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A FRAMEWORK FOR MAINTAINING FORMATIONS BASED ON RIGIDITY

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Abstract: In this paper, a framework for maintaining formations of large number of mobile autonomous vehicles based on rigidity is proposed. The aim of this paper is to explore strategies for maintaining formations with more limited communication/sensing requirements. An inductive construction method for provably rigid formations is proposed, and a method for optimum angle measures between vehicles is developed. The method scales with the number of vehicles and is flexible to support many rigid formation shapes.

Keywords: Autonomous Vehicles, Coordination, Geometric Approaches, Graphs, Graph Theory,

1. INTRODUCTION

Current interest in the problem of coordinating the motion of large number of mobile autonomous vehicles by means of distributed control has raised a number of issues about forming, maintaining and real-time modification of vehicle formations of all types. By a formation is usually meant a collection of vehicles moving through real two or three dimensional space in such a way that the distance between every pair of vehicles remains unchanged over time, at least under ideal conditions. One way to retain a formation over time is for each vehicle to maintain a fixed distance between itself and every other vehicle in the formation. For all but the smallest of formations such a strategy would obviously be impractical, since it would require each vehicle to sense its distance from each other vehicle in the formation. It is thus of interest to explore strategies for maintaining formations with more limited communication/sensing requirements. To do this in a systematic manner, it is necessary to develop a framework for modelling vehicle formations which characterizes the communication/sensing links needed to maintain the formation. The aims of this paper are to suggest such a framework, to draw attention to a large existing literature within mathematics which is applicable to problems of this type, and to develop some specific new results along these lines. So far in the literature, formations with small number of vehicles have been addressed (see for example (Desai, *et al.*, 2001) and (Pappas, *et al.*, 2001)).

In §2 we define the concept of a "point formation" and explain how it can be used as a high-level model of a formation of vehicles. We explain what it means for a point formation to be rigid. The ideas of a point formation and a rigid point formation are essentially the same as the concepts of a "framework" and a "rigid framework" studied in mathematics as well as within the theory of structures in mechanical and civil engineering (see for example (Roth, 1981), (Whiteley and Tay, 1985)). The slightly different terminology used in this paper more closely describes the applications we have in mind. In §3 we outline two well-known conditions for testing the rigidity of a formation. The first involves evaluating the generic rank of a parameter dependent matrix while the second requires one to enumerate certain properties of a graph. In §4 we turn to the issue of constructing rigid formations for a given set of points. We explain the Henneberg construction in §4.1, point out some of the construction limitations, and then in §4.2 develop a way around these limitations using a new construction based on the Delaunay triangulation. Finally in §4.3 we develop a method based on polygonal triangulations to handle the constraints imposed by objects other than vehicles.

2. POINT FORMATIONS

By a d-dimensional *point formation* at $p \triangleq$ column $\{p_1, p_2, \ldots, p_n\}$, written $\mathbb{F}_p$, is meant a set of n points $\{p_1, p_2, \ldots, p_n\}$ in $\mathbb{R}^d$ together with a set $\mathcal{L}$ of m *maintenance links*, labelled (i, j), where i and j are distinct integers in $\{1, 2, \ldots, n\}$; the *length* of link (i, j) is the Euclidean distance between points p_i and p_j. By a *trajectory* of $\mathbb{F}_p$ is meant a continuously parameterized, one-parameter family of points $\{q(t) : t \geq 0\}$ in $\mathbb{R}^{nd}$ which contains p.

