Not All Vinnicombe Metric Neighbourhoods are Homotopically Connected

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Abstract- We prove by counterexample that even for two transfer functions which are close in the Vinnicombe (Nu-gap) metric, there does not necessarily exist a Vinnicombe metric homotopy from one transfer function to the other, such that intermediate transfer functions in the homotopy remain close to the transfer function at the beginning of the homotopy. This implies that the Vinnicombe metric neighbourhoods of some transfer functions in L-infinity space, are not connected.

Keywords– Vinnicombe Metric, Nu-gap Metric, Scalar Homotopy, Subunitary Homotopy, Monotonic Homotopy.

1. INTRODUCTION

Motivated particularly by robustness issues arising in control systems, several metrics have been introduced to measure the distance between two transfer function matrices. These include the graph metric, the gap metric (El-Sakkary, 1985) and the ν-gap (Nu-gap or Vinnicombe gap) metric (Vinnicombe, 1993, 1999).

One generic question of interest about metrics is the following. Suppose that P_α and P_Ω are rational transfer functions of two plants for which $d(P_\alpha, P_\Omega) = \eta < 1$, where $d(\cdot,\cdot)$ is a metric function. Does there exist a (continuous) homotopy P_λ of rational transfer functions, $\lambda \in [0,1]$, from P_α to P_Ω, such that $d(P_\alpha, P_\lambda)$ moves monotonically from 0 to η as λ moves from 0 to 1? We pose and analysed this question for the ν-gap metric.

By relaxing the monotonicity constraint, but still requiring that $d(P_\alpha, P_\lambda)$ remains less than unity, an alternative, less demanding question can also be posed. Note that the maximum ν-gap distance between any two plants is unity. Hence, we suppose that P_α and P_Ω are rational transfer functions of two plants for which $\delta_\nu(P_\alpha, P_\Omega) = \eta < 1$, where $\delta_\nu(\cdot,\cdot)$ denotes the ν-gap metric, and ask if there exists a homotopy of rational transfer functions P_λ, such that $\delta_\nu(P_\alpha, P_\lambda)$ moves continuously from 0 to η as λ moves from 0 to 1, and remains less than unity.

While (Vinnicombe, 1993) does not precisely answer this question, it turns out that the answer to this question is negative. The key result here demonstrates the existence of transfer functions $\hat{P}_\alpha$ and $\hat{P}_\Omega$ for which $\delta_\nu(\hat{P}_\alpha, \hat{P}_\Omega)$ is arbitrarily small, but for which no homotopy $\hat{P}_\lambda$ of rational transfer functions exists such that $\delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda)$ varies continuously and never attains the value unity. This is a counter-intuitive conclusion.

The structure of this paper is as follows. In Section 2, we review the Vinnicombe metric, restricting our attention, for convenience, to scalar plants. In Section 3, we present two first-order plants with small Vinnicombe gap distance between them, and show that there is no homotopy of degree one transfer functions with $\delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda)$ remaining less than unity. In Section 4, we extend the argument to consider all rational homotopies $\hat{P}_\lambda$ where the numerator and denominator individually vary continuously. A conclusion appears as Section 5.

2. THE VINNICOMBE GAP METRIC

For a fully detailed exposition of the Vinnicombe Gap metric, consult (Vinnicombe, 1993, 1999). Here, we briefly summarise some basic properties. Let P_γ and P_ζ be two rational scalar transfer functions. Define the *chordal distance* between P_γ and P_ζ at frequency ω by

$$\kappa(P_\gamma, P_\zeta, \omega) = \lim_{\hat{\omega}\to\omega} \frac{|P_\zeta(j\hat{\omega}) - P_\gamma(j\hat{\omega})|}{\sqrt{[1+|P_\zeta(j\hat{\omega})|^2][1+|P_\gamma(j\hat{\omega})|^2]}}. \quad (1)$$

The chordal distance is obviously continuous in ω and bounded by unity. Now define

$$\bar{\kappa}(P_\gamma, P_\zeta) = \operatorname*{ess\,sup}_{\omega} \kappa(P_\gamma, P_\zeta, \omega)$$

The Vinnicombe metric $\delta_\nu(P_\gamma, P_\zeta)$ is defined as

$$\delta_\nu(P_\gamma, P_\zeta) = \bar{\kappa}(P_\gamma, P_\zeta),$$

provided the following two conditions are satisfied:

$$\text{for all } \omega \quad 1 + P_\gamma(j\omega)P_\zeta^*(j\omega) \neq 0 \quad (2)$$

and $\quad \text{wno}[1 + P_\gamma(j\omega)P_\zeta^*(j\omega)] + \breve{\eta}(P_\gamma) - \bar{\eta}(P_\zeta) = 0.$

In the above, $P^*(s)$ denotes $P(-s)^T$ and $\text{wno}(X)$ is the winding number of the transfer function X, that is, the number of counterclockwise encirclements of the origin by $X(s)$ as s moves around the standard (clockwise) Nyquist D-contour. Finally $\breve{\eta}(X)$ and $\bar{\eta}(X)$ denote the number of poles of X in the open and closed right half planes

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respectively, where the closed right hand plane also includes the point at infinity.

