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# On the influence of weight modification in $H_\infty$ control design[1]

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## Abstract

$H_\infty$ control design is generally performed iteratively. At each iteration, the weights constraining the desired closed loop transfer functions are adapted. The way in which the weights are adapted is generally purely heuristic. It is consequently very interesting to build some insights about the influence of a weight modification on the obtained (central) controller and, more importantly, on the obtained closed-loop transfer functions. In this paper, we analyze this influence in the case of a classical two-block problem under the assumption of "small" modifications in the weights. The concept *small modification* must be understood in the sense of small enough to allow first order approximation.

## 1 Introduction

In recent years, $H_\infty$ control design has become a well-known method to design a model-based controller satisfying a number of constraints expressed by amplitude bounds (weights) on the "to-be-designed" closed-loop transfer functions. This method whose theoretical basis can be found in the works [1, 6] has known numerous applications for control design on real-life systems (see e.g. [2]). The design of a controller using $H_\infty$ control design generally follows an iterative procedure. In a first step, only the sensitivity function is effectively constrained (i.e. the constraints on the other transfer functions are chosen in such a way that they remain ineffective). A first controller is obtained in this way. However, this controller has generally an unsatisfactory performance with respect to the closed-loop transfer functions for which the constraints were (in this first step) too loose to be effective. Consequently, in a second step, the weights on these closed-loop transfer functions are adapted in order to improve the closed-loop behaviour of the controller and a second controller is computed using these adapted weights. This procedure is pursued until the obtained controller is judged satisfactory enough. The way with which the weights are adapted at each "iteration" is generally purely heuristic. It is consequently very interesting to build some insights about the influence of a weight modification on the obtained (central) controller and, more importantly, on the obtained closed-loop transfer functions. In this paper, we analyze this influence in the case of a classical two-block problem (for a scalar model) under the assumption of "small" modifications in the weights. The concept *small modification* must be understood in the sense of small enough to allow first order approximation.

In order to solve an $H_\infty$ control design problem, two different methods are available. The most attractive one with respect to computational efficiency is the one developed in [1], and relies on a state-space formulation. The second one found for example in [3, 4, 5] is based on the $J$-spectral factorization of the augmented plant. This second method is less attractive computationally speaking. However, it can of course be used for analysis purposes. In order to develop our analysis, we will use the second method since this is based on frequency domain expressions which are important for our purpose. In particular, we contemplate variation of weight functions that may change their McMillan degree, and yet in frequency domain terms are small. Such variations almost certainly could not easily be cast as small for state-space descriptions[1], involving as they do degree change.

In this paper, the first contribution is to give a frequency domain approximation of the change in the central controller due to a (small) weight modification after a first $H_\infty$ control design step. This approximation is a function of the weight modification frequency response and of the variables involved in the initial control design problem, and is therefore computable before performing the new control design step (i.e. the one with the modified weights). We show also that the modification in the central controller persists outside the frequency band where the weight modification is mainly located. A last contribution is to analyze the influence of this modification of the central controller on the modified closed-loop transfer functions.

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[1]used in the method of [1]

## 2 $J$-spectral factorization and $H_\infty$ control design

In this section, we first recall how we can solve a classical two-block $H_\infty$ control design problem (for a scalar model $G_{mod}$) using $J$-spectral factorization (see also [3, 4, 5] for more details). For this purpose let us first introduce the notion of homographic transformation. Consider a scalar transfer function $Q(s)$ and a partitioned transfer matrix $H(s) \triangleq \begin{pmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{pmatrix}$. In the partitioning, $H_{21}$ and $H_{22}$ are scalars. Define the homographic transformation $HOM(H,Q)$ as follows:

$$HOM(H,Q) \triangleq \frac{H_{11}Q + H_{12}}{H_{21}Q + H_{22}} \qquad (1)$$

**Proposition 2.1** ([5]) *Consider the two-block $H_\infty$ control problem consisting of finding a stabilizing controller $C$ for the model $G_{mod}$ such that:*

$$\left\| \begin{matrix} W_1(s)\ T_1(G_{mod}, C) \\ W_2(s)\ T_2(G_{mod}, C) \end{matrix} \right\|_\infty < 1 \qquad (2)$$

*where $W_1$, $W_2$ are two stable and inversely stable weights and $T_1$, $T_2$ are two closed-loop transfer functions of the to-be-designed loop $[C\ G_{mod}]$. Suppose that this control design has a solution. Then the central controller $C_c$ corresponding to this control design problem is given by: $C_c = HOM(\Theta, 0)$ where $\Theta$ is a $2\times 2$ stable and inversely stable transfer matrix whose inverse $\Pi(s) \triangleq \Theta^{-1}$ is obtained via the $J$-spectral factorization of the augmented plant $H(s)$ i.e.*

$$H^* \overbrace{\begin{pmatrix} I_2 & 0 \\ 0 & -1 \end{pmatrix}}^{\triangleq J_3} H = \Pi^* \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}^{\triangleq J_2} \Pi \qquad (3)$$