A point formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$ provides a natural high-level model for a set of n-vehicles moving in formation in real 2 or 3 dimensional space. In this context, the points p_i represent the positions of the vehicles in $\mathbb{R}^d$ at time t and the links in $\mathcal{L}$ correspond to distances between specific vehicles which are to be maintained over time. In practice, actual vehicle groups cannot be expected to move exactly in formation because of sensing errors, vehicle modelling errors, etc. The ideal benchmark point formation against which the performance of an actual vehicle formation is to be measured is called a *reference formation*. Such a formation is said to undergo *rigid motion* along a trajectory $q([0, \infty)) \triangleq \{\text{column } \{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ if the Euclidean distance between each pair of points $q_i(t)$ and $q_j(t)$ remains constant all along the trajectory. Let us note that $\mathbb{F}_p$ undergoes rigid motion along a trajectory $q([0, \infty))$ just in case each pair of points $q(t_1), q(t_2) \in q([0, \infty))$ are *congruent* in the sense that there exists a distance preserving map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T(q_i(t_1)) = q_i(t_2), i \in \{1, 2, \ldots, n\}$. The set of points $\mathcal{M}_p$ in $\mathbb{R}^{dn}$ which are congruent to p is known to be a smooth manifold. It is clear that any trajectory along which $\mathbb{F}_p$ undergoes rigid motion must lie completely within $\mathcal{M}_p$; conversely any trajectory of $\mathbb{F}_p$ which lies within $\mathcal{M}_p$ is one along which $\mathbb{F}_p$ undergoes rigid motion. It can be shown that in the generic case when the set of points $\{p_1, p_2, \ldots, p_n\}$ is not contained in any proper hyperplane within $\mathbb{R}^d$, the dimension of $\mathcal{M}_p$ is $\frac{d(d+1)}{2}$ (Roth, 1981). Thus under this condition the dimension of $\mathcal{M}_p$ is 6 for $d = 3$ and 3 for $d = 2$.

A formation $\mathbb{F}_p$ is said to be *rigid* if rigid motion is the only kind of motion it can undergo along any trajectory on which the lengths of all links in $\mathcal{L}$ remain constant. Thus if $\mathbb{F}_p$ is rigid, it is possible to "keep formation" by making sure that the lengths of formation's maintained links do not change as the formation moves. It is possible to associate with any point formation $\mathbb{F}_p$ a graph $\mathbb{G}_{\mathbb{F}} \triangleq \{\mathcal{V}, \mathcal{L}\}$ whose vertex set is the set of labels of the points in $\mathbb{F}_p$ and whose edge set is $\mathcal{L}$. The definition of rigid formation implies that every formation $\mathbb{F}_p$ whose graph $\mathbb{G}_{\mathbb{F}}$ is complete is automatically rigid. The converse however is not generally true.

In practice, autonomously functioning vehicles would typically be expected to maintain not only relative distance to nearby vehicles but also relative heading. More often than not desired heading and relative position specifications can be converted into desired position constraints with respect to a suitably defined "virtual vehicle". For example, three vehicles v_1, v_2, v_3 all heading north in $\mathbb{R}^2$ with a maintained distance of 100 meters between nearest neighbors plus the constraint that the trailing two vehicles each maintain a relative heading of 30^o west with respect to its nearest neighbor, determines a straight line formation as shown in Figure 1a. The same formation can equivalently be specified in terms of lengths alone by introducing two virtual vehicles and four additional links as shown in Figure 1b. For analysis purposes, such equivalent point formation representation enables one to address rigidity questions, etc. within the framework outlined above.

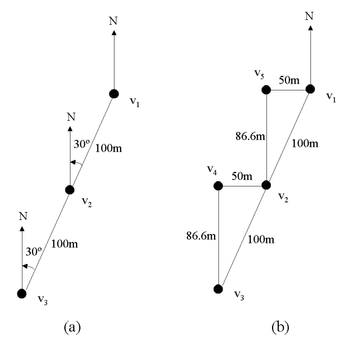

Fig. 1. Line Formation (a) Heading Determined Formation (b)Equivalent Distance Determined Formation

3. CONDITIONS FOR RIGIDITY

The question of whether or not a given formation is rigid has been studied for a long time. One approach to the question starts by examining what happens to a given formation $\mathbb{F}_p = (\{p_1, p_2, \ldots, p_n\}, \mathcal{L})$, along the trajectory $\{q_1(t), q_2(t), \ldots, q_n(t)\} : t \geq 0\}$ on which the Euclidean distances $d_{ij} \triangleq ||p_i - p_j||$ between pairs of points (q_i, q_j) for which (i, j) is a link, are constant. Thus along such a trajectory

$$(q_i - q_j)'(q_i - q_j) = d_{ij}^2, \qquad (i,j) \in \mathcal{L}, \quad t \geq 0 \quad (1)$$