In the case that either condition in (2) fails then

$$\delta_\nu(P_\gamma, P_\zeta) = 1.$$

The appellation "metric" (implying that particular properties, such as the triangle inequality, for example, hold) is justified in (Vinnicombe, 1993). The metric fulfils some intuitively reasonable expectations. For example, any two *stable* transfer functions whose Nyquist diagrams are close have a small ν-gap metric distance which gets smaller as the Nyquist diagrams get closer. For unstable plants however, the issues are more subtle. The chordal distance of equation (1) is a distance measure corresponding frequencies on two Nyquist diagrams. Equations (2) are related to the Nyquist Stability Criterion: involving the avoidance of the -1 point, conditions involving the number of encirclements unstable (open loop) pole counts. Note, for example, that the three plants $P_{-\epsilon} = \frac{1}{s-\epsilon}$, $P_0 = \frac{1}{s}$ and $P_\epsilon = \frac{1}{s+\epsilon}$ (with ϵ small) are separated from each other by a Vinnicombe distance of $\mathcal{O}(\epsilon)$.

There are alternative equivalent formulations of the Vinnicombe metric. For exmple, let $P_\gamma = \frac{n_\gamma}{m_\gamma}$ and $P_\zeta = \frac{n_\zeta}{m_\zeta}$, where n_γ; m_γ and n_ζ; m_ζ are each polynomials, coprime over polynomials (Vidyasagar, 1985). Let r_γ and r_ζ be Hurwitz polynomials such that $r_\gamma^* r_\gamma = n_\gamma^* n_\gamma + m_\gamma^* m_\gamma$ and $r_\zeta^* r_\zeta = n_\zeta^* n_\zeta + m_\zeta^* m_\zeta$. Then P_γ and P_ζ have *normalised coprime fractional descriptions* $P_\gamma = \frac{n_\gamma/r_\gamma}{m_\gamma/r_\gamma}$, and $P_\zeta = \frac{n_\zeta/r_\zeta}{m_\zeta/r_\zeta}$, where the numerators n_ξ/r_ξ and denominators m_ξ/r_ξ are each rational transfer functions, which are coprime over proper stable transfer functions. We set

$$G_\xi = \begin{bmatrix} n_\xi/r_\xi \\ m_\xi/r_\xi \end{bmatrix} \quad \text{and} \quad \tilde{G}_\xi = \begin{bmatrix} -m_\xi/r_\xi & n_\xi/r_\xi \end{bmatrix}.$$

Note that $\tilde{G}_\xi G_\xi = 0$ and that $G_\xi^* G_\xi = 1$. An equivalent expression for equation (1) is thus $\kappa(P_\gamma, P_\zeta, \omega) = |\tilde{G}_\gamma(j\omega) G_\zeta(j\omega)| = |\tilde{G}_\zeta(j\omega) G_\gamma(j\omega)|$. Equivalent expressions for equations (2) are

$$\forall\omega \quad [G_\gamma^* G_\zeta](j\omega) \neq 0 \text{ and } \quad \text{wno}[G_\gamma^* G_\zeta] = 0. \quad (3)$$

3. FIRST ORDER PLANTS WITH NO FIRST ORDER HOMOTOPY

Let us define two particular first-order plants

$$\hat{P}_\alpha = \frac{\epsilon}{s-1} \quad \text{and} \quad \hat{P}_\Omega = -\frac{\epsilon}{s-1}$$

and suppose that $0 < \epsilon < 1$. The normalised coprime fractional descriptions are given by

$$G_\alpha = \begin{bmatrix} \dfrac{\epsilon}{s+\sqrt{\epsilon^2+1}} \\ \dfrac{s-1}{s+\sqrt{\epsilon^2+1}} \end{bmatrix}, \quad (4)$$

$$\text{and} \quad G_\Omega = \begin{bmatrix} \dfrac{-\epsilon}{s+\sqrt{\epsilon^2+1}} \\ \dfrac{s-1}{s+\sqrt{\epsilon^2+1}} \end{bmatrix}, \quad (5)$$

$$\text{that} \quad \tilde{G}_\alpha G_\Omega = \frac{2\epsilon(s-1)}{(s+\sqrt{\epsilon^2+1})\,(s+\sqrt{\epsilon^2+1})},$$

$$\text{and} \quad G_\alpha^* G_\Omega = \frac{-s^2+1-\epsilon^2}{(-s+\sqrt{\epsilon^2+1})\,(s+\sqrt{\epsilon^2+1})}.$$

Note that $|\tilde{G}_\alpha G_\Omega(j\omega)|$ achieves its supremum, namely $2\epsilon/(1+\epsilon^2)$ at $\omega = 0$ and that $G_\alpha^* G_\Omega$ has zero winding number. Thus $\delta_\nu(\hat{P}_\alpha, \hat{P}_\Omega) = \frac{2\epsilon}{1+\epsilon^2}$. An obvious proposal is for a first order homotopy $\hat{P}_\lambda$ linking $\hat{P}_\alpha$ and $\hat{P}_\Omega$ is $\hat{P}_\lambda = \frac{\epsilon(1-2\lambda)}{s-1}$ so that $\lambda = 0$ and $\lambda = 1$ correpond to $\hat{P}_\alpha$ and $\hat{P}_\Omega$.

