponding to the derivative of parallel rigid trajectory: translations along each two axes and one scaling. If the system of equations in (2) has no other solutions, i.e.

$$\operatorname{rank} T(\mathbb{F}_p) = 2n - 3$$

then the rigidity of formation would be implied.

The two problems of infinitesimally rigid point formations with distance constraints in the plane and parallel rigid point formations with direction constraints in the plane are 'dual', and the characterizations and methods for infinitesimally rigid point formations can be translated to parallel rigid point formations [8], [10]. A rigid point formation with minimum number of constraints is called a *minimally rigid* point formation. A direction constraint between two points is sometimes denoted with a tick mark denoting the normal constraint.

A point formation $\mathbb{F}_p$ is *generically rigid* if it is rigid and if there is an open neighborhood of points about p in $\mathbb{R}^{dn}$ at which $\mathbb{F}_p$ is also rigid. The generic rigidity question can be posed solely in terms of the graph $\mathbb{G}_{\mathbb{F}_p} \triangleq (\mathcal{V}, \mathcal{D})$ without any reference to $\mathbb{F}_p$'s actual points. The translation of Laman's conditions [4] for generic direction formations in the plane can be given as follows.

Theorem 1. *A graph $\mathbb{G} = (\mathcal{V}, \mathcal{D})$ where $|\mathcal{V}| \geq 2$ is a generically minimally rigid direction formation if and only if, (i) $|\mathcal{D}| = 2|\mathcal{V}| - 3$; (ii) for every $\mathbb{G}' = (\mathcal{V}', \mathcal{D}') \subseteq \mathbb{G} = (\mathcal{V}, \mathcal{D})$, where $|\mathcal{V}'| \geq 2$, $|\mathcal{D}'| \leq 2|\mathcal{V}'| - 3$.*

where $|\mathcal{V}|$ and $|\mathcal{D}|$ denote the number of elements in the sets $\mathcal{V}$ and $\mathcal{D}$ respectively.

There are sequential techniques to generate rigid classes of graphs both in the plane and in 3-space based on 0 and 1 extension and vertex splitting operations. We explain and use only the first two operations in the sequel. We refer the reader to [11] for the vertex splitting operation. Before explaining these operations and sequences, we introduce some additional terminology which will be useful. If (i, j) is an edge then we say that i and j are *neighbors* of each other. The *degree* or *valency* of a vertex i is the number of neighbors of i. If a vertex has k neighbors, it is called a vertex of degree k or a k-valent vertex. Now we are ready to present the following operations.

0-Extension (Vertex Addition): Let $\mathbb{G} = \{\mathcal{V}, \mathcal{L}\}$ be a graph with a vertex i of degree d in d-space; let $\mathbb{G}^* = \{\mathcal{V}^*, \mathcal{L}^*\}$ denote the subgraph obtained by deleting i and the edges incident with it. Then $\mathbb{G}$ is generically minimally rigid if and only if $\mathbb{G}^*$ is generically minimally rigid.

1-Extension (Edge Splitting): Let $\mathbb{G} = \{\mathcal{V}, \mathcal{L}\}$ be a graph with a vertex i of degree $d+1$, let $\mathcal{V}_i$ be the set of vertices adjacent to i, and let $\mathbb{G}^* = \{\mathcal{V}^*, \mathcal{L}^*\}$ be the subgraph obtained by deleting i and its $d+1$ incident edges. Then $\mathbb{G}$ is generically minimally rigid if and only if there is a pair j, k of vertices of $\mathcal{V}_i$ such that the edge (j, k) is not in $\mathcal{L}^*$ and the graph $\mathbb{G}' = (\mathcal{V}^*, \mathcal{L}^* \bigcup (j, k))$ is generically minimally rigid.

Applied in sequence, these operations generate all generically minimally rigid point formations in the plane based on directions. These methods translate the Henneberg sequences used for formations with distance constraints in [4], [8], [12] and can be given as follows. Starting from two vertices connected by an edge denoted by $\mathbb{G}_2$, the underlying graphs of point formations $\mathbb{G}_3, \mathbb{G}_4, \ldots, \mathbb{G}_n$ are created by either (i) adding a new vertex to $\mathbb{G}_i$ with two edges, or (ii) splitting an edge in $\mathbb{G}_i$ by removing it and adding a new vertex connected to the end points of the removed edge and another third vertex. At each step, a generically minimally rigid point formation is created. Moreover, all generically minimally rigid graphs are created by such sequences. A visual example of the analogous Henneberg sequence in 3-space is given in the next section.

