15th Triennial World Congress, Barcelona, Spain

FREQUENCY-WEIGHTED BALANCING RELATED CONTROLLER REDUCTION

A. Varga *, **B.D.O. Anderson** **

* *German Aerospace Center, DLR - Oberpfaffenhofen*
Institute of Robotics and Mechatronics
D-82234 Wessling, Germany
Andras.Varga@dlr.de

** *Research School of Information Sciences and Engineering*
Australian National University
Canberra ACT 0200, Australia
Brian.Anderson@anu.edu.au

Abstract: The efficient solution of a class of controller approximation problems by using frequency-weighted balancing related model reduction approaches is considered. It is shown that for certain standard performance and stability enforcing frequency-weights, the computation of the frequency-weighted controllability and observability grammians can be done by solving reduced order Lyapunov equations regardless the controller itself is stable or unstable. The new approach can be used in conjunction with accuracy enhancing square-root and balancing-free techniques recently developed by the authors for the frequency-weighted balancing related model reduction.

Keywords: Controller order reduction, frequency-weighted model reduction, balanced realization, linear systems, numerical methods.

1. INTRODUCTION

Let $\mathbf{G} := (A, B, C, D)$ be a given n-th order state-space model with the *transfer-function matrix* (TFM) $G(\lambda) = C(\lambda I - A)^{-1}B + D$, where $\lambda = s$ is the complex Laplace-transform variable in the case of a continuous-time system or $\lambda = z$ is the complex Z-transform variable in the case of a discrete-time system. Let $\mathbf{K} := (A_c, B_c, C_c, D_c)$ be an n_c-th order stabilizing controller, resulted from controller synthesis methods as the LQG-, H_2- or H_∞-design methodologies. Since these methodologies lead typically to controllers whose orders are often too large for a practical implementation, it is frequently necessary to perform controller reduction by determining a lower order approximation $\mathbf{K}_r$ of the original controller $\mathbf{K}$.

Notation. Throughout the paper, the following notational convention is used. The bold-notation $\mathbf{G}$ is used to denote a state-space system having the TFM $G(\lambda)$ or G. This notation is used consistently to denote systems corresponding to particular TFMs: $\mathbf{G}_1\mathbf{G}_2$ denotes the series coupling of two systems having the TFM $G_1(\lambda)G_2(\lambda)$, $\mathbf{G}_1 + \mathbf{G}_2$ represents the (additive) parallel coupling of two systems with TFM $G_1(\lambda) + G_2(\lambda)$, $\mathbf{G}^{-1}$ denotes the inverse system corresponding to the inverse TFM $G^{-1}(\lambda)$.

The controller reduction problem is frequently formulated as a *frequency-weighted model reduction* (FWMR) problem (Anderson and Liu, 1989) to find $\mathbf{K}_r$, an r_c-th order approximation of $\mathbf{K}$ having the same number of unstable poles as $\mathbf{K}$, such that a weighted error of the form

$$\|W_o(K - K_r)W_i\|_\infty, \tag{1}$$

is minimized, where W_o and W_i are suitably chosen weighting TFMs. To enforce closed-loop stability, one-sided weights of the form

$$W_i = I, \quad W_o = (I + GK)^{-1}G \qquad (2)$$

or

$$W_i = G(I + KG)^{-1}, \quad W_o = I \qquad (3)$$

can be used, while performance-preserving considerations lead to two-sided weights

$$W_o = (I + GK)^{-1}G, \quad W_i = (I + GK)^{-1} \qquad (4)$$

Balancing related FWMR techniques which attempt to minimize (1) can be also used to determine reduced order controllers. The following *frequency-weighted controller reduction* (FWCR) procedure is based on the FWMR approach proposed by Enns (1984):

FWCR Procedure.