For $\lambda \neq \frac{1}{2}$ we have

$$G_\lambda = \begin{bmatrix} \dfrac{\epsilon(1-2\lambda)}{s+\sqrt{\epsilon^2(1-2\lambda)^2+1}} \\ \dfrac{s-1}{s+\sqrt{\epsilon^2(1-2\lambda)^2+1}} \end{bmatrix}, \quad (6)$$

so that

$$\tilde{G}_\alpha G_\lambda = \frac{2\lambda\epsilon(s-1)}{(s+\sqrt{\epsilon^2+1})(s+\sqrt{\epsilon^2(1-2\lambda)^2+1})} \quad \text{and}$$

$$G_\alpha^* G_\lambda = \frac{-s^2+1+\epsilon^2(1-2\lambda)}{(-s+\sqrt{\epsilon^2+1})(s+\sqrt{\epsilon^2(1-2\lambda)^2+1})}$$

$$\text{and hence} \quad \delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda) = \frac{2\lambda\epsilon}{\sqrt{[1+\epsilon^2(1-2\lambda)][1+\epsilon^2]}}.$$

However $G_{\frac{1}{2}}$ cannot be obtained by setting $\lambda = \frac{1}{2}$ in (6), since $\hat{P}_{\frac{1}{2}} = 0 = 0/1$, so that

$$G_{\frac{1}{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{G}_\alpha G_{\frac{1}{2}} = \frac{\epsilon}{s+\sqrt{\epsilon^2+1}} \quad (7)$$

$$\text{and} \quad G_\alpha^* G_{\frac{1}{2}} = \frac{s+1}{s-\sqrt{\epsilon^2+1}}. \quad (8)$$

Now although equation (8) satisfies the first condition in (3), it does not satisfy the second since $\text{wno}(G_\alpha^* G_{\frac{1}{2}}) \neq 0$. Hence $\delta_\nu(\hat{P}_\alpha, \hat{P}_{\frac{1}{2}}) = 1$. The winding number formula in (2) (with $P_\gamma = \hat{P}_\alpha$ and $P_\zeta = \hat{P}_\lambda$) lends further insight, since $\bar{\eta}(\hat{P}_\lambda)$ has a discontinuity at $\lambda = \frac{1}{2}$ and the second condition of (2) is not satisfied.

A still different perspective results if we investigate Cauchy indices. This is defined (Gantmacher, 1959) (See in particular Chapter XV, Section 2) for a real rational function $f(s)$ over an interval $[l, u]$ of the real line (where either or both l or u can be at infinity) as

$$\mathcal{I}_l^u[f(s)] = \tilde{p}_l^u[f(s)] - \tilde{n}_l^u[f(s)]$$

where $\tilde{p}_l^u$ is the number of (positive) jumps that $f(s)$ makes from $-\infty$ to $+\infty$ as s increases in the open interval (l,u) and $\tilde{n}_l^u$ is the number of (negative) jumps from $+\infty$ to $-\infty$. For a rational $f(s)$ with $f(\infty) < \infty$ and denominator of degree k in a coprime polynomial fraction, the Cauchy index over any interval is an integer in the set $\{-k, -k+2, \ldots, k-2, k\}$. One key result (Brockett, 1976) is that if $f_\alpha(s)$ and $f_\Omega(s)$ are two k^{th} degree transfer functions, with (polynomial) coprime descriptions n_α/m_α and n_Ω/m_Ω, there exist homotopies n_λ, m_λ with $\lambda \in [0,1]$ taking f_α to f_Ω such that (n_λ, m_λ) are coprime polynomials for all λ if and only if

$$\mathcal{I}_{-\infty}^{+\infty}[f_\alpha(s)] = \mathcal{I}_{-\infty}^{+\infty}[f_\Omega(s)].$$

The Cauchy indices over $(0,\infty)$ for $\hat{P}_\alpha$ and $\hat{P}_\Omega$ differ, and $\lambda = \frac{1}{2}$ is both where the Vinnicombe metric distance reverts to unity in the proposed homotopy, and also where the Cauchy index of $\hat{P}_\lambda$ changes. It is no accident that the two quantities, the Vinnicombe distance and the Cauchy index, have a discontinuity at the same value of the homotopy parameter λ.

We shall now generalise the above result. Consider two plants $\tilde{P}_\alpha = \frac{K_\alpha}{s+p_\alpha}$ and $\tilde{P}_\lambda = \frac{K_\lambda}{s+p_\lambda}$.

It is not hard to verify that provided $K_\alpha K_\lambda \neq 0$,

$$\tilde{G}_\lambda G_\alpha = \frac{(K_\lambda - K_\alpha)s + (K_\lambda p_\alpha - K_\alpha p_\lambda)}{\left(s + \sqrt{K_\alpha^2 + p_\alpha^2}\right)\left(s + \sqrt{p_\lambda^2 + K_\lambda^2}\right)} \quad \text{and}$$

$$G_\lambda^* G_\alpha = \frac{-s^2 + (p_\lambda - p_\alpha)s + K_\alpha K_\lambda + p_\alpha p_\lambda}{\left(-s + \sqrt{K_\lambda^2 + p_\lambda^2}\right)\left(s + \sqrt{K_\alpha^2 + p_\alpha^2}\right)}$$

and that the winding number condition is fulfilled if and only if $K_\alpha K_\lambda + p_\alpha p_\lambda > 0$. However, if $K_\lambda = 0$ ie $\tilde{P}_\lambda = 0$, then $G_\lambda = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and

$$G_\alpha^* G_\lambda = \frac{-s + p_\alpha}{-s + \sqrt{K_\alpha^2 + p_\alpha^2}}.$$

If $(-p_\alpha) \geq 0$, then the winding number condition fails and if $(-p_\alpha) < 0$, then it holds. Now let us regard $K_\alpha > 0$ and p_α as fixed with K_λ, p_λ free and consider two cases: one where $(-p_\alpha)$ is in the (open) right half plane (RHP) and one where $(-p_\alpha)$ is in the (open) left hand plane (LHP).

