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STATE ESTIMATION VIA THE WHITENING FILTER*

by

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ABSTRACT

The concept of the whitening filter applicable to a nonstationary covariance is used to give a derivation of the Kalman filter. The derivation requires specification of the whitening filter in state space terms, and the solution of a Wiener-Hopf equation of such a type that evaluation of a single covariance yields a solution.

I. INTRODUCTION

Since the original appearance of Wiener's work on filtering, there have been numerous attempts to generalize the material to cope with such extensions as finite observation times, nonstationary stochastic processes, and time-varying systems. Among the more successful of these generalizations has been the so-called Kalman filter, [1], which combined the various desired extensions to the Wiener theory with the idea of describing dynamical systems in state-space terms.

The processes through which the Kalman filter is derived are not straightforward, and, in the first instance, require such unpleasant features as integrating a distribution containing a delta function over an interval, an end-point of which coincides with the discontinuity of the delta function. It is the aim of this paper to derive the Kalman filter as a logical, and we believe simple, follow-on from the whitening filter [2], [3].

The whitening filter, it will be recalled, has coloured noise as its input, and white noise for its output. It was possibly originally popularised in the paper of Bode and Shannon [2], which gave much insight into the mathematical manipulations of the Wiener filtering theory. Bode and Shannon's work however contained the same limitations as the Wiener theory; thus the systems considered were time-invariant, the stochastic processes were stationary, and filtering operation had to commence infinitely far back in the past. Recently though a development of whitening filters has been presented in state-space terms which allows time-variation, lack of stationarity, and finite initial time of operation, [3]. The development will be applied in this paper to yield the Kalman filter.

To state the filtering problem, we consider a system described by the equations

$$\dot{x} = Fx + Gu \tag{1a}$$

$$z = H'x + v \tag{1b}$$

Here, u is the input to the system, x is the system state, v is additive noise, and z is the system output. The input u is white noise of zero mean and covariance

$$E[u(t)u'(\tau)] = Q(t)\delta(t-\tau) \tag{2}$$

It is assumed that $Q(t)$ is bounded. The additive noise at the output is also of zero mean, with covariance

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$$E[v(t)v'(\tau)] = R(t)\delta(t-\tau) \qquad (3)$$

where

$$\alpha_2 I \geq R(t) \geq \alpha_1 I > 0 \qquad (4)$$

for some positive constants α_1, α_2 and for all t. The processes u and v are uncorrelated; the same is thus true of x and v.

The filtering problem is of course to use the measurements z to estimate x in an optimal fashion. More precisely, the measurement of z commences at some time t_0, ($t_0 = -\infty$ is allowed) at which time $\mathrm{cov}(x(t_0),x(t_0))$ is known; from a knowledge of $z(\tau)$ for $t_0 \leq \tau \leq t$, an estimate $\hat{x}(t)$ of $x(t)$ is required which is unbiased and is such that the expected squared error in estimating any linear function of the message is minimized.

It is well known that the estimate $\hat{x}(t)$ may be generated using an operator $A(\cdot,\cdot)$ via

$$\hat{x}(t) = \int_{t_0}^{t} A(t,\tau)z(\tau)d\tau \qquad (5)$$

where $A(\cdot,\cdot)$ satisfies the Wiener-Hopf equation

$$\mathrm{cov}[x(t),z(\sigma)] = \int_{t_0}^{t} A(t,\tau)\mathrm{cov}[z(\tau),z(\sigma)]d\tau; \qquad t_0 \leq \sigma < t \qquad (6)$$

Thus the filtering problem reduces to evaluating the covariances in (6), solving for the unknown $A(t,\tau)$, and finally interpreting the solution in state-space terms.