1. Compute the additive stable-unstable spectral decomposition $\mathbf{K} = \mathbf{K}_s + \mathbf{K}_u$, where $\mathbf{K}_s$, of order n_{cs}, contains the stable poles of $\mathbf{K}$ and $\mathbf{K}_u$, of order $n_c - n_{cs}$, contains the unstable poles of $\mathbf{K}$.
2. Compute the controllability grammian of $\mathbf{K}_s\mathbf{W}_i$ and the observability grammian of $\mathbf{W}_o\mathbf{K}_s$ and define according to (Enns, 1984), appropriate n_{cs} order frequency-weighted controllability and observability grammians P_E and Q_E, respectively.
3. Using P_E and Q_E in place of standard grammians of $\mathbf{K}_s$, determine a reduced order approximation $\mathbf{K}_{sr}$ by applying, for example, the *balanced truncation* (BT) method (Moore, 1981; Tombs and Postlethwaite, 1987).
4. Form $\mathbf{K}_r = \mathbf{K}_{sr} + \mathbf{K}_u$.

Note that this procedure ensures that the resulting reduced order controller has exactly the same number of unstable poles as the original one. Several enhancements of this procedure have been recently proposed by the authors in the context of a general FWMR approach (Varga and Anderson, 2001).

In this paper the efficient and numerically accurate computation of low order controllers is considered by applying the **FWCR Procedure** to the case of the three particular stability and performance enforcing weights defined in (2), (3) and (4). The main computational burden in this procedure is the computation of the two grammians at Step 2. Typically, controller synthesis methods based on the LQG- or H_∞-design methodologies lead to a controller order $n_c = n$. Thus apparently, for these controllers the computation of grammians involves the solutions of one or two $3n$ order Lyapunov equations. By exploiting the problem structure, Liu *et al.* (1990) have shown that for a *stable* state-feedback and full-order estimator based controller, it is possible to solve Lyapunov equations of order only $2n$. In this paper the results of Liu *et al.* (1990) are extended to the case of a general, possibly unstable controller and it is shown that the grammians can be determined by solving Lyapunov equations of order at most $n + n_c$. Further, a generalization of the results in (Liu *et al.*, 1990) to the case of arbitrary stabilizing state-feedback and observer-based controllers follows as a corollary of our general result. In a separate section, the direct computation of the Cholesky factors of the frequency-weighted grammians is discussed. This is a prerequisite for the applicability of the square-root and balancing-free accuracy-enhancing techniques to controller reduction.

2. EFFICIENT SOLUTION OF SOME CONTROLLER REDUCTION PROBLEMS

This section addresses the specific aspects of computing the frequency-weighted controllability and observability grammians to be employed in the **FWCR Procedure**. To simplify the discussions it is assumed for the beginning that the controller $\mathbf{K} = (A_c, B_c, C_c, D_c)$ is stable and the two weights W_o and W_i are also stable TFMs. In the case of an unstable controller, the discussion applies for the stable part $\mathbf{K}_s$ of the controller.

Consider the minimal realizations of the frequency weights

$$\mathbf{W_o} = (A_o, B_o, C_o, D_o), \quad \mathbf{W_i} = (A_i, B_i, C_i, D_i)$$

and construct the realizations of $\mathbf{KW}_i$ and $\mathbf{W_oK}$:

$$\mathbf{KW_i} = \left[\begin{array}{c|c} \overline{A}_i & \overline{B}_i \\ \hline \overline{C}_i & \overline{D}_i \end{array}\right] =: \left[\begin{array}{cc|c} A_c & B_cC_i & B_cD_i \\ 0 & A_i & B_i \\ \hline C_c & D_cC_i & D_cD_i \end{array}\right] \qquad (5)$$

$$\mathbf{W_oK} = \left[\begin{array}{c|c} \overline{A}_o & \overline{B}_o \\ \hline \overline{C}_o & \overline{D}_o \end{array}\right] =: \left[\begin{array}{cc|c} A_o & B_oC_c & B_oD_c \\ 0 & A_c & B_c \\ \hline C_o & D_oC_c & D_oD_c \end{array}\right] \qquad (6)$$

Let $\overline{P}_i$ and $\overline{Q}_o$ the controllability grammian of $\mathbf{KW_i}$ and the observability grammian of $\mathbf{W_oK}$, respectively. According to the system type, continuous-time (c) or discrete-time (d), $\overline{P}_i$ and $\overline{Q}_o$ satisfy the corresponding Lyapunov equations

$$\begin{array}{ll} \left\{ \begin{array}{ll} \overline{A}_i\overline{P}_i + \overline{P}_i\overline{A}_i^T + \overline{B}_i\overline{B}_i^T & = 0 \\ \overline{A}_o^T\overline{Q}_o + \overline{Q}_o\overline{A}_o + \overline{C}_o^T\overline{C}_o & = 0 \end{array} \right. & (c) \\ \left\{ \begin{array}{ll} \overline{A}_i\overline{P}_i\overline{A}_i^T + \overline{B}_i\overline{B}_i^T & = \overline{P}_i \\ \overline{A}_o^T\overline{Q}_o\overline{A}_o + \overline{C}_o^T\overline{C}_o & = \overline{Q}_o \end{array} \right. & (d) \end{array} \qquad (7)$$