If $\tilde{P}_\alpha$ is unstable then $(-p_\alpha) > 0$. Figure 1 shows the set of K_λ, p_λ values for which $\delta_\nu(\tilde{P}_\alpha, \tilde{P}_\lambda) < 1$. The negative p_λ axis is excluded, as is the upward sloping line through the origin. There are two non-intersecting regions with intersecting boundaries.

If $\tilde{P}_\alpha$ is stable then $(-p_\alpha) < 0$. Figure 2 depicts $\hat{P}_\lambda$ for which $\delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda) < 1$. The positive p_λ axis is included, but the line of negative slope through the origin is not.

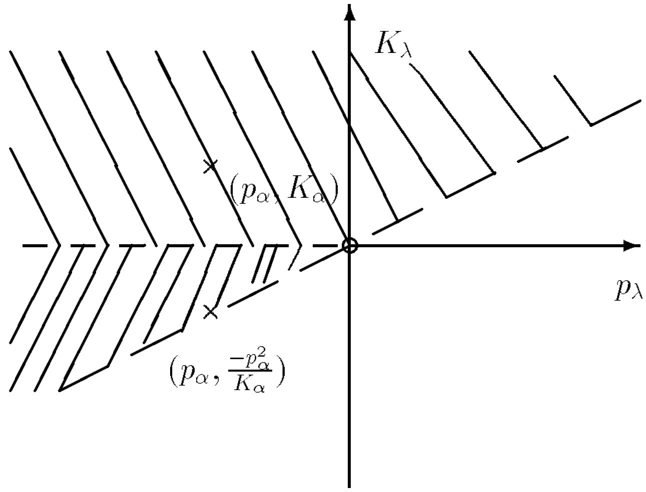

Fig. 1. Region for $\delta_\nu(\tilde{P}_\alpha, \tilde{P}_\lambda) < 1$, Unstable $\tilde{P}_\alpha$

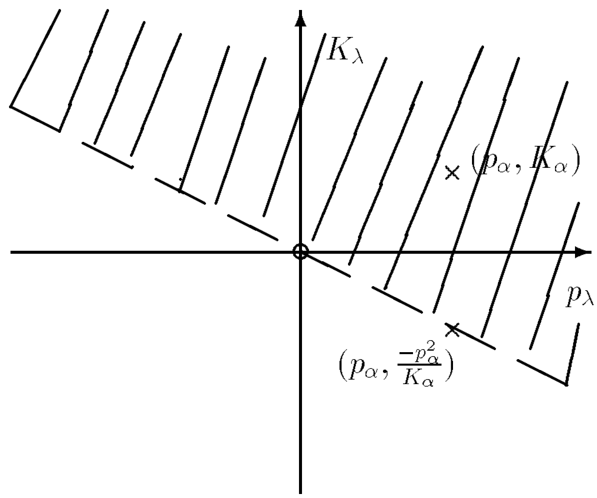

Fig. 2. Region for $\delta_\nu(\tilde{P}_\alpha, \tilde{P}_\lambda) < 1$, Stable $\tilde{P}_\alpha$

It can now be seen from Figure 1 (the unstable case), that there is no path from $\hat{P}_\alpha = \frac{\epsilon}{s-1}$ to $\hat{P}_\Omega = -\frac{\epsilon}{s-1}$ via some homotopy $\tilde{P}_\lambda = \frac{K_\lambda}{s+p_\lambda}$ with varying K_λ, p_λ and such that $\delta_\nu(\hat{P}_\alpha, \tilde{P}_\lambda) < 1$ for all λ, because $\hat{P}_\alpha$ is above and $\hat{P}_\Omega$ is below the negative p_λ axis in Figure 1.

However, for stable $\check{P}_\alpha = \frac{\epsilon}{s+1}$ and $\check{P}_\Omega = -\frac{\epsilon}{s+1}$, with $0 < \epsilon < 1$, Figure 2 shows there *is* a continuous connecting homotopy, eg $\check{P}_\lambda = \frac{\epsilon(1-2\lambda)}{s+1}$.

4. MORE GENERAL HOMOTOPIES

In this section we shall consider homotopies $\hat{P}_\lambda$ between $\hat{P}_\alpha = \frac{\epsilon}{s-1}$ and $\hat{P}_\Omega = -\frac{\epsilon}{s-1}$, which are rational, but not necessarily first order. Thus we suppose that $\hat{P}_\lambda = \frac{b_\lambda(s)}{a_\lambda(s)}$ and that $\hat{P}_\alpha = \frac{b_\alpha(s)}{a_\alpha(s)}$, $\hat{P}_\Omega = \frac{b_\Omega(s)}{a_\Omega(s)}$, with

$$b_\alpha(s) = \epsilon u_\alpha(s) \qquad a_\alpha(s) = (s-1)u_\alpha(s)$$
$$b_\Omega(s) = -\epsilon u_\Omega(s) \qquad a_\Omega(s) = (s-1)u_\Omega(s)$$

and $u_\alpha(s), u_\Omega(s)$ each Hurwitz polynomials. We now justify the restriction on $u_\alpha(s)$ to be Hurwitz. If $u_\alpha(s)$ were not Hurwitz, then homotopies in the numerator and denominator polynomials which do not preserve the right hand plane pole-zero cancellation will not translate to Vinnicombe metric homotopies of the transfer functions.