B. Direction Constraints in 3-space

While the direction of a line in the plane can be uniquely determined by a normal vector, two normal vectors are needed for determining the direction of a line in 3-space. One normal vector only determines a plane in 3-space. Hence for a given pair of points p_i and p_j, the direction constraint leads to two linear equations involving the two normals $(p_i - p_j)^{N_1}$ and $(p_i - p_j)^{N_2}$. To get a nice mathematical theory, with one equation per constraint, we will split the 'direction constraint' into such a pair, and permit only one of the pair to occur as a constraint. We recognize that it is unlikely, but not impossible that the maintenance links will measure only one component of a direction, but that might be true, with either the horizontal or vertical component selected.

In the theory which follows, drawn from parallel drawings [9], [10], $\mathcal{D}$ will become a multigraph which can have two 'edges' between a pair of vertices. The constraint equations take the form:

$$\begin{aligned} (p_i - p_j)^{N_1} \cdot (q_i(t) - q_j(t)) &= 0, \quad (i,j) \in \mathcal{D}, \quad t \geq 0 \\ (p_i - p_j)^{N_2} \cdot (q_i(t) - q_j(t)) &= 0, \quad (i,j) \in \mathcal{D}, \quad t \geq 0 \end{aligned}$$

The selected constraint equations have to be satisfied along the trajectories. If we take the derivative of these equations, we obtain

$$\begin{aligned} (p_i - p_j)^{N_1} \cdot (\dot{q}_i(t) - \dot{q}_j(t)) &= 0, \quad (i,j) \in \mathcal{D}, \quad t \geq 0 \\ (p_i - p_j)^{N_2} \cdot (\dot{q}_i(t) - \dot{q}_j(t)) &= 0, \quad (i,j) \in \mathcal{D}, \quad t \geq 0 \end{aligned} \tag{4}$$

Of course we can choose to only impose one normal for a given pair, rather than two independent normals. Which normal we choose would have to be specified (for example as the normal to the plane defined with a third point, when we are comparing with angles). These equations can be rewritten in matrix form as in (3). If the formation contains at least 4 points which are not contained in any proper hyperplane within $\mathbb{R}^3$, then there are always four independent solutions of (4) corresponding the derivatives of trajectories of a parallel point formation, where there are translations along each three axes and one scaling. If the system of equations in (4) has no other solutions, *i.e.*

$$\operatorname{rank} T(\mathbb{F}_p) = 3n - 4$$

then the rigidity of formation would be implied. Laman type conditions are still necessary and sufficient for rigid graphs in 3-space [9], [10]. A sequence for generating generically minimally rigid direction based point formations in 3-space can be given as follows.

Theorem 2. *Starting from two vertices connected by normal constraints N_1N_2, denoted by $\mathbb{G}_1$, the underlying graphs of formations $\mathbb{G}_2, \mathbb{G}_3, \ldots, \mathbb{G}_n$ are created by one of three steps:*

1) *connecting a new vertex by three normal constraints to the existing vertices;*
2) *removing an edge in the current formation and adding a new vertex connecting it to the end-points of the removed edge with one normal constraint to each and then two more normal constraints to any of the vertices without exceeding two normal constraints to each vertex;*
3) *removing two edges in the current formation and adding a new vertex connecting to the four end-points of the removed edges with one normal constraint to each (and two edges if the removed edges both contacted a given vertex) and then two more normal N constraints to any of the vertices without exceeding two normal constraints to each vertex.*

Then at each step, a minimally rigid formation is created. Moreover, all generically minimally rigid formations are created in this way.

This theorem is a direct corollary of the recent results of [6] and the older results of [9]. An example of such a sequence is shown in Fig. 1.

This complete combinatorial theory, with corresponding fast algorithms for independence and parallel rigidity, stands in contrast to the long-standing unsolved problem of finding a polynomial time algorithm for which graphs are generically rigid in 3-space. Put simply, direction constraints are easier to handle than distance constraints.

Note that if at least one distance constraint is added to a parallel rigid point formation, then only translations of the point formation will be the trivial point formations.