If $\hat{z}(\cdot)$ is some other variable related to $z(\cdot)$ via a deterministic linear relation in such a way that knowledge of $z(\cdot)$ up till time t yields $\hat{z}(t)$, and knowledge of $\hat{z}(\cdot)$ up till time t yields $z(t)$, then we may consider attempting to find $\hat{x}(t)$ from $\hat{z}(\cdot)$ rather than from $z(\cdot)$, via a relation of the form

$$\hat{x}(t) = \int_{t_0}^{t} \hat{A}(t,\tau)\hat{z}(\tau)d\tau \qquad (7)$$

where now $\hat{A}(\cdot,\cdot)$ will satisfy the equation

$$\mathrm{cov}[x(t),\hat{z}(\sigma)] = \int_{t_0}^{t} \hat{A}(t,\tau)\mathrm{cov}[\hat{z}(\tau),\hat{z}(\sigma)]d\tau; \qquad t_0 \leq \sigma < t \qquad (8)$$

For at least one particular $\mathrm{cov}[\hat{z}(\tau),\hat{z}(\sigma)]$, this equation becomes very easy to solve. For if $\hat{z}$ is a white noise process, i.e.

$$\mathrm{cov}[\hat{z}(\tau),\hat{z}(\sigma)] = I\delta(\tau-\sigma) \qquad (9)$$

then the solution of (8) is simply

$$\hat{A}(t,\tau) = \mathrm{cov}[x(t),\hat{z}(\tau)] \qquad (10)$$

To achieve the relation (9) with an appropriate deterministic connection between z and $\hat{z}$, a whitening filter is used, the input to the whitening filter being z (*which is coloured as distinct from white noise*), and the output of the whitening filter being white noise $\hat{z}$.

Thus solution of the filtering problem requires the following sequence of operations: (a) specification of the whitening filter, (b) evaluation of $\mathrm{cov}[x(t),\hat{z}(\sigma)]$, and (c) interpretation of the cascade of the whitening filter and a system realizing $\hat{A}(t,\tau)$ [see (10)] in state-space terms.

Section 2 reviews the whitening filter theory, following [3]; section 3 carries out steps (b) and (c) above, and section 4 is a rapprochement between the development of the earlier sections and the filter theory of [1].

2. THE WHITENING FILTER, [3]

To develop a whitening filter for the process $z(\cdot)$, knowledge of $\mathrm{cov}[z(t),z(\tau)]$ is required.

Starting with eqs. (1a) and (2), it is straightforward to verify that

$$\mathrm{cov}[x(t),x(\tau)] = \Phi(t,\tau)P(\tau)1(t-\tau) + P(t)\Phi'(\tau,t)1(\tau-t) \quad (11)$$

where $1(t)$ is the unit step function at time t, $\Phi(\cdot,\cdot)$ is the transition matrix of $\dot{x} = Fx$, and P is defined by

$$\dot{P} = PF' + FP + GQG' \quad (12a)$$

$$P(t_0) = \mathrm{cov}[x(t_0),x(t_0)] \quad (12b)$$

It follows from eqs (1b), (3) and (11) that

$$\mathrm{cov}[z(t),z(\tau)] = R(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)P(\tau)H(\tau)1(t-\tau) + H'(t)P(t)\Phi'(\tau,t)H(\tau)1(\tau-t) \quad (13)$$

References [4] and [5] examine the problem of determining ways of generating the covariance (13) by driving a system with white noise. Of course, the equations (1) present one such way, but for our purposes, we use a result that the scheme of Figure 1 may be used to generate the covariance (13). The key quantity to specify is $\tilde{K}$, and the procedure for specifying it is as follows:

The equations

$$\dot{\Pi}_s = \Pi_s(F'-HR^{-1}H'P) + (F-PHR^{-1}H')\Pi_s + \Pi_s HR^{-1}H'\Pi_s + PHR^{-1}H'P \quad (14a)$$

$$\Pi_s(t_0) = 0 \quad (14b)$$

are used to define $\Pi_s(t)$ for all $t \geq t_0$. The fact that (14) has no finite escape time is important and is discussed further below. Then Π_s is used to define $\tilde{K}$ by

$$\tilde{K} = (P - \Pi_s)HR^{-\frac{1}{2}} \quad (15)$$

Actually, the matrix Π_s has significance other than merely in the definition of $\tilde{K}$; it turns out that

$$\mathrm{cov}[x_s(t),x_s(t)] = \Pi_s(t) \quad (16)$$

The initial condition for the system of Fig. 1 is $x_s(t_0) = 0$.