Partition $\overline{P}_i$ and $\overline{Q}_o$ in accordance with the structure of matrices $\overline{A}_i$ and $\overline{A}_o$, respectively, i.e.

$$\overline{P}_i = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad \overline{Q}_o = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \qquad (8)$$

where P_{11} and Q_{22} are $n_c \times n_c$ matrices. The approach proposed by Enns (1984) defines

$$P_E = P_{11}, \quad Q_E = Q_{22} \qquad (9)$$

as the frequency-weighted controllability and observability grammians, respectively. Although successfully employed in many application, the stability of the reduced controller is not guaranteed in the case of two-sided weighting, unless either $W_o = I$ or $W_i = I$. Occasionally, quite poor approximations result even for one-sided weighting. Alternative choices of grammians guaranteeing stability have been proposed by Wang *et al.* (1999) for continuous-time systems. These choices have been recently improved by Varga and Anderson (2001) by reducing the gap to Enns' choice and also extended to discrete-time systems.

For all these choices of the frequency-weighted grammians, the solution of the controller reduction problems for the special weights defined in (2), (3), or (4) involves the solution of Lyapunov equations of order $n + 2n_c$, where n is the order of the open-loop system $\mathbf{G} = (A, B, C, D)$ and n_c is the order of the controller $\mathbf{K}$. Controller synthesis methods based on the LQG- or H_∞-design methodologies lead typically to a controller order $n_c = n$, so that apparently, for these controllers the solutions of $3n$ order Lyapunov equations are necessary. It is shown in what follows that it is always possible to solve Lyapunov equations of order at most $n + n_c$ to compute the frequency-weighted controllability and observability grammians for the special weights (2), (3), or (4). The following theorem extends the result of Liu *et al.* (1990) for an estimator-based controller to the case of an arbitrary stabilizing controller:

Theorem 1. For a given n-th order system $\mathbf{G} = (A, B, C, D)$ assume that $\mathbf{K} = (A_c, B_c, C_c, D_c)$ is an n_c-th order stabilizing controller with $I + DD_c$ nonsingular. Then the frequency-weighted controllability and observability grammians for Enns' method (Enns, 1984) applied to frequency-weighting controller reduction problems with weights defined in (2), (3), or (4) can be computed by solving Lyapunov equations of order at most $n + n_c$.

Proof: Assume at the beginning that the controller $\mathbf{K}$ is stable and consider first the performance preserving input weighting $W_i = (I + GK)^{-1}$. The computation of the controllability grammian for the system $\mathbf{KW_i} = \mathbf{K(I + GK)^{-1}}$ involves the solution of the appropriate continuous-time (c) or discrete-time (d) Lyapunov equation

$$\begin{array}{ll} A_{wi}P_{wi} + P_{wi}A_{wi}^T + B_{wi}B_{wi}^T = 0 & (c) \\ A_{wi}P_{wi}A_{wi}^T + B_{wi}B_{wi}^T = P_{wi} & (d) \end{array} \qquad (10)$$

where A_{wi} and B_{wi} are the state and input matrices of a state-space realization of $\mathbf{K(I + GK)^{-1}}$. A_{wi} and B_{wi} can be constructed in the form

$$A_{wi} = \begin{bmatrix} A_c & -B_cR^{-1}C & -B_cR^{-1}DC_c \\ 0 & A - BD_cR^{-1}C & B\widetilde{R}^{-1}C_c \\ 0 & -B_cR^{-1}C & A_c - B_cR^{-1}DC_c \end{bmatrix},$$

$$B_{wi} = \begin{bmatrix} B_cR^{-1} \\ BD_cR^{-1} \\ B_cR^{-1} \end{bmatrix}$$

with $R = I + DD_c$ and $\widetilde{R} = I + D_cD$. Consider the transformation matrix T

$$T = \begin{bmatrix} I_{n_c} & 0 & I_{n_c} \\ 0 & I_n & 0 \\ 0 & 0 & I_{n_c} \end{bmatrix}$$