Assuming a smooth trajectory, we can differentiate to get

$$(q_i - q_j)'(\dot{q}_i - \dot{q}_j) = 0, \qquad (i,j) \in \mathcal{L}, \quad t \geq 0 \quad (2)$$

These equations can be evaluated at p and rewritten in matrix form as

$$R(q)\dot{q} = 0 \qquad (3)$$

where $\dot{q}$ = column $\{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\}$, and R is a specially structured $m \times dn$ array called a *rigidity matrix*. Because any trajectory of $\mathbb{F}_p$ which lies within $\mathcal{M}_p$ is one along which $\mathbb{F}_p$ undergoes rigid motion, (2) automatically holds along any trajectory which lies within $\mathcal{M}_p$. From this it follows that the tangent space to $\mathcal{M}_p$ at q, written $\mathcal{T}_q$ must be contained in the kernel of $R(q)$. Since p must be on any such trajectory, it must be true that $\mathcal{T}_p \subset$ kernel $R(p)$. In the generic case when $\{p_1, p_2, \ldots, p_n\}$ is not contained in any proper hyperplane within $\mathbb{R}^d$, the dimension of $\mathcal{M}_p$ and thus $\mathcal{T}_p$ is $\frac{d(d+1)}{2}$ (Roth, 1981). Thus under this condition the dimension of kernel $R(p)$ is at least 6 for $d = 3$ and at least 3 for $d = 2$. In other words, if $\{p_1, p_2, \ldots, p_n\}$ is not contained in any proper hyperplane within $\mathbb{R}^d$, then

$$\text{rank } R(p) \leq \begin{cases} 3n - 6 \text{ if } d = 3 \\ 2n - 3 \text{ if } d = 2 \end{cases}$$

Since trajectories which lie totally within $\mathcal{M}_p$ are trajectories along which $\mathbb{F}_p$ moves rigidly, it is natural to expect that rigidity of $\mathbb{F}_p$ would be implied if the preceding were to hold with equality. Unfortunately there are counterexamples which prove that this is not the case. Fortunately in the "generic case," the preceding does indeed lead to a test for rigidity. By the generic case we mean the case when (1) $\{p_1, p_2, \ldots, p_n\}$ is not contained in any proper hyperplane within $\mathbb{R}^d$ *and* (2) p is a point in $\mathbb{R}^{dn}$ at which the rank of $R(q)$ is maximal over all $q \in \mathbb{R}^{dn}$. Points which maximize $R(q)$ are called regular points and the rank of $R(q)$ at any such point is called R's *generic* or maximal rank. We can now state the following theorem (Roth, 1981):

Theorem 1. Let $\mathbb{F}_p$ be a formation whose point set $\{p_1, p_2, \ldots, p_n\}$ is not contained in any proper hyperplane within $\mathbb{R}^d$. Suppose p is point in $\mathbb{R}^{dn}$ at which the rank of $R(q)$ achieves its maximal rank. Then $\mathbb{F}_p$ is a rigid formation if and only if

$$\text{rank } \{R(p)\} = \begin{cases} 3n - 6 \text{ if } d = 3 \\ 2n - 3 \text{ if } d = 2 \end{cases}$$

The hypotheses of the preceding theorem lead naturally one to the following concept. A formation $\mathbb{F}_p$ is said to be *generically rigid* if it is rigid and if there is an open neighborhood of points about p in $\mathbb{R}^{dn}$ at which $\mathbb{F}_p$ is also rigid. The concept of generic rigidity does not depend on the distances between the points of $\mathbb{F}_p$ and for this reason, it is a desirable specialization of the definition of a rigid formation for our purposes. For generic rigidity, there is the following simplification of Theorem 1.