A whitening filter for the $z(\cdot)$ process is now defined by manipulating the system of Fig. 1; as is explained in [3], the whitening filter has the form of Fig. 2, where

$$K = \tilde{K}R^{-\frac{1}{2}} \quad (17a)$$

$$= (P - \Pi_s)HR^{-1} \quad (17b)$$

and

$$x_w(t_0) = 0 \quad (18)$$

Summarising: there is a filter which whitens the $z(\cdot)$ process of eqs (1); the filter has the form of Fig. 2, where K is given by (17), P is given by (12) and Π_s is given by (14).

The systems of Figs. 1 and 2 are actually antecedal inverses of one another, so that if the two systems are cascaded in either of the two possible orders, the impulse response of the cascade system is $I\delta(t-\tau)$.

To conclude this section, we give conditions for the existence of solutions to the differential equation (14), and thus conditions for the stability of the whitening filter.

With $F(\cdot)$, $H(\cdot)$ and $P(\cdot)$ finite valued on $[t_0,t_1]$ then Π_s, the solution of (14) exists and is well defined on $[t_0,t]$ if EITHER

(i) an $F(\cdot)$, $H(\cdot)$ and $P(\cdot)$ can be defined on an interval $[t_1,t_1+T_1]$ such that the right hand side of (13) is a covariance on $[t_0,t_1+T_1]$ and $[F,H']$ is completely observable in the sense that $x(t_1)$ may be determined from a knowledge of $H'(t)x(t)$ over $[t_1,t_1+T_1]$ when $u(t)$ is zero on the interval

OR

(ii) (and (ii) implies (i)) that $R-\eta I_n$ is positive definite on $[t_0, t_1]$ for some positive constant η.

For the limiting case as $t_0 \to -\infty$, the above is true with the addition to (ii) of the requirements that $P(\cdot)$, $H(\cdot)$ and $F(\cdot)$ be bounded on $(-\infty, t_1]$, that $|\Phi(\cdot,\cdot)|$ be bounded by a decaying exponential on $(-\infty, t_1]$, and the replacement of t_0 by $-\infty$ in (i) and (ii).

For the limiting case as $t_1 \to \infty$, sufficient conditions are EITHER that $[F, H']$ is uniformly completely observable with $F(\cdot)$, $H(\cdot)$, $P(\cdot)$ and $R(\cdot)$ bounded on $[t_0, \infty)$ OR that $R-\eta I_n$ is positive definite on $[t_0, \infty)$ for some positive constant η and F is asymptotically stable with $F(\cdot)$, and $P(\cdot)H(\cdot)$ *bounded on* $[t_0, \infty)$.

For the limiting case as both $t_1 \to \infty$ and $t_0 \to -\infty$, sufficient conditions as for the case $[t_0, \infty)$ exist with t_0 replaced by $-\infty$.

3. SOLUTION OF THE WIENER-HOPF EQUATION

In this section, we solve the Wiener-Hopf equation to determine how to find $\hat{x}$, the optimal estimate of x, from $\hat{z}$. As explained earlier, the Wiener-Hopf equation becomes nothing more than

$$\hat{A}(t,\tau) = \mathrm{cov}[x(t), \hat{z}(\tau)] \qquad (10)$$

where $\hat{z}$ is the output of the whitening filter. The essential problem is thus to evaluate the covariance in (10), and interpret the resulting $\hat{A}(\cdot,\cdot)$ in state-space terms.