It is easy to see that the controllability grammian $\widetilde{P}_{wi}$ for the transformed pair

$$(\widetilde{A}_{wi}, \widetilde{B}_{wi}) := (T^{-1}A_{wi}T, T^{-1}B_{wi})$$

has the form

$$\widetilde{P}_{wi} = \begin{bmatrix} 0 & 0 \\ 0 & P_i \end{bmatrix}$$

where P_i satisfies the appropriate Lyapunov equation

$$\begin{array}{ll} A_iP_i + P_iA_i^T + B_iB_i^T = 0 & (c) \\ A_iP_iA_i^T + B_iB_i^T = P_i & (d) \end{array} \qquad (11)$$

with

$$A_i = \begin{bmatrix} A - BD_cR^{-1}C & B\widetilde{R}^{-1}C_c \\ -B_cR^{-1}C & A_c - B_cR^{-1}DC_c \end{bmatrix},$$

$$B_i = \begin{bmatrix} BD_cR^{-1} \\ B_cR^{-1} \end{bmatrix}$$

With P_i partitioned in accordance with the structure of A_i

$$P_i = \begin{bmatrix} P_{22} & P_{23} \\ P_{23}^T & P_{33} \end{bmatrix}, \qquad (12)$$

the grammian in the original coordinate basis results as

$$P_{wi} = T\widetilde{P}_{wi}T^T = \begin{bmatrix} P_{33} & P_{23}^T & P_{33} \\ P_{23} & P_{22} & P_{23} \\ P_{33} & P_{23}^T & P_{33} \end{bmatrix}$$

Thus the frequency-weighted grammian according to Enns method is $P_E = P_{33}$, the trailing $n_c \times n_c$ block of P_i in (12).

For the stability preserving input weighting $W_i = G(I + KG)^{-1}$, the computation of the controllability grammian for the system $\mathbf{KW_i} = \mathbf{KG(I + KG)^{-1}}$ is very similar to the approach given above. The grammian P_i is computed using the same A_i but with a different B_i, namely

$$B_i = \begin{bmatrix} -B\widetilde{R}^{-1} \\ B_cD\widetilde{R}^{-1} \end{bmatrix}$$

Thus for both choices of the input frequency weight, the frequency-weighted controllability grammian P_E

is the trailing $n_c \times n_c$ block of P_i, the controllability grammian of the frequency weight W_i (assuming a particular realization with the same A_i is used) and it can be computed by solving the $n+n_c$ order Lyapunov equation (11).

For the stability preserving output weighting $W_o = (I+GK)^{-1}G$, the computation of the observability grammian of the system $\mathbf{W_o K} = (\mathbf{I}+\mathbf{GK})^{-1}\mathbf{GK}$ involves the solution of the appropriate Lyapunov equation

$$\begin{array}{ll} A_{wo}^T Q_{wo} + Q_{wo} A_{wo} + C_{wo}^T C_{wo} = 0 & (c) \\ A_{wo}^T Q_{wo} A_{wo} + C_{wo}^T C_{wo} = Q_{wo} & (d) \end{array} \quad (13)$$

where A_{wo} and C_{wo} are the state and output matrices of a state-space realization of $(I+GK)^{-1}GK$. The matrices A_{wo} and C_{wo} can be constructed in the form

$$A_{wo} = \begin{bmatrix} A - BD_cR^{-1}C & B\widetilde{R}^{-1}C_c & -B\widetilde{R}^{-1}C_c \\ -B_cR^{-1}C & A_c - B_cR^{-1}DC_c & B_cR^{-1}DC_c \\ 0 & 0 & A_c \end{bmatrix}$$

$$C_{wo} = \begin{bmatrix} -R^{-1}C & -R^{-1}DC_c & D\widetilde{R}^{-1}C_c \end{bmatrix}$$