Theorem 2. A formation $\mathbb{F}_p$ is generically rigid if and only if

$$\begin{aligned} \text{rank } \{R(p)\} &= \text{generic rank } \{R(p)\} \\ &= \begin{cases} 3n - 6 \text{ if } d = 3 \\ 2n - 3 \text{ if } d = 2 \end{cases} \end{aligned}$$

The generic rigidity question thus can be reduced to developing a test for computing the generic rank of a matrix. The generic rank question is very closely related to the question of structural controllability treated some time ago in (Corfmat and Morse, 1976) and (Anderson and Hong, 1982). In particular, in the latter, explicit constructions are given for determining the generic rank of an arbitrary matrix net of the form

$$M(p) = M_0 + p_1 M_1 + \cdots + p_n M_n$$

over the linear space of all n-vectors p = column $\{p_1, p_2, \cdots, p_n\}$. Unfortunately these constructions are computationally complex and reveal little about the structure of generically rigid formations.

As noted above the concept of generic rigidity does not depend on the precise distances between the points of $\mathbb{F}_p$. It is perhaps not surprising then, that the generic rigidity question can be posed solely in terms of the graph $\mathbb{G}_{\mathbb{F}_p}$ without any reference to $\mathbb{F}_p$'s actual points. The following theorem settles the generic rigidity question for $d = 2$ in strictly graph theoretic terms. To state the theorem, we need the following idea. A graph $\mathbb{G} \triangleq \{\mathcal{V}, \mathcal{L}\}$ with n vertices is said to be a *generically rigid graph* for $\mathbb{R}^d$ if there is an open dense set of points $p \in \mathbb{R}^{dn}$ at which $\mathbb{F}_p$ is a rigid formation with graph $\mathbb{G}$.

Theorem 3. (Laman, 1970) A graph $\mathbb{G} \triangleq \{\mathcal{V}, \mathcal{L}\}$ with n vertices is a generically rigid graph on $\mathbb{R}^2$ if and only if $\mathcal{L}$ contains a subset $\mathcal{E}$ consisting of $2n - 3$ edges with the property that for any nonempty subset $\bar{\mathcal{E}} \subset \mathcal{E}$, the number of edges in $\bar{\mathcal{E}}$ cannot exceed $2k - 3$ where k is the number of vertices of $\mathbb{G}$ which are endpoints of edges in $\bar{\mathcal{E}}$.

The generalization of Laman's theorem to higher dimensions, including most especially $d = 3$ has been proved to be quite elusive. At present, this is the most general result known for characterizing generic rigidity in graph theoretic terms.

4. CONSTRUCTION METHODS FOR GENERICALLY RIGID POINT FORMATIONS

In this section we turn to the question of how to construct a generically rigid point formation. We begin by reviewing inductive constructions that applies to *isostatic point formations*. By isostatic point formation is meant a rigid point formation such that removing any maintenance link gives a non-rigid point formation. Similarly a *generically isostatic point formation* is a generically rigid point formation such that removal of any maintenance link gives a generically non-rigid point formation. A *generically isostatic graph* is defined in a similar manner. *Generically d-isostatic* means isostatic condition in d-dimension. The *degree* of a vertex i of a graph is the number of incident vertices to i.

4.1 *The Henneberg Construction of Formations*

In the sequel we explain the Henneberg construction method which is an inductive approach that creates generically d-isostatic point formations both in real 2 and 3 dimensional space while maintaining rigidity. The construction makes use of the following theorems (Whiteley and Tay, 1985).

Theorem 4. Let $\mathbb{G} \triangleq \{\mathcal{V}, \mathcal{L}\}$ be a a generic graph with a vertex i of degree d; let $\mathbb{G}^* \triangleq \{\mathcal{V}^*, \mathcal{L}^*\}$ denote the subgraph obtained by deleting i and the edges incident with it. Then G is generically d-isostatic if and only if G^* is generically d-isostatic.