It proves convenient to work with the variable $\hat{y}$ in Fig. 2; observing that $\hat{y} = z - H'x_w$, and $\hat{z} = R^{-\frac{1}{2}}(z - H'x_w)$, it is evident that

$$\hat{y} = R^{\frac{1}{2}} \hat{z} \qquad (19)$$

Now for $t > \tau$

$$\begin{aligned} E[x(t)\hat{y}'(\tau)] &= E[x(t)z'(\tau)] \\ &\quad - E[x(t)x_w'(\tau)H(\tau)] \\ &= E[x(t)x'(\tau)H(\tau)] \\ &\quad - E[x(t)x_w'(\tau)H(\tau)] \\ &= \Phi(t,\tau)P(\tau)H(\tau) \\ &\quad - E[x(t)x_w'(\tau)]H(\tau) \end{aligned}$$

The first equality follows by noting $\hat{y} = z - H'x_w$, the second by noting that $z = H'x + w$ and using the fact that $x(t)$ and $w(\tau)$ are obviously independent random variables, and the third by using (11).

By inspection of Fig. 2, and the fact that $x_w(t_0) = 0$,

$$x_w(\tau) = \int_{t_0}^{\tau} \Phi(\tau,\sigma)K(\sigma)\hat{y}(\sigma)d\sigma$$

and thus

$$E[x(t)\hat{y}'(\tau)] = \Phi(t,\tau)P(\tau)H(\tau) - \int_{t_0}^{\tau} E[x(t)\hat{y}'(\sigma)]K'(\sigma)\Phi'(\tau,\sigma)d\sigma H(\tau) \qquad (20)$$

Now subtract and add $\Phi(t,\tau)\int_{t_0}^{\tau} \Phi(\tau,\sigma)K(\sigma)R(\sigma)K'(\sigma)\Phi'(\tau,\sigma)d\sigma$ on the right side of (20) to get

$$\begin{aligned} E[x(t)\hat{y}'(\tau)] &= \Phi(t,\tau)P(\tau)H(\tau) \\ &\quad - \Phi(t,\tau)\int_{t_0}^{\tau} \Phi(\tau,\sigma)K(\sigma)R(\sigma)K'(\sigma)\Phi'(\tau,\sigma)d\sigma H(\tau) \\ &\quad - \int_{t_0}^{\tau} \{E[x(t)\hat{y}'(\sigma)] \\ &\quad - \Phi(t,\sigma)K(\sigma)R(\sigma)\}K'(\sigma)\Phi'(\tau,\sigma)d\sigma H(\tau) \end{aligned} \qquad (21)$$

From Fig. 1 and the interpretation of $\Pi_s(t)$ as $\mathrm{cov}[x_s(t), x_s(t)]$, together with (17a), it follows that

$$\Pi_s(\tau) = \int_{t_0}^{\tau} \Phi(\tau,\sigma)K(\sigma)R(\sigma)K'(\sigma)\Phi'(\tau,\sigma)d\sigma$$

and thus in (21),

$$E[x(t)\hat{y}'(\tau)] = \Phi(t,\tau)[P(\tau) - \Pi_s(\tau)]H(\tau)$$

$$- \int_{t_0}^{\tau} \{E[x(t)\hat{y}'(\sigma)]$$

$$- \Phi(t,\sigma)K(\sigma)R(\sigma)\}K'(\sigma)\Phi'(\tau,\sigma)d\sigma H(\tau)$$

or, using (17b)

$$\{E[x(t)\hat{y}'(\tau)] - \Phi(t,\tau)K(\tau)R(\tau)\}$$

$$= - \int_{t_0}^{\tau} \{E[x(t)\hat{y}'(\sigma)]$$

$$- \Phi(t,\sigma)K(\sigma)R(\sigma)\}K'(\sigma)\Phi'(\tau,\sigma)d\sigma H(\tau) \tag{22}$$