By employing the transformation matrix T

$$T = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{n_c} & I_{n_c} \\ 0 & 0 & I_{n_c} \end{bmatrix}$$

the observability grammian $\widetilde{Q}_{wo}$ of the transformed pair

$$(\widetilde{A}_{wo}, \widetilde{C}_{wo}) := (T^{-1}A_{wo}T, C_{wo}T)$$

has the form

$$\widetilde{Q}_{wo} = \begin{bmatrix} Q_o & 0 \\ 0 & 0 \end{bmatrix}$$

where Q_o satisfies the appropriate Lyapunov equation

$$\begin{array}{ll} A_o^T Q_o + Q_o A_o + C_o^T C_o = 0 & (c) \\ A_o^T Q_o A_o + C_o^T C_o = Q_o & (d) \end{array} \quad (14)$$

with

$$A_o = \begin{bmatrix} A - BD_cR^{-1}C & B\widetilde{R}^{-1}C_c \\ -B_cR^{-1}C & A_c - B_cR^{-1}DC_c \end{bmatrix},$$

$$C_o = \begin{bmatrix} -R^{-1}C & -R^{-1}DC_c \end{bmatrix}$$

With Q_o partitioned in accordance with the structure of A_o

$$Q_o = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \quad (15)$$

the grammian in original coordinates results as

$$Q_{wo} = T^{-T}\widetilde{Q}_{wo}T^{-1} = \begin{bmatrix} Q_{11} & Q_{12} & -Q_{12} \\ Q_{12}^T & Q_{22} & -Q_{22} \\ -Q_{12}^T & -Q_{22} & Q_{22} \end{bmatrix}$$

Thus the frequency-weighted grammian according to Enns' method is $Q_E = Q_{22}$, the trailing $n_c \times n_c$ block of Q_o.

In the case of an unstable controller, only the stable part of the controller is reduced and a copy of the unstable part is kept in the reduced controller. Consider a state-space representation of the controller with A_c already reduced to a block-diagonal form

$$\mathbf{K} = \left[\begin{array}{cc|c} A_{c1} & 0 & B_{c1} \\ 0 & A_{c2} & B_{c2} \\ \hline C_{c1} & C_{c2} & D_c \end{array}\right], \quad (16)$$

where $\Lambda(A_{c1}) \subset \mathbb{C}^+$ and $\Lambda(A_{c2}) \subset \mathbb{C}^-$. Here $\mathbb{C}^-$ denotes the open left complex plane of $\mathbb{C}$ in a continuous-time setting or the interior of the unit circle in a discrete-time setting, while $\mathbb{C}^+$ denotes the complement of $\mathbb{C}^-$ in $\mathbb{C}$. The above form corresponds to an additive decomposition of the controller $\mathbf{K} = \mathbf{K}_u + \mathbf{K}_s$, where $\mathbf{K}_u = (A_{c1}, B_{c1}, C_{c1})$ contains the unstable poles of $\mathbf{K}$ and $\mathbf{K}_s = (A_{c2}, B_{c2}, C_{c2}, D_c)$ contains the stable poles of $\mathbf{K}$. Assume the order of $\mathbf{K}_s$ is n^-.

As before, P_i and Q_o are obtained by solving the $(n+n_c)$-order Lyapunov equations (11) and (14), respectively, where A_i and A_o are stable (the controller being stabilizing by assumption). The frequency-weighted controllability grammian P_E of $\mathbf{K}_s\mathbf{W}_i$ can be identified as the trailing $n^- \times n^-$ part of P_{33} in (12) and the frequency-weighted observability grammian Q_E of $\mathbf{W}_o\mathbf{K}_s$ can be identified as the trailing $n^- \times n^-$ part of Q_{22} in (15). □

Some simplifications arise in the case of a state-feedback and full-order observer based controller

$$\mathbf{K} = (A + BF + LC + LDF, F, L, 0) \quad (17)$$

where it is assumed that $A+BF$ and $A+LC$ are stable. The following result is an extension of *Lemma 1* of (Liu *et al.*, 1990) to the case of possibly unstable controllers.

Corollary 2. For a given n-th order system $\mathbf{G} = (A, B, C, D)$ and observer based controller $\mathbf{K}$ (17), suppose F is a state feedback gain and L is a state estimator gain, such that $A+BF$ and $A+LC$ are stable. Then the frequency-weighted controllability and observability grammians for Enns' method (Enns, 1984) applied to the frequency-weighting controller reduction problems with weights defined in (2), (3), or (4) can be computed by solving Lyapunov equations of order at most $2n$.