Theorem 5. Let $\mathbb{G} \triangleq \{\mathcal{V}, \mathcal{L}\}$ be a graph with a vertex i of degree $d + 1$, let $\mathcal{V}_i$ be the set of vertices incident to i, and let $\mathbb{G}^* \triangleq \{\mathcal{V}^*, \mathcal{L}^*\}$ be the subgraph obtained by deleting i and its $d+1$ incident edges. Then $\mathbb{G}$ is generically d-isostatic if and only if there is a pair j, k of vertices of $\mathcal{V}_i$ such that the edge (j, k) is not in $\mathcal{L}^*$ and the graph $\mathbb{G}' = (\mathcal{V}^*, \mathcal{L}^* \bigcup (j, k))$ is generically d-isostatic.

This inductive construction starts from a single maintenance link at the first step and then it either adds a new vertex with d maintenance links to the existing point formation, or it removes an existing maintenance link by adding a new vertex connecting it to the end points of the removed maintenance link and to other $d - 1$ vertices in the existing point formation. The resulting point formations that we get after each step are generically rigid. A sample point formation in 3 dimensional space created by this construction method is depicted in Figure 2.

4.2 *The Delaunay Triangulation of Formations*

While the Henneberg construction method creates rigid point formations, it lacks imposing geometrical constraints while creating them. Some of the

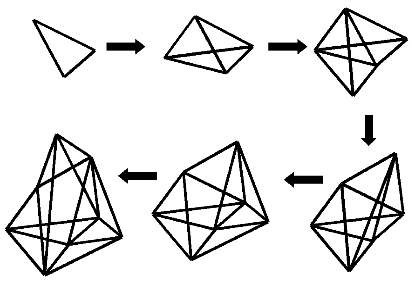

Fig. 2. Creating an isostatic point formation in 3 dimensional space

formations created with this method will not be of much practical use. Because the method does not consider the lengths of maintenance links and the angles between them. Now we are going to develop a way around these limitations using a new construction based on the *Delaunay triangulation*.

A *triangulation* in the plane is a partition of a point set into triangles that meet only at shared maintenance links. A triangulation in 3-space is a partition of a points set into tetrahedra that meet only at shared faces. Figure 3 depicts the triangulation of a ten-vehicle formation. Now that we have inductive construction methods for isostatic point formations, we will investigate the rigidity of triangulated point formations. First we will investigate a simpler triangulation method, triangulation by the plane sweep method, which we will use to create the Delaunay triangulation.

Triangulated Point Formations by the Plane Sweep Method. In this well-known method, in order to sort a set of points in the plane, an imaginary vertical line sweeps through them. The points of $\{p_1, p_2, \ldots, p_n\}$ are added by x-coordinate order (using y-order to break ties) for sweeping. The triangles for a new vertex p_i are constructed using the boundary maintenance links of the current triangulated point formation visible from p_i. To prove the rigidity of the triangulations created by the plane sweep method, we will use the following lemma from rigidity theory.

Lemma 6. (Whiteley and Tay, 1985) If $\mathbb{G}_{\mathbb{F}_1} = (\mathcal{V}_1, \mathcal{L}_1)$ and $\mathbb{G}_{\mathbb{F}_2} = (\mathcal{V}_2, \mathcal{L}_2)$ are generically rigid graphs sharing at least d vertices, then $\mathbb{G} = (\mathcal{V}_1 \bigcup \mathcal{V}_2, \mathcal{L}_1 \bigcup \mathcal{L}_2)$ is generically d-rigid.

Theorem 7. Assume $\mathbb{G}$ is the resulting graph of triangulation constructed by the plane sweep method. Then $\mathbb{G}$ is generically rigid.