Regarding this as an integral equation for $w(t,\tau) = E[x(t)\hat{y}'(\tau)] - \Phi(t,\tau)K(\tau)R(\tau)$, it becomes evident that

$$E[x(t)\hat{y}'(\tau)] - \Phi(t,\tau)K(\tau)R(\tau) = 0 \tag{23}$$

Using (10) and (19), evidently

$$\hat{A}(t,\tau) = \Phi(t,\tau)K(\tau)R^{\frac{1}{2}}(\tau)$$

and using (19) again, evidently

$$\hat{x}(t) = \int_{t_0}^{t} \Phi(t,\tau)K(\tau)\hat{y}(\tau)d\tau \tag{24}$$

Hence by comparison with Fig. 2, we see that $x_w = x$. This is in a sense accidental; we would have expected that a system simulating $\hat{A}(t,\tau)$ would have been cascaded with that of Fig. 2 to yield $\hat{x}$. Such a cascade is not necessary here because of the structure of the whitening filter.

Finally, let us note that the direct feedthrough part of Fig. 2 (the block with unity gain I), and the scaling block $R^{-\frac{1}{2}}$ at the output, now have no relevance to the generation of $\hat{x}$, i.e. x_w is equally well determined by the scheme of Fig. 3.

4. RAPPROCHEMENT WITH THE KALMAN FILTER

The schematic of Fig. 3 bears close resemblance to the Kalman filter; indeed reference to [1] will show that it is identical in all but possibly one respect: the gain K of Fig. 3 has not been demonstrated to be the same gain as that in the Kalman filter. That the two gains are the same will now be shown.

The procedure for finding K has already been described, using eqs. (12), (14) and (17b). The Kalman gain, call it K* for the moment, is given by

$$K^* = \Pi_k H R^{-1} \tag{25}$$

where Π_k is given from the matrix Riccati differential equation

$$\dot{\Pi}_k = \Pi_k F' + F\Pi_k - \Pi_k H R^{-1} H' \Pi_k + GQG' \tag{26a}$$

$$\Pi_k(t_0) = \mathrm{cov}[x(t_0), x(t_0)] \tag{26b}$$

Comparison of (17b) and (25) shows that K* will equal K if

$$\Pi_k = P - \Pi_s \tag{27}$$

To see that (27) holds, first subtract the differential equation (14a) for Π_s from that for P, viz. (12a); this yields after a little manipulation

$$(\dot{P} - \dot{\Pi}_s) = (P - \Pi_s)F' + F(P - \Pi_s)$$

$$- (P - \Pi_s)HR^{-1}H'(P - \Pi_s) + GQG' \tag{28}$$

Equations (12b), (14b) and (26b) guarantee that (27) holds at t_0, and since (28) and (26a) are the same equation, it follows that (27) holds for all $t \geq t_0$.

It is of interest to observe that the conditions stated earlier which guarantee asymptotic stability of the optimal filter are similar to those of [1]. We note also a point that seems to have hitherto escaped attention: the fact that immediately to the right of the left hand subtracting element of the Kalman filter white noise of covariance $R\delta(t-\tau)$ [see (19)] is observed.

For the more general case when there is correlation between the additive noise v and the input u to the system i.e. when

$$E[u(t),v'(t)] = S(t)\delta(t-\tau) \quad (29)$$

where $S(t)$ is a known bounded nonzero rectangular matrix, then the above calculations are readily extended to give the state estimator as shown in Fig. 3 with a modified K denoted by $\bar{K}$. In (13), (14), (15), and (17), PH is replaced by $PH + GS$ and in (25) and (26), K^* and Π_K are replaced by $\bar{K}^*$ and $\bar{\Pi}_K$ where

$$\bar{K}^* = (\bar{\Pi}_K H + GS)R^{-1} \quad (30)$$

and

$$\dot{\bar{\Pi}}_K = \bar{\Pi}_K F' + F\bar{\Pi}_K - (\bar{\Pi}_K H + GS)R^{-1}(\bar{\Pi}_K H + GS)' + GQG' \quad (31a)$$

$$\bar{\Pi}_K(t_0) = \text{cov}[x(t_0),x(t_0)] \quad (31b)$$

This result was originally given in [6] as the limiting result of the solution of the equivalent problem set up for the discrete time case.