Proof: For $\mathbf{W_i} = (\mathbf{I}+\mathbf{GK})^{-1}$, the matrices A_i and B_i appearing in (11) are

$$A_i = \begin{bmatrix} A & BF \\ -LC & A+BF+LC \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ L \end{bmatrix}$$

while for $\mathbf{W_i} = \mathbf{G}(\mathbf{I}+\mathbf{GK})^{-1}$, A_i is the same and

$$B_i = \begin{bmatrix} -B \\ LD \end{bmatrix}$$

For $\mathbf{W_o} = (\mathbf{I} + \mathbf{GK})^{-1}\mathbf{G}$, the matrices A_o and C_o appearing in (14) are

$$A_o = A_i, \quad C_o = \begin{bmatrix} -C & -DF \end{bmatrix}$$

Thus in both cases, the computation of grammians involves the solution of appropriate Lyapunov equations of orders at most $2n$. □

In the case of state feedback and observer based controllers important computational effort saving results by further exploiting the structure of A_i. Consider

$$T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

and compute

$$\widetilde{A}_i = T^{-1}A_iT = \begin{bmatrix} A+BF & BF \\ 0 & A+LC \end{bmatrix}$$

and $\widetilde{B}_i = T^{-1}B_i$. If $\widetilde{P}_i$ is the solution of

$$\begin{array}{ll} \widetilde{A}_i\widetilde{P}_i + \widetilde{P}_i\widetilde{A}_i^T + \widetilde{B}_i\widetilde{B}_i^T = 0 & (c) \\ \widetilde{A}_i\widetilde{P}_i\widetilde{A}_i^T + \widetilde{B}_i\widetilde{B}_i^T = \widetilde{P}_i & (d) \end{array}$$

and is partitioned as

$$\widetilde{P}_i = \begin{bmatrix} \widetilde{P}_{22} & \widetilde{P}_{23} \\ \widetilde{P}_{23}^T & \widetilde{P}_{33} \end{bmatrix}$$

then $P_i = T\widetilde{P}_iT^T$ and the trailing $n \times n$ block P_{33} of the partitioned P_i in (12) is obtained as

$$P_{33} = \widetilde{P}_{22} + \widetilde{P}_{23} + \widetilde{P}_{23}^T + \widetilde{P}_{33}$$

Similarly, consider the transformed matrices $\widetilde{A}_o = \widetilde{A}_i$, and $\widetilde{C}_o = C_oT$. The corresponding transformed observability grammian $\widetilde{Q}_o$ satisfies

$$\begin{array}{ll} \widetilde{A}_o^T\widetilde{Q}_o + \widetilde{Q}_o\widetilde{A}_o + \widetilde{C}_o^T\widetilde{C}_o = 0 & (c) \\ \widetilde{A}_o^T\widetilde{Q}_o\widetilde{A}_o + \widetilde{C}_o^T\widetilde{C}_o = \widetilde{Q}_o & (d) \end{array}$$

and the grammian Q_o in the original coordinates is $Q_o = T^{-T}\widetilde{Q}_oT^{-1}$. With $\widetilde{Q}_o$ partitioned as

$$\widetilde{Q}_o = \begin{bmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{12}^T & \widetilde{Q}_{22} \end{bmatrix},$$

the frequency-weighted observability grammian Q_{22} in (15) results as

$$Q_{22} = \widetilde{Q}_{22}$$

The computational saving arises from the need to reduce A_i to a *real Schur form* (RSF) when solving the corresponding Lyapunov equations (11) and (14). Instead of reducing the $2n \times 2n$ matrix A_i to a RSF, only two smaller $n \times n$ matrices $A+BF$ and $A+LC$ are needed to be reduced to obtain $\widetilde{A}_i$ in a RSF. This means a 4 times speedup of computations for this step.