*where $X^*(s)$ denotes the adjoint of $X(s)$ i.e. $X^T(-s)$. The augmented plant $H(s)$ is here defined as the transfer matrix such that*

$$HOM(H,C) = \begin{pmatrix} W_1(s)\ T_1(G_{mod}, C) \\ W_2(s)\ T_2(G_{mod}, C) \end{pmatrix} \qquad (4)$$

If we add a classical condition [1] on the state-space realization of the augmented plant, then the central controller has the following property:

$$C_c(j\infty) = 0. \qquad (5)$$

In the sequel, the central controllers will always be assumed strictly proper.

## 3 Considered problem

As stated in the introduction, the objective in this paper is to analyze the influence of a (small) modification in the weights on the obtained central controller $C_c$ and, more importantly, on the obtained closed-loop transfer functions $T_1(G_{mod}, C_c)$ and $T_2(G_{mod}, C_c)$. In the sequel, we will restrict attention to one particular (and classical) two-block problem. This problem is the one for which $T_1 = C/(1+CG_{mod})$ and $T_2 = 1/(1+CG_{mod})$ in expression (2). For this particular two-block problem, we will analyze the following (classical) situation. We assume that, in a first step, a central controller $C_c$ has been designed using the criterion (2) with an appropriate constraint $W_2$ on the sensitivity function, but a very loose constraint $W_1$ on the other closed-loop transfer function. For instance, in this first step, the weight $W_1$ could have been chosen equal to a very small constant. However, this is of course not a requirement. In a second step, the weight $W_1$ is adapted and becomes $W_{1,bis} = W_1 + \Delta W_1$, a second $H_\infty$ design procedure is achieved with the adapted weights $W_{1,bis} = W_1 + \Delta W_1$ and $W_2$, and a new central controller $C_{c,bis} = C_c + \Delta C$ is obtained delivering modified closed-loop transfer functions $T_{1,bis} = (C_c+\Delta)/(1+(C_c+\Delta)G_{mod}) = T_1 + \Delta T_1$ and $T_{2,bis} = 1/(1+(C_c+\Delta C)G_{mod}) = T_2 + \Delta T_2$.

The final objective of our research is to fully understand the link existing between a change in the weights and the obtained closed-loop transfer functions. In this paper, we will nevertheless restrict attention to the following tasks: find a frequency domain approximation $\Delta C_{appr}$ for $\Delta C$, as a function of $\Delta W_1$ and of the variables involved in the initial two-block problem (i.e. the one with $W_1$ and $W_2$); and then, knowing $\Delta C_{appr}$, analyze the influence of this modification on the modified closed-loop transfer functions $T_{1,bis}$ and $T_{2,bis}$. For this purpose, we will need to assume that the change $\Delta W_1$ is small (so we may use first order approximations) and since we are interested in frequency domain expressions, we will use the frequency domain expression (3) as basis of our analysis. It is to be noted that all approximations given in this paper will therefore only be relevant in the frequency domain.

In effect, we will construct a mapping $\Delta W_1 \longrightarrow \Delta C_{appr} \longrightarrow \Delta T_1$ and $\Delta T_2$. The smallness assumption means that the mapping is linear, but we will show that in general it is not memoryless, i.e. $\Delta C_{appr}$ evaluated at a frequency $\omega_1$ does not just depend on $\Delta W_1(j\omega_1)$, but on $\Delta W_1(j\omega)$ for, in principle, $\omega \in [0\ \infty)$. In practice, $\Delta C_{appr}(j\omega_1)$ can nevertheless be expected to depend essentially on $\Delta W_1(j\omega)$ for $\omega$ confined to an interval around $\omega_1$.

## 4 Modification of the transfer matrix $\Theta$

In order to find a frequency domain approximation of $\Delta C$, we will first need to find an expression for the modification $\Delta\Theta$ in the transfer matrix $\Theta = \Pi^{-1}$ which defines the central controller as shown in Proposition 2.1. Beforehand let us introduce the expression of the augmented plant $H(s)$ for the particular two-block problem that we analyze in this paper (i.e. with $T_1 = C/(1+CG_{mod})$ and $T_2 = 1/(1+CG_{mod})$):

$$H(s) = \begin{pmatrix} \begin{pmatrix} W_1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ W_2 \end{pmatrix} \\ G_{mod} & 1 \end{pmatrix} \tag{6}$$