Specifically, suppose that $v_\lambda(s), w_\lambda(s)$ are polynomials with coefficients very close to those of non-Hurwitz $u_\alpha(s)$ and no common right hand plane zeros, and set $\hat{P}_\lambda = \frac{\epsilon v_\lambda(s)}{(s-1)w_\lambda(s)}$. Although $\bar{\kappa}(\hat{P}_\alpha, \hat{P}_\lambda)$ will be very small, the winding number condition of equation (2) will not hold.

Indeed with ϵ suitably small, there will hold $|\hat{P}_\alpha \hat{P}_\lambda^*| < 1$ for all $j\omega$ and so $\text{wno}[1 + \hat{P}_\alpha \hat{P}_\lambda^*] = 0$. Then the left side of the second equation in (2) will evaluate to $\breve{\eta}(\hat{P}_\alpha) - \bar{\eta}(\hat{P}_\lambda) = 1 - [1 + \mathcal{Z}(w_\lambda)] \neq 0$, where $\mathcal{Z}(w_\lambda)$ is the number of RHP zeroes of $w_\lambda(s)$. Conversely, if $u_\alpha(s)$ is Hurwitz, and $v_\lambda(s), w_\lambda(s)$ are small perturbations of $u_\alpha(s)$ as described above, (with of course, no common right hand plane zeros), then $\bar{\kappa}(\hat{P}_\alpha, \hat{P}_\lambda)$ will again be very small as before, but the winding number condition will hold and so $\delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda)$ will be small.

Note that there is no requirement for $b_\lambda(s)$ and $a_\lambda(s)$ to have a constant degree for all λ. The leading coefficient(s) of either polynomial may vanish at a point within, or even on a subinterval of $[0, 1]$.

We now show that, in general, the existence of a (continuous) P_λ obtained as a juxtaposition of homotopies, each defined as a ratio of polynomial homotopies, implies the existence of single numerator and denominator homotopies n_λ and m_λ such that $P_\lambda = n_\lambda/m_\lambda$ (The notational change emphasises that the following holds not only for $\hat{P}_\alpha$ and $\hat{P}_\Omega$ above). Hence, assume that $\bar{n}_\lambda$ and $\bar{m}_\lambda$, each continuous on $(\bar{\mu}_l, \mu]$ defines a rational $P_\lambda = \bar{n}_\lambda/\bar{m}_\lambda$; and that $\tilde{n}_\lambda$ and $\tilde{m}_\lambda$, each continuous on $[\mu, \tilde{\mu}_u)$ defines $P_\lambda = \tilde{n}_\lambda/\tilde{m}_\lambda$ such that $\lim_{\lambda\to\mu^-} P_\lambda = \lim_{\lambda\to\mu^+} P_\lambda = P_\mu$.

Assume that although the numerator and denominator polynomials $\bar{n}_\lambda, \bar{m}_\lambda$ and $\tilde{n}_\lambda, \tilde{m}_\lambda$ are continuous on the respective open intervals, $(\bar{\mu}_l, \mu)$ and $(\mu, \tilde{\mu}_u)$, they are not continuous at μ, that is

$$\lim_{\lambda\to\mu^-} \bar{n}_\lambda(s) \neq \lim_{\lambda\to\mu^+} \tilde{n}_\lambda(s),$$
$$\lim_{\lambda\to\mu^-} \bar{m}_\lambda(s) \neq \lim_{\lambda\to\mu^+} \tilde{m}_\lambda(s).$$

Even so, we can manufacture homotopies n_λ, m_λ of both numerator and denominator polynomials with continuous coefficients such that $P_\lambda = n_\lambda/m_\lambda$. Specifically, we know that

$$P_\lambda(s) = \frac{\bar{n}_\lambda(s)}{\bar{m}_\lambda(s)} \quad \text{for } 0 \le \lambda < \mu,$$
$$P_\mu(s) = \frac{\bar{n}_\mu(s)}{\bar{m}_\mu(s)} = \frac{\tilde{n}_\mu(s)}{\tilde{m}_\mu(s)}$$
$$\text{and} \quad P_\lambda = \frac{\tilde{n}_\lambda(s)}{\tilde{m}_\lambda(s)} \quad \text{for } \mu \le \lambda \le 1,$$

where $\bar{n}_\lambda, \bar{m}_\lambda, \tilde{n}_\lambda, \tilde{m}_\lambda$ are each homotopies. Since P_λ is continuous at μ there must exist some $p(s), q(s), \bar{u}_\mu(s), \tilde{u}_\mu(s)$ such that $\bar{n}_\mu = p(s)\bar{u}_\mu(s)$, $\bar{m}_\mu = q(s)\bar{u}_\mu(s)$, $\tilde{n}_\mu = p(s)\tilde{u}_\mu(s)$, and $\tilde{m}_\mu = q(s)\tilde{u}_\mu(s)$. Then we can define

$$n_\lambda = \begin{cases} \bar{n}_\lambda(s)\tilde{u}_\mu(s) & \text{for } \bar{\mu}_l \le \lambda < \mu \\ \tilde{n}_\lambda(s)\bar{u}_\mu(s) & \text{for } \mu \le \lambda < \tilde{\mu}_u \end{cases},$$
$$m_\lambda = \begin{cases} \bar{m}_\lambda(s)\tilde{u}_\mu(s) & \text{for } \bar{\mu}_l < \lambda < \mu \\ \tilde{m}_\lambda(s)\bar{u}_\mu(s) & \text{for } \mu \le \lambda < \tilde{\mu}_u \end{cases},$$