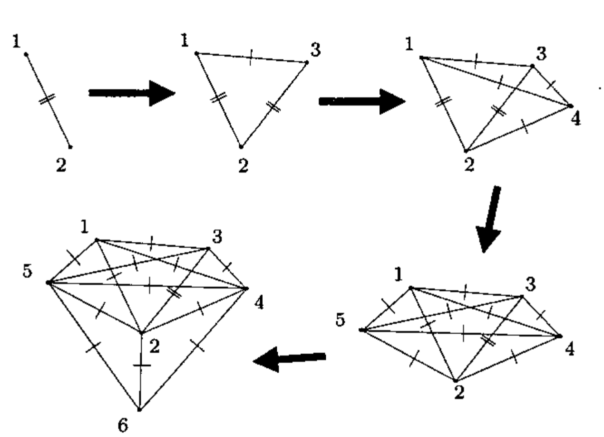


Fig. 1.

Formations with a combination of distance and direction constraints having only translations as trivial motions are called *translation rigid* formations. If a formation is translation rigid with $dn - d$ constraints, it is called a *minimally translation rigid formation* in d-space. We refer the reader to [8] which characterizes generically translation rigid formations of distances and directions in the plane.

For moving formations, pure directions in a fixed coordinate system are too restrictive. However, if there is a moving reference coordinate system, established for example by a core object or subconfiguration, then the directions can be held relative to this reference coordinate system. The relevant theory will still apply without change.

IV. POINT FORMATIONS BASED ON ANGLES

In practice, one might measure the angle between two maintenance links. In the most general form, this problem becomes much more complex and there is no complete theory. In fact, it is conjectured that there is no polynomial time algorithm to detect generically minimally rigid configurations of angles. What can be said is that, in the same sense that directions and distances in the plane generate isomorphic theories, the first-order theory of angles between segments defined by pairs of points are isomorphic to the first-order theory of ratios of lengths between the same segments. Here, we give some initial results.

A. A Matrix for First-order Angle Constraints.

The following analysis builds on the preliminary work in [2]. For an angle between two lines p_1p_2 and p_3p_2 meeting at the vertex p_2, the cosine of the angle satisfies the equation:

$$\big[(p_1 - p_2)\cdot(p_1 - p_2)\big]\big[(p_3 - p_2)\cdot(p_3 - p_2)\big]\cos^2(\alpha) = \big[(p_1 - p_2)\cdot(p_3 - p_2)\big]^2$$

If we assume that the points are functions of t, but the angle is constant, then implicit differentiation gives:

$$\begin{aligned}2\cos^2(\alpha)\Big\{&\big[(p_1 - p_2)\cdot(\dot{p}_1 - \dot{p}_2)\big]\big[(p_3 - p_2)\cdot(p_3 - p_2)\big]\\ &+\big[(p_1 - p_2)\cdot(p_1 - p_2)\big]\big[(p_3 - p_2)\cdot(\dot{p}_3 - \dot{p}_2)\big]\Big\}\\ = 2\big[(p_1 - p_2)&\cdot(p_3 - p_2)\big]\big[(\dot{p}_1 - \dot{p}_2)\cdot(p_3 - p_2)\\ &+(p_1 - p_2)\cdot(\dot{p}_3 - \dot{p}_2)\big]\end{aligned}$$

For compactness, we write $|p_1 - p_2| = l_{12}$ and $|p_2 - p_3| = l_{23}$. This equation can be simplified to:

$$\left[\frac{(p_1 - p_2)^{\perp}}{l_{12}^2}\right]\cdot\dot{p}_1 + \left[\frac{(p_2 - p_1)^{\perp}}{l_{12}^2} + \frac{(p_3 - p_2)^{\perp}}{l_{23}^2}\right]\cdot\dot{p}_2 + \left[\frac{(p_2 - p_3)^{\perp}}{l_{23}^2}\right]\cdot\dot{p}_3 = 0$$

In matrix form this linear equation gives the row

$$\begin{pmatrix} v_1 & v_2 & v_3 \\ \frac{(p_1-p_2)^{\perp}}{l_{12}^2} & \frac{(p_2-p_1)^{\perp}}{l_{12}^2} + \frac{(p_3-p_2)^{\perp}}{l_{23}^2} & \frac{(p_2-p_3)^{\perp}}{l_{23}^2}\end{pmatrix}$$

where v_i is vertex i and is just placed at the top of the row to point the entry corresponding to this vertex.