3. SQUARE-ROOT TECHNIQUES

The square-root technique for model reduction has been introduced by Tombs and Postlethwaite (1987). This accuracy enhancing technique relies exclusively on the Cholesky factors of the frequency-weighted grammians, computed in the form $P_E = S_ES_E^T$ and $Q_E = R_E^TR_E$. The method of Hammarling (Hammarling, 1982) can be generally employed to solve (11) directly for the Cholesky factor S_i of $P_i = S_iS_i^T$. In the case of an unstable controller, a state-space realization for $\mathbf{K}$ as in (16) is assumed with the $n^- \times n^-$ matrix A_{c2} containing the stable eigenvalues of A_c. By partitioning S_i in the form

$$S_i = \begin{bmatrix} S_{22} & S_{23} \\ 0 & S_{33} \end{bmatrix}$$

with S_{33} $n^- \times n^-$, the Cholesky factor of the trailing block P_{33} in (12) corresponding to the stable part of $\mathbf{K}$ is simply $S_E = S_{33}$.

Similarly, (14) can be solved directly for the Cholesky factor R_o of $Q_o = R_o^TR_o$. By partitioning R_o in the form

$$R_o = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \tag{18}$$

with a $n^- \times n^-$ trailing block R_{22}, it follows that the Cholesky factor R_E of the trailing block Q_{22} in (15) satisfies $R_E^TR_E = R_{22}^TR_{22} + R_{12}^TR_{12}$. Thus the computation of R_E involves an additional QR-decomposition of $\begin{bmatrix} R_{22} \\ R_{12} \end{bmatrix}$ and can be computed using standard updating techniques (Gill *et al.*, 1974).

In the case of one-sided weight $W_o = (I+GK)^{-1}G$, an alternative state-space realization of $\mathbf{W_o}$ can be used with the matrices A_o and C_o having permuted row/column blocks

$$\begin{aligned} A_o &= \begin{bmatrix} A_c - B_cR^{-1}DC_c & -B_cR^{-1}C \\ B\widetilde{R}^{-1}C_c & A - BD_cR^{-1}C \end{bmatrix}, \\ C_o &= \begin{bmatrix} -R^{-1}DC_c & -R^{-1}C \end{bmatrix} \end{aligned}$$

Further, the controller $\mathbf{K}$ in (16) is realized such that the leading $n^- \times n^-$ diagonal block A_{c1} contains the stable eigenvalues of A_c. If the Cholesky factor R_o of $Q_o = R_o^TR_o$ is partitioned as in (18) with R_{11} $n^- \times n^-$, then $R_E = R_{11}$, and the updating of the QR-decomposition can be avoided when computing R_E. Still, in the case of two-sided weighting with $W_o = (I+GK)^{-1}G$ and $W_i = (I+GK)^{-1}$, the approach used in the proof of the theorem with W_i and W_o sharing the same state matrix (i.e., $A_i = A_o$)

is to be preferred, because the computation of both grammians can be done with a single reduction of a $(n+n_c)\times(n+n_c)$ matrix to the RSF. In this case the cost to compute the two grammians is practically the same as for one grammian.

For a state-feedback and full-order observer based controller, let $\widetilde{S}_i$ be the Cholesky factor of $\widetilde{P}_i$ partitioned as

$$\widetilde{S}_i = \begin{bmatrix} \widetilde{S}_{22} & \widetilde{S}_{23} \\ 0 & \widetilde{S}_{33} \end{bmatrix}$$

The $n^-\times n^-$ Cholesky factor S_E corresponding to the trailing $n^-\times n^-$ part of P_{33} is the trailing $n^-\times n^-$ block of a matrix $\widehat{S}_{33}$ which satisfies

$$\widehat{S}_{33}\widehat{S}_{33}^T = \widetilde{S}_{11}\widetilde{S}_{11}^T + (\widetilde{S}_{12}+\widetilde{S}_{22})(\widetilde{S}_{12}+\widetilde{S}_{22})^T$$

$\widehat{S}_{33}$ can be computed from the RQ-decomposition of $\left[\, \widetilde{S}_{11} \;\; \widetilde{S}_{12}+\widetilde{S}_{22} \,\right]$ using standard factorization updating formulas (Gill *et al.*, 1974). Since $\widetilde{Q}_{22} = Q_{22}$ no difference appears in the computation of the Cholesky factor R_E.