in order to obtain polynomial homotopies on $[0, 1]$ with the property that $P_\lambda = n_\lambda/m_\lambda$ for all $\lambda \in [0, 1]$. The numerator and denominator polynomials n_λ and m_λ defined above are continuous, that is,

$$\lim_{\lambda\to\mu^-} \bar{n}_\lambda(s)\tilde{u}_\mu(s) = \lim_{\lambda\to\mu^+} \tilde{n}_\lambda(s)\bar{u}_\mu(s),$$
$$\lim_{\lambda\to\mu^-} \bar{m}_\lambda(s)\tilde{u}_\mu(s) = \lim_{\lambda\to\mu^+} \tilde{m}_\lambda(s)\bar{u}_\mu(s).$$

The main result is now as follows.

Theorem 4.1. Let $\hat{P}_\alpha = \frac{\epsilon}{s-1}$ and $\hat{P}_\Omega = -\frac{\epsilon}{s-1}$ for some $0 < \epsilon < 1$. Let $u_\alpha(s)$ and $u_\Omega(s)$ be Hurwitz polynomials and let $b_\alpha(s) = \epsilon u_\alpha(s)$, $a_\alpha(s) = (s-1)u_\alpha(s)$ and $b_\Omega(s) = -\epsilon u_\Omega(s)$, $a_\Omega(s) = (s-1)u_\Omega(s)$. Let $b_\lambda(s)$, $a_\lambda(s)$ be homotopies on $[0, 1]$ continuously linking b_α to $b_\Omega(s)$ and $a_\alpha(s)$ to $a_\Omega(s)$, such that setting $\hat{P}_\lambda = \frac{b_\lambda(s)}{a_\lambda(s)}$ gives a proper, but not necessarily strictly proper $\hat{P}_\lambda$. Then for some $\lambda \in (0, 1)$ there holds

$$\delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda) = 1.$$

We first establish a simple lemma.

Lemma 4.2. Let $n_\lambda(s)$ $m_\lambda(s)$ be two homotopies on $[0, 1]$ continuously linking $n_\alpha(s)$ to $n_\Omega(s)$ and $m_\alpha(s)$ to $m_\Omega(s)$. Suppose that $\frac{n_\alpha(s)}{m_\alpha(s)}$ and $\frac{n_\lambda(s)}{m_\lambda(s)}$ have no pole-zero cancellations in the closed RHP. Then, with $P_\alpha = \frac{n_\alpha}{m_\alpha}$, $P_\lambda = \frac{n_\lambda}{m_\lambda}$ the two conditions

$$\lim_{\lambda\to\bar{\lambda}^-} \delta_\nu(P_\alpha, P_\lambda) = 1$$
$$\text{and} \quad \delta_\nu(P_\alpha, P_\lambda) < 1 \quad \text{for all } \lambda < \bar{\lambda}$$

will hold true if and only if $n_\alpha^* n_\lambda + m_\alpha^* m_\lambda$ is non-zero for all $s = j\omega$ with $0 < \lambda < \bar{\lambda}$, and $n_\alpha^* n_{\bar{\lambda}} + m_\alpha^* m_{\bar{\lambda}}$ is zero for some $s = j\omega$.

Proof : Let r_α and r_λ be (the) Hurwitz polynomials with positive leading coefficients satisfying $r_\alpha^* r_\alpha = n_\alpha^* n_\alpha + m_\alpha^* m_\alpha$, $r_\lambda^* r_\lambda = n_\lambda^* n_\lambda + m_\lambda^* m_\lambda$. Even if n_α, m_α or n_λ, m_λ have a Hurwitz common factor then

$$\kappa(\hat{P}_\alpha, P_\lambda, \omega) = \left| \frac{n_\alpha(j\omega) m_\lambda(j\omega) - n_\lambda(j\omega) m_\alpha(j\omega)}{r_\alpha(j\omega) r_\lambda(j\omega)} \right| \quad (9)$$

and $\delta_\nu(\hat{P}_\alpha, P_\lambda) = \sup_\omega \kappa(\hat{P}_\alpha, P_\lambda, \omega)$ provided that

$$\text{for all } \omega \quad \frac{n_\alpha(j\omega)^* n_\lambda(j\omega) + m_\alpha(j\omega)^* m_\lambda(j\omega)}{r_\alpha(j\omega)^* r_\lambda(j\omega)} \neq 0$$

$$\text{and} \quad \text{wno} \left[\frac{n_\alpha(s)^* n_\lambda(s) + m_\alpha(s)^* m_\lambda(s)}{r_\alpha(s)^* r_\lambda(s)} \right] = 0.$$

One can verify that for all ω

$$\begin{aligned} &|n_\alpha(j\omega) m_\lambda(j\omega) - n_\lambda(j\omega) m_\alpha(j\omega)|^2 \\ &\quad + |n_\alpha(j\omega)^* n_\lambda(j\omega) + m_\alpha(j\omega)^* m_\lambda(j\omega)|^2 \quad (10) \\ &\quad = |r_\alpha(j\omega) r_\lambda(j\omega)|^2 . \end{aligned}$$