For a set of points and an angle set, we now have a *first-order angle matrix* $M_{\mathbf{F}_p}$, with two columns for each vertex since each p_i has x and y coordinates, and a row for each angle. The task is to give conditions under which the rows of this matrix are independent (or have maximal rank for the number of vertices). The reader can immediately see that the row for an angle is precisely a linear combination of the rows for the two directions of the segments at the angle.

B. Dependence of Angle Polygons.

The trivial motions of a plane angle formation are the translations, the rotations, and the scaling. If we have at least three points, these form a space of dimension 4. In a triangle, we have three possible angles - but the counts indicate we can only have $2|\mathcal{V}|-4 = 6-4 = 2$ independent angles. As we already know from geometry that the three angles are dependent: $\theta_{1,2,3}+\theta_{2,3,1}+\theta_{3,1,2} = \pi$. This basic dependence will be reflected in a dependence of the rows for the matrix.

Theorem 3. *Angle Polygons Given a formation with angle constraints with an angle polygon (a cycle in the angle diagram)* $L_1, \theta_{1,2}, L_2, \theta_{2,3}, \ldots, \theta_{k-1,k}, L_k, \theta_{k,1}, L_1$, *the rows of the constraint matrix* $M_{\mathbf{F}_p}$ *corresponding to the polygon form a minimal dependent set.*

One can offer some necessary counting properties of minimally rigid sets of angles in a generic formation $\mathbf{F}_p$ with a pair of sets of vertices and angles $(\mathcal{V}, \mathcal{A})$.

1) $|\mathcal{A}| = 2|\mathcal{V}| - 4$;
2) for all non-empty subsets of angles $\mathcal{A}' \subset \mathcal{A}$, on vertices $\mathcal{V}'$, $|\mathcal{A}'| \leq 2|\mathcal{V}'| - 4$;
3) for the set of angles $\mathcal{A}_0$ centered at v_0, with end points $\mathcal{V}_0$, $|\mathcal{A}_0| \leq |\mathcal{V}_0| - 1$.

Unfortunately, these conditions are not sufficient, as the general polygonal result above demonstrates. It is conjectured that there is no combinatorial characterization of all independent sets of angles which can be checked in polynomial time [12].

C. Sequential Techniques.

For plane angles we have similar extension operations to those of the previous sections. Again, the analysis builds on the preliminary work in [2].

1) 0-extension with two angles: The process adds two columns (for the new vertex) and two rows for the two added angles.

Theorem 4. *The 0-extension with two angles (Fig. 2) take a generically minimally rigid formation* $\mathbb{H} = (\mathcal{V}, \mathcal{A}, h)$ *to a generically minimally rigid formation* $\mathbb{H}' = (\mathcal{V}', \mathcal{A}', h')$.

2) 1-extension with angles: These cover the additional of one new vertex, removal of one angle and the addition of three angles. However, with even four angles and four points, we may have $3 \times 4 + 3 = 15$ nonzero blocks - so

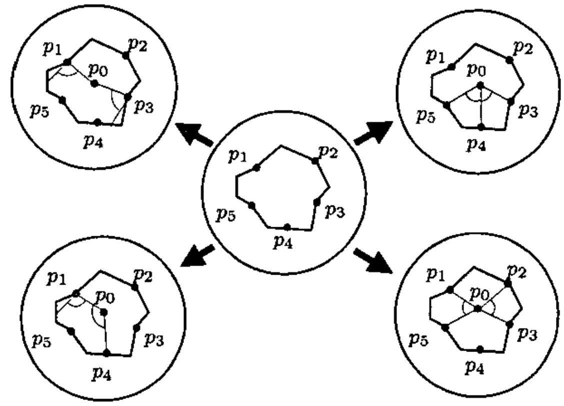


Fig. 2.

each column may have at least three active rows. Therefore the sequential techniques must include methods for creating new vertices with three active rows.

Theorem 5. *The 1-extensions with angles illustrated in Fig. 3 take a generically minimally rigid formation* $(\mathcal{V}, \mathcal{A}, h)$ *to a generically minimally rigid formation* $(\mathcal{V}', \mathcal{A}', h')$.

We note that these two sets of extensions do not generate all possible generically minimally homothetic rigid graphs. To accomplish that, we would need operations for inserting vertices of valence up to 5, and we have stopped with a last vertex of valence 2 or 3. There are some additional conjectured sequential steps, such as forms of vertex splitting, which await further work. However, the conjectured sequential steps will not fill this gap. We do not anticipate a set of sequential constructions for plane angles which will generate all generically minimally homothetic rigid graphs for angles. However these techniques do generate useful rigid subsets, whose inclusion in larger sets will guarantee rigidity.