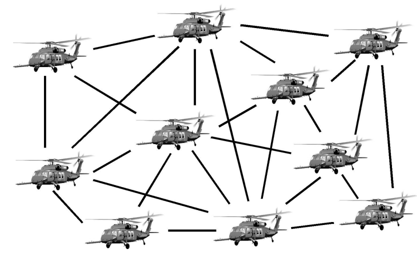

Fig. 3. Triangulation of a ten-vehicle formation

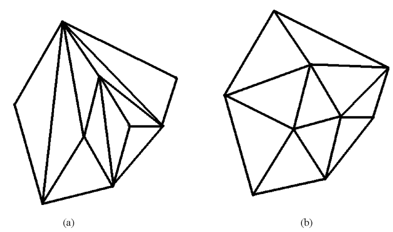

Fig. 4. Triangulated point formations by (a) the plane sweep method (b) the Delaunay method

PROOF. $\mathbb{G}_2$ is a single maintenance link on two vertices. Assume that $\mathbb{G}_n$ is generically rigid. $\mathbb{G}_{n+1}$ is constructed from $\mathbb{G}_n$ by adding triangle(s) to the existing graph. The triangles for a new vertex i are constructed using the boundary edges of the current triangulation visible from i. Each added triangle t_{ijk} and the existing graph $\mathbb{G}_n$ share two vertices j and k. The resulting graph is generically rigid by Lemma 6. $\square$

Neither the Henneberg construction method nor Lemma 6 consider the the angles between maintenance links. We want to avoid skinny triangles with small angles created by the plane sweep method as shown in Figure 4a. To meet our goals, we apply maintenance link flipping on triangulated point formations so that the *circumcircle* of any triangle/tetrahedron in the point formation does not include any other vertex in it. By circumcircle, we mean the circle/sphere passing through the vertices of a triangle/tetrahedron. This maintenance link flipping results in a special kind of triangulation which is known to be the Delaunay triangulation. We prove that the maintenance link flipping preserves the rigidity of the initial triangulated point formation. The Delaunay triangulation maximizes the minimum angle of all triangulations/tetrahdra of a given vertex set.

Creating the Delaunay Triangulated Point Formation. Let $\mathcal{V}$ be a set of $n \geq 3$ vertices in $\mathbb{R}^2$. We can compute the Delaunay triangulated point formation as follows: First we determine some triangulation of $\mathcal{V}$. The plane sweep method can be used for initial triangulation. Then, while there are two opposite triangles t_{ijk} and t_{ikl} that are not locally Delaunay, we flip the diagonal, that is, we replace the two triangles with triangles t_{ijl} and t_{jkl}. The Delaunay triangulation of the same vertex set of Figure 4a is shown in Figure 4b.

Lemma 8. Maintenance link flipping applied on a triangulated point formation created by the plane sweep method preserves generic rigidity.

PROOF. We first show that the maintenance link that is flipped can not be shared by two triangles t_{ijk} and t_{ijl} that are added at the same step when the vertex i was added in the plane sweep method. Assume that i is added to the graph $\mathbb{G}_n$ and we get the graph $\mathbb{G}_{n+1}$. Since t_{ijk} and t_{ijl} are added at the same step, the links $\{j,k\}$ and $\{j,l\}$ are visible from the vertex i. Since $\mathbb{G}_n$ is convex, the angle $\widehat{kjl}$ in $\mathbb{G}_n$ is more than $180\,^\circ$. Therefore the quadrilateral $ijkl$ is not convex and the Delaunay diagonal flip can not be applied on (i,j). This completes the first part of the proof. Now assume that the maintenance link that is flipped is shared by two triangles t_{ijk} and t_{lik} that are added at different steps when the vertices i and l were added in the plane sweep method respectively. Assume that (i,k) that is flipped is a diagonal in the quadrilateral $ijkl$. Flipping the diagonal changes the order of the vertices that were added. Since Lemma 6 is still satisfied, maintenance link flipping preserves rigidity. □

Theorem 9. The Delaunay triangulated point formation is rigid.

PROOF. The proof follows immediately from Theorem 7 and Lemma 8. □

Remark 10. The Delaunay triangulated point formations and its rigidity properties can be extended to 3 dimensional space.