When $\lambda < \bar{\lambda}$ the condition $\delta_\nu(P_\alpha, P_\lambda) < 1$ implies that the winding number condition holds and therefore $|n_\alpha(j\omega)^* n_\lambda(j\omega) + m_\alpha(j\omega)^* m_\lambda(j\omega)| \neq 0$ for all ω. When $\lambda \to \bar{\lambda}^-$ we have that $\delta_\nu(P_\alpha, P_\lambda) \to 1$. This can only be because $\kappa(P_\alpha, P_\lambda, \omega) \to 1$ at some ω, that is $|n_\alpha(j\omega) m_\lambda(j\omega) - n_\lambda(j\omega) m_\alpha(j\omega)| \to |r_\alpha(j\omega) r_\lambda(j\omega)|$ at some ω, or in the light of equation (9) that $|n_\alpha(j\omega)^* n_\lambda(j\omega) + m_\alpha(j\omega)^* m_\lambda(j\omega)| \to 0$ at some ω, or both. (The winding number cannot change without $n_\alpha(j\omega)^* n_\lambda(j\omega) + m_\alpha(j\omega)^* m_\lambda(j\omega)$ having a zero pass from the ORHP to the OLHP). Inspection of (10) shows that both conditions causing $\delta_\nu(P_\alpha, P_\lambda) = 1$ operate simultaneously, at the one frequency. The argument is easily reversible. □

We are now in a position to prove Theorem 4.1

Proof : The proof will proceed by contradiction. Therefore, suppose that a homotopy exists with $\delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda) < 1$ for all $\lambda \in [0, 1]$. For convenience let us suppose, without loss of generality, that $b_\alpha(0) > 0$. At $\lambda = 0$ and $\lambda = 1$, there is precisely one zero of $a_\lambda(s)$ in $(0, \infty)$. If this holds true for all λ, then we can argue that there will necessarily be a $\bar{\lambda} \in [0, 1]$ at which $b_\lambda(s)$ and $a_\lambda(s)$ will have a common real zero in $(0, \infty)$. This conclusion follows from the fact that $\hat{P}_\alpha$ and $\hat{P}_\Omega(s)$ have different Cauchy indices (Brockett, 1976). By arguments presented at the beginning of this section, this is seen to lead to $\delta_\nu(\hat{P}_\alpha, \hat{P}_{\bar{\lambda}}) = 1$.

Hence, if there were a homotopy, it would necessarily include, as λ varies, changes in the number of real zeros of $a_\lambda(s)$ on $(0, \infty)$. This number can change in different ways: as two complex conjugate zeros in $a_\lambda(s)$ turn into a double zero on the positive real axis (or conversely), as a zero moves from $+\infty$ to $-\infty$ (or conversely), or if a zero passes through the origin.

We consider the effect of each of these possibilities on two quantities: the Cauchy index $\mathcal{I}_0^\infty \hat{P}_\lambda(s)$ and the winding number wno$[1+\hat{P}_\alpha \hat{P}_\lambda^*]$. We shall show it is not possible to change the Cauchy index from $\mathcal{I}_0^\infty \hat{P}_\lambda(s) = 1$ to $\mathcal{I}_0^\infty \hat{P}_\lambda(s) = -1$ without violating the winding number condition.

Complex Zero becoming Real

When two complex zeros of $a_\lambda(s)$ hit the real axis at say $s = p$ as λ varies and then separate, if there is no zero of $b_\lambda(s)$ at $s = p$ then the Cauchy index is initially unchanged. If $b_\lambda(s)$ does have a zero at $s = p$, then $b_\lambda(s)/a_\lambda(s)$ acquires an unstable pole-zero cancellation at p and $\delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda) = 1$.

Zero passing through Infinity

If a zero of $a_\lambda(s)$ passes through infinity at some $\bar{\lambda}$, then the leading coefficient c_λ must pass through zero, and $a_\lambda(s)$ will decrease in degree at $\bar{\lambda}$. We will show that as λ passes through $\bar{\lambda}$, the winding number must change.

Specifically, suppose that $a_\lambda(s) = c_\lambda s^n + \bar{a}_\lambda(s)$ where $\deg[\bar{a}_\lambda(s)] < n$ and consider

$$\begin{aligned} &b_\alpha(s)^* b_\lambda(s) + a_\alpha(s)^* a_\lambda(s) = \\ &\quad b_\alpha(-s) b_\lambda(s) + a_\alpha(-s) \bar{a}_\lambda(s) + c_\lambda a_\alpha(-s) s^n \end{aligned}$$

Recall that, since $\hat{P}_\lambda$ is proper, we must have that $\deg[b_\lambda(s)] \leq \deg[a_\lambda(s)]$ and that $\deg[b_\alpha] = \deg[a_\alpha] - 1$ so that clearly (for $c_\lambda \neq 0$) $\deg[b_\alpha(-s) b_\lambda(s) + a_\alpha(-s) \bar{a}_\lambda(s)] < \deg[c_\lambda a_\alpha(-s) s^n]$.

When $\lambda = \bar{\lambda}$ the polynomial $b_\alpha^* b_\lambda + a_\alpha^* a_\lambda$ necessarily drops in degree, and as λ passes through $\bar{\lambda}$ its leading coefficient changes sign so that a zero of $b_\alpha^* b_\lambda + a_\alpha^* a_\lambda$ also necessarily passes from positive to negative infinity or vice versa. Hence wno$[(b_\alpha^* b_\lambda + a_\alpha^* a_\lambda)/(r_\alpha^* r_\lambda)]$, changes as λ passes through $\bar{\lambda}$. We can argue similarly that if a_λ acquires a zero from infinity as λ increases, then there is again a change of winding number as λ changes from $\bar{\lambda}$ to a larger value.