The computation of reduced order approximation at Step 3 of the **FWCR Procedure** can be done for the BT method using a projection formulation with the help of two truncation matrices L and T. Assuming the controller is stable, the matrices of the reduced order controller $\mathbf{K}_r = (A_{cr}, B_{cr}, C_{cr}, D_{cr})$ can be computed as

$$(A_{cr}, B_{cr}, C_{cr}, D_{cr}) = (LA_cT, LB_c, C_cT, D_c)$$

For an unstable controller the same computation is performed on the stable part of the controller. The computation of L and T relies on the singular value decomposition

$$R_ES_E = \left[\, U_1 \;\; U_2 \,\right] \mathrm{diag}(\Sigma_1,\Sigma_2) \left[\, V_1 \;\; V_2 \,\right]^T$$

where

$$\begin{aligned} \Sigma_1 &= \mathrm{diag}(\sigma_1,\ldots,\sigma_{r_c}) \\ \Sigma_2 &= \mathrm{diag}(\sigma_{r_c+1},\ldots,\sigma_{n_c}) \end{aligned}$$

and $\sigma_1 \geq \ldots \geq \sigma_{r_c} > \sigma_{r_c+1} \geq \ldots \geq \sigma_{n_c} \geq 0$.

The *square-root* methods determine L and T as (Tombs and Postlethwaite, 1987)

$$L = \Sigma_1^{-1/2}U_1^TR_E, \qquad T = S_EV_1\Sigma_1^{-1/2}.$$

This approach is usually numerically very accurate for well-equilibrated systems. However if the original system is highly unbalanced, potential accuracy losses can be induced in the reduced model if either of the truncation matrices L or T is ill-conditioned (i.e., nearly rank deficient). To avoid ill-conditioned truncation matrices, *balancing-free* approaches can be used, as for example, the balancing-free square-root algorithm for the BT introduced by Varga (1991).

4. CONCLUSIONS

An efficient and numerically reliable balancing related computational approach has been proposed for the FWCR with special frequency weights enforcing closed-loop stability and performance. By solving lower order Lyapunov equations for computing the grammians, the new procedure is more efficient than the standard frequency-weighted balancing based reduction approach. The grammians can be determined directly in Cholesky factored forms to facilitate the application of square-root and balancing-free accuracy enhancing techniques. For the newly developed method, robust numerical software is available in the FORTRAN 77 library SLICOT, together with user friendly interfaces to the computational environments MATLAB and `Scilab`.

5. REFERENCES

Anderson, B. D. O. and Y. Liu (1989). Controller reduction: concepts and approaches. *IEEE Trans. Autom. Control* **34**, 802–812.

Enns, D. (1984). Model Reduction for Control Systems Design. PhD thesis. Dept. Aeronaut. Astronaut., Stanford Univ., Stanford, CA.

Gill, P. E., G. H. Golub, W. Murray and M. A. Saunders (1974). Methods for modifying matrix factorizations. *Math. Comput.* **28**, 505–535.

Hammarling, S. J. (1982). Numerical solution of the stable, non-negative definite Lyapunov equation. *IMA J. Numer. Anal.* **2**, 303–323.

Liu, Y., B. D. O. Anderson and U. L. Ly (1990). Coprime factorization controller reduction with Bezout identity induced frequency weighting. *Automatica* **26**, 233–249.

Moore, B. C. (1981). Principal component analysis in linear system: controllability, observability and model reduction. *IEEE Trans. Autom. Control* **26**, 17–32.

Tombs, M. S. and I. Postlethwaite (1987). Truncated balanced realization of a stable non-minimal state-space system. *Int. J. Control* **46**, 1319–1330.

Varga, A. (1991). Efficient minimal realization procedure based on balancing. In: *Prepr. of IMACS Symp. on Modelling and Control of Technological Systems* (A. El Moudni, P. Borne and S. G. Tzafestas, Eds.). Vol. 2. pp. 42–47.

Varga, A. and B. D. O. Anderson (2001). Square-root balancing-free methods for the frequency-weighted balancing related model reduction. In: *Proc. of CDC'2001, Orlando, Florida*.

Wang, G., V. Sreeram and W. Q. Liu (1999). A new frequency-weighted balanced truncation method and error bound. *IEEE Trans. Autom. Control* **44**, 1734–1737.