5. REFERENCES

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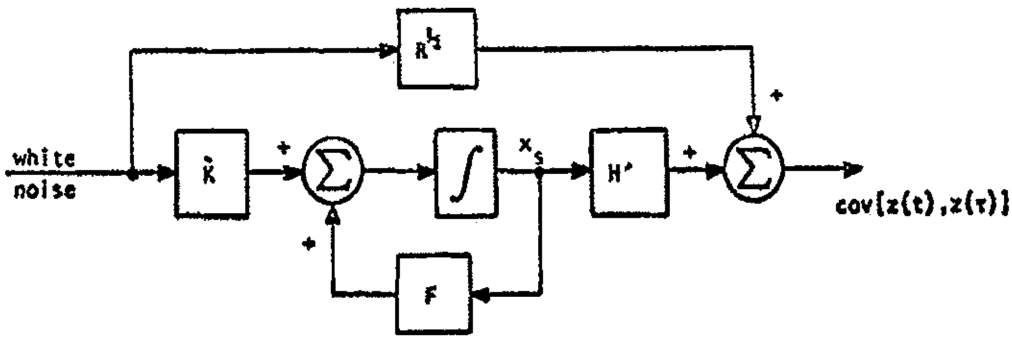
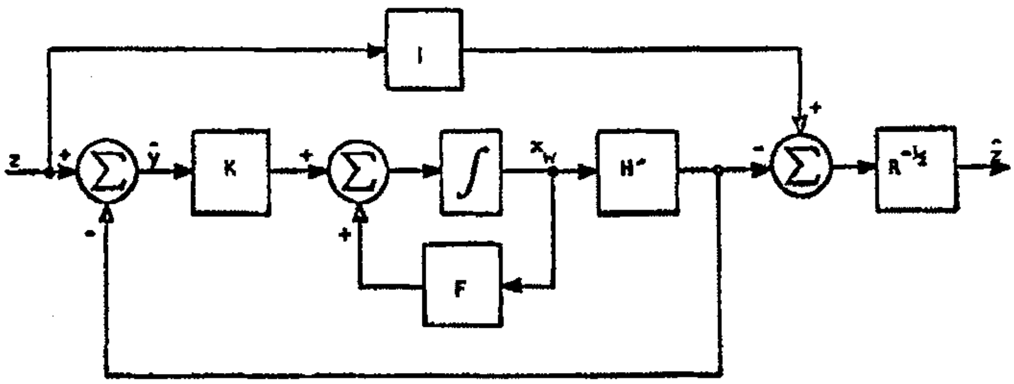
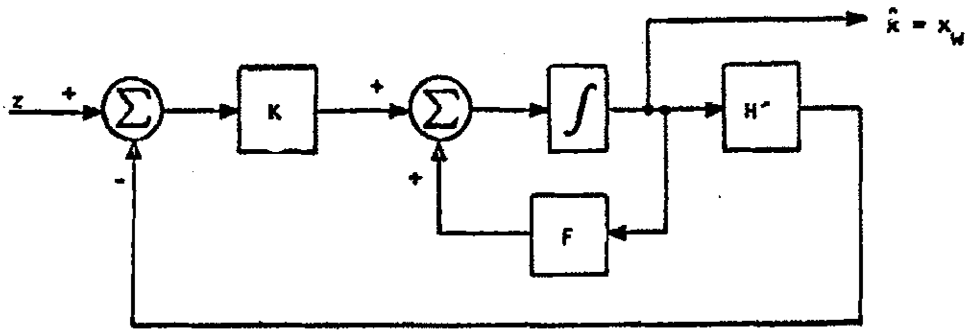


[1] System generating cov{z(t),z(τ)}



[2] Whitening Filter



[3] State Estimator