Hence we can rule out the possibility that the number of real zeros of $a_\lambda(s)$ on $[0, \infty)$ changes by the acquisition or loss of zeros at infinity.

Zero moving through the Origin

Suppose that a zero of $a_\lambda(s)$ moves, as λ varies, through the origin, either from the negative to the positive real axis or vice versa. Evidently, $b_\lambda(0)$ will vanish at some $\bar{\lambda}$. Now by Lemma 4.2, the winding number condition requires that $b_\alpha(0)b_\lambda(0) + a_\alpha(0)a_\lambda(0)$ remains positive for all λ. So if $a_\alpha(0)a_\lambda(0)$ vanishes at $\lambda = \bar{\lambda}$, then by continuity we must have that $b_\alpha(0)b_\lambda(0) > 0$ for λ both at $\bar{\lambda}$ and in a neighbourhood of $\bar{\lambda}$. Since we have assumed, without loss of generality, that $b_\alpha(0) > 0$, it follows that $b_{\bar{\lambda}}(0) > 0$. Also, since $b_\alpha(0) > 0$ and $b_\alpha(0)/a_\alpha(0) = -\epsilon$ we can conclude that $a_\alpha(0) < 0$.

Now suppose that the set of locations where a zero of odd multiplicity passes through the origin is composed of discrete values of λ, say $\bar{\lambda}_1 < \bar{\lambda}_2 < \bar{\lambda}_3 < \dots$. As λ moves through each $\bar{\lambda}_i$ the quantity $a_\lambda(0)$ changes sign.

For values of λ in the semi-open interval $[0, \bar{\lambda}_1)$, the quantity $a_\lambda(0)$ retains its sign, that is $a_\lambda(0) < 0$. (This is true even if $a_\lambda(s)$ acquired real zeros in the open RHP by conjugate zeros becoming real.) If a zero of odd multiplicity moves from the right of the origin to the left at $\bar{\lambda}_1$ then the sign constraint on $a_\lambda(0)$ and $b_\lambda(0)$ means that the Cauchy index must decrement by one. If a zero of odd multiplicity moves in the opposite direction from the left to the right of the origin at $\bar{\lambda}$, then similar reasoning again shows that the Cauchy index must decrement by one. (Note that if an even multiplicity zero were to cross through the origin at some $\tilde{\lambda}$, then neither the Cauchy index nor the sign of $a_\lambda(0)$ in the neighbourhood of $\tilde{\lambda}$ would change although it would hold that $a_{\tilde{\lambda}}(0) = 0$).

Now in the open interval $(\bar{\lambda}_1, \bar{\lambda}_2)$, we now have that $a_\alpha(0)a_\lambda(0) < 0$ since $a_\lambda(0)$ changes from being negative in $[0, \bar{\lambda}_1)$ to positive for $\lambda > \bar{\lambda}$ and $a_\alpha(0) < 0$. By appealing once again to the winding number condition and Lemma 4.2, we see that it must hold that $b_\alpha(0)b_\lambda(0) > 0$ for λ in the open interval $(\bar{\lambda}_1, \bar{\lambda}_2)$. Since $a_\lambda(0) = 0$ for each of $\lambda = \bar{\lambda}_1$ and $\bar{\lambda}_2$ we can conclude that $b_{\bar{\lambda}}(0) \neq 0$ and hence that $b_\alpha(0)b_\lambda(0) > 0$ for λ in the closed inteval $[\bar{\lambda}_1, \bar{\lambda}_2]$. As in the previous paragraph, examination of the sign of $b_\lambda(0)/a_\lambda(0)$ shows that a passage of a zero of a_λ of odd multiplicity through the origin in either direction as λ increases through $\bar{\lambda}_2$ leads to an increment of the Cauchy index by one.

This demonstrates that provided that the winding number condition required for $\delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda) < 1$ holds, and that $b_\lambda(s)$ does not so change as to produce an unstable pole zero cancellation with $a_\lambda(s)$, the resultant total variation of the Cauchy index of $b_\lambda(s)/a_\lambda(s)$ over $(0, \infty)$ due to a series of zeroes passing backwards and forwards through the origin can only be zero or one.

This remains true even in the case that $a_\lambda(s)$ acquires positive real zeros due to the conversion of pairs of complex conjugate zeros during the homotopy. Consequently, it will never occur that $\mathcal{I}_0^\infty \frac{b_\lambda}{a_\lambda} = -1$. This demonstrates that the homotopy from $\hat{P}_\alpha$ to $\hat{P}_\Omega$ satisfying the subunitary condition $\delta_\nu(\hat{P}_\alpha, \hat{P}_\lambda) < 1$ does not exist. □

5. CONCLUSION

We have demonstrated by counterexample that it is possible for two transfer functions to be (arbitrarily) close in the Vinnicombe metric, and yet to not have a subunitary connecting homotopy between them. This is a counter-intuitive result.

The result in this paper was demonstrated for *scalar* transfer functions only, and exploited properties arising from the fact that the candidate transfer functions for the homotopy end-points had different Cauchy indices on the interval $(0, \infty)$. As such, we do not expect that this result will hold in the multivariable case, since the Cauchy index is essentially only well-defined for scalar transfer functions. Admittedly, however, it is true that the Cauchy index of a *symmetric* multivariable transfer function can be defined (Bitmead and Anderson, 1977). If P_α and P_Ω were symmetric and symmetric P_λ were sought, similar results to the scalar case could be expected.

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