There are special sets of angles in the plane which are well enough connected that after choosing one line of sight (one pair for a direction) then the remaining angle constraints simply become additional fixed directions

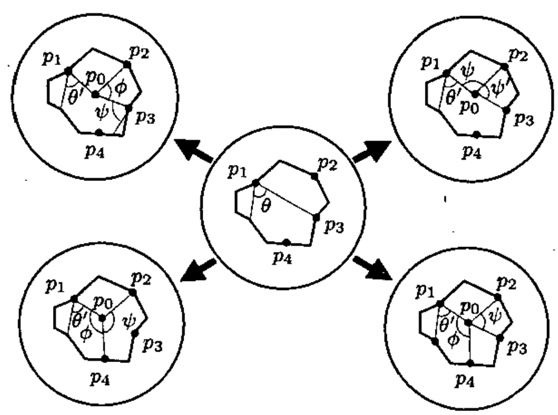
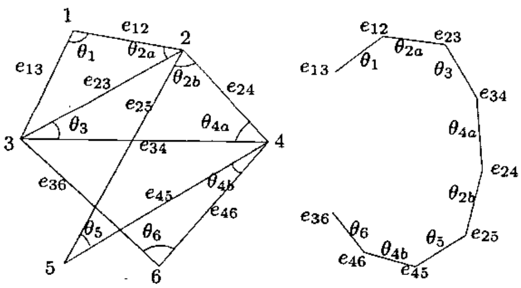


Fig. 3.



(a) (b)

Fig. 4.

(relative to the original choice). Consider an angle constraint assigned to each vertex so that in the corresponding formation, each agent has a set point reference to follow. As it turns out, the abstract graph $\mathbb{G}(\mathcal{V}, \mathcal{E})$ of such a formation can be considered in an alternative graph form that has vertex set $\mathcal{E}$ and two vertices $e, f \in \mathcal{E}$ are adjacent if and only if they are the two sides of an angle $\theta \in \Theta$ where Θ is the set of angles. Note that the abstract graph of edges as vertices and angles as edges in Fig. 4a is connected. Hence the angle information can be converted to directions, by arbitrarily freezing out the rotations by assigning a direction to one of the edges as shown in Fig. 4b. Whenever that is true, we might use the direction results.

D. Angles in 3-space

We can create a similar matrix for angles in 3-space.What we need is the perpendicular to $(p_i - p_j)$ within the plane of the three vertices of the angle p_i, p_j, p_k. This selected normal $(p_i - p_j)^{\perp_{ijk}}$ will then work in a fashion similar to the normals used in direction diagrams or parallel drawing in 3-space. In matrix form this gives the row

$$\begin{pmatrix} \overset{v_1}{\frac{(p_1-p_2)^{\perp 123}}{l_{12}^2}} & \overset{v_2}{\frac{(p_2-p_1)^{\perp 123}}{l_{12}^2} + \frac{(p_3-p_2)^{\perp 123}}{l_{23}^2}} & \overset{v_3}{\frac{(p_2-p_3)^{\perp 123}}{l_{23}^2}} \end{pmatrix}$$

It is no longer true that a polygon of angles is dependent, unless the polygon happens to lie in a plane, as a triangle would.

It is not difficult to state at least some situations in which 0-insertion of a new vertex, with three attached angles works in 3-space, and 1-insertion of a new vertex in 3-space replacing one angle with four angles. However, this work is very preliminary. Among the special cases of angles in 3-space will be the angles for pairs of edges out from a central vertex. Effectively, these angles are distances on the unit sphere, and as such are isomorphic under central projection into a plane, with distance constraints in the plane [12].

V. CONCLUDING REMARKS

The topologies that we have analyzed consist of pure direction, bearing and angle information between agents. Combinations of these information, such as distance-direction, distance-bearing, distance-angle are practically used in formations of autonomous agents. The presented techniques in this paper can be extended to analyze such topologies with mixed constraints (see for example [8]).

It is clear from the developments in this and previous papers [4], [5] that ideas from rigidity, discrete geometry and CAD programming can play a central role in both the analysis and synthesis of sensor and network topologies of provably rigid formations of mobile autonomous agents.

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