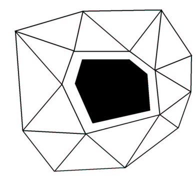

Fig. 5. Polygonal triangulated point formation around an object

4.3 *Rigid Formations with Visibility Constraints*

Now we extend triangulated point formations with occlusions imposed by objects other than vehicles, i.e. some vehicles may be carrying an object while others are escorting. This will impose additional constraints on maintenance links, such as some maintenance links have to be included in the point formation while some others have to be excluded. To solve this problem, we propose the use of triangulations of *simple polygons* and *polygons with holes* from computer graphics.

In the triangulation of a simple polygonal point formation, the boundary forms a simple, polygonal, closed curve. In the case of polygonal point formation with holes, the boundary maintenance links may form several disjoint polygonal Jordan curves. The triangulation must use the links of the boundary as maintenance links in the triangulation. Thereafter we are going to use the term polygon for both simple polygon and polygon with holes. The following lemma applies to any polygon, and hence every polygonal point formation can be triangulated.

Lemma 11. (Fortune, 1995) Every polygon with more than three sides has a diagonal.

Theorem 12. Triangulated polygonal point formations are generically rigid.

PROOF. Let $\mathbb{F}_p$ denote a polygonal point formation with only boundary maintenance links. We can find the remaining maintenance links as follows. By Lemma 11, once we have found a single diagonal of $\mathbb{F}_p$, we can split the polygon into two, and recursively triangulate each part. Let $t_1, ..., t_n$ denote the resulting triangles. Each triangle shares a maintenance link at least with one other triangle, where the shared maintenance link is a diagonal of $\mathbb{F}_p$. Since each t_i $(i = 1, ..., n)$ is sharing at least 2 vertices with the rest of the point formation, starting from t_1 and adding $t_2, \ldots, t_n$ generates a generically rigid point formation at each step by Lemma 6. □

With this method we are able to impose the polygonal boundary maintenance links surrounding an object as constraints as seen in Figure 5.

5. CONCLUDING REMARKS

In this paper, we developed methods to construct rigid point formations for modelling communication/sensing links needed to maintain formations of autonomous vehicles. We proposed inductive construction methods which are provably rigid at each step of construction without a need to test at every step. While the Henneberg construction method creates rigid point formations, it lacks imposing constraints while creating them. To overcome this difficulty, we used the Delaunay triangulation to optimize angle measures and to avoid large maintenance links. We extended triangulated point formations to polygonal triangulations to handle the obstructions imposed by objects other than vehicles. Significant features of the developed model are that it scales with the number of vehicles, it is flexible to support many rigid formation shapes, and robust in maintaining rigid shapes with visibility constraints in the environment. Future research will focus on extending these models to real-time modifications in point formations such as splitting a formation, maintaining rigidity in case of vehicle removals and creating the Delaunay triangulations without redundant links.

References

Anderson, B.D.O and H-M. Hong (1982). Structural controllability and matrix nets. *International Journal of Control*, **35**, 397–416.

Corfmat, J.P. and A.S. Morse (1976). Structurally controllable and structurally canonical systems. *IEEE Transactions on Automatic Control*, **21**, 129–131, 1976.

Desai, J., V. Kumar and J. Ostrowski (2001). Control of changes in formation for a team of mobile robots. *IEEE Transactions on Robotics and Automations*, submitted.

Fortune, S. (1995). Voronoi Diagrams and Delaunay Triangulations. In *Computing in Euclidean Geometry*,(D.Z. Du, F. Hwang. (Ed).). World Scientific Publishing Co., Singapore.

Laman, G. (1970). On Graphs and Rigidity of Plane Skeletal Structures. *Journal of Engineering Mathematics*, **4**, 331-340.

Pappas, G.J., P. Tabuada and P. Lima (2001). Feasible Formations of Multi-Agent Systems. *Proceedings of the American Control Conference*, Arlington, Virginia.

Roth, B. (1981). Rigid and Flexible Frameworks. *American Mathematical Monthly*, **88**, 6-21.

Whiteley, W. and T. Tay (1985). Generating Isostatic Frameworks. *Structural Topology*, **11**, 21-69.