**Proposition 4.1** *Consider the two-block $H_\infty$ problem defined in (2) with $T_1 = C/(1+CG_{mod})$ and $T_2 = 1/(1+CG_{mod})$. Let $\Theta$ denote the matrix defining the central controller obtained using the weights $W_1$ and $W_2$ i.e. $C_c = HOM(\Theta, 0)$. Consider now the modified two-block problem with the weights $W_{1,bis} = W_1 + \Delta W_1$ and $W_2$. Then, if $\Delta W_1$ is small, the matrix $\Theta_{bis}$ defining the modified central controller $C_{c,bis} = HOM(\Theta_{bis}, 0)$ can be approximated by:*

$$\Theta_{bis} \approx \Theta - \Theta J_2 \Psi \tag{7}$$

*where $\Psi$ is a stable transfer matrix such that*

$$\Psi + \Psi^* = (W_1^* \Delta W_1 + W_1 \Delta W_1^*) \begin{pmatrix} \theta_{11}^* \theta_{11} & \theta_{11}^* \theta_{12} \\ \theta_{12}^* \theta_{11} & \theta_{12}^* \theta_{12} \end{pmatrix}, \tag{8}$$

*and such that*

$$HOM(\Theta - \Theta J_2 \Psi, 0) \text{ is strictly proper.} \tag{9}$$

*$\theta_{ij}$ are the entries of the "initial" transfer matrix $\Theta$.*

**Proof.** Denote by $H$ and $H_{bis}$ the augmented plants corresponding to the two-block problems with $W_1$ (and $W_2$) and with $W_1 + \Delta W_1$ (and $W_2$), respectively (see (6)). Then

$$H_{bis}^* J_3 H_{bis} \approx H^* J_3 H + \begin{pmatrix} W_1^* \Delta W_1 + W_1 \Delta W_1^* & 0 \\ 0 & 0 \end{pmatrix} \tag{10}$$

since $\Delta W_1$ is small. Let us now denote the difference between $\Pi_{bis}$ and $\Pi$ by $\Delta\Pi$ (i.e. $\Pi_{bis} = \Pi + \Delta\Pi$). We have then the following:

$$\Pi_{bis}^* J_2 \Pi_{bis} \approx \Pi^* J_2 \Pi + \Delta\Pi^* J_2 \Pi + \Pi^* J_2 \Delta\Pi \tag{11}$$

Since $\Pi$ and $\Pi_{bis}$ are the solutions of (3) for $H$ and $H_{bis}$, respectively, we have that $H^* J_3 H = \Pi^* J_2 \Pi$ and $H_{bis}^* J_3 H_{bis} = \Pi_{bis}^* J_2 \Pi_{bis}$. Consequently, from (10) and (11), we can deduce the following approximation of $\Delta\Pi$ as a function of $\Delta W_1$:

$$\begin{pmatrix} W_1^* \Delta W_1 + W_1 \Delta W_1^* & 0 \\ 0 & 0 \end{pmatrix} \approx \Delta\Pi^* J_2 \Pi + \Pi^* J_2 \Delta\Pi \tag{12}$$

Let us now pre-multiply (12) by $\Pi^{-*} = \Theta^*$ and post-multiply the same expression by $\Pi^{-1} = \Theta$:

$$\Theta^* \begin{pmatrix} W_1^* \Delta W_1 + W_1 \Delta W_1^* & 0 \\ 0 & 0 \end{pmatrix} \Theta \approx \Theta^* \Delta\Pi^* J_2 + J_2 \Delta\Pi\Theta \tag{13}$$

Since $\Pi$ and $\Pi_{bis} = \Pi + \Delta\Pi$ are both stable and inversely stable, the right hand side of (13) is the sum of a stable transfer matrix $J_2 \Delta\Pi\Theta$ and its complex conjugate $\Theta^* \Delta\Pi^* J_2$. Let us now decompose the left hand side of (13) as in (8). From (8) and (13), it is then obvious that: $J_2 \Delta\Pi\Theta \approx \Psi$. We can thus write successively the following: $\Pi_{bis} = \Pi + \Delta\Pi \approx (I_2 + J_2\Psi)\Pi$ and $\Theta_{bis} = \Pi_{bis}^{-1} \approx \Pi^{-1}(I_2 - J_2\Psi) = \Theta - \Theta J_2 \Psi$. Note that, in order to invert $\Pi_{bis}$, we have made use of the fact that the changes are small. ■

## 5 Modification in the central controller

Proposition 4.1 gives us a (frequency domain) approximation of the modified transfer matrix $\Theta_{bis}$. This approximation is a function of $\Delta W_1$ and the variables involved in the initial two-block problem (i.e. the one with $W_1$ and $W_2$). This result will now allow us to deduce an approximate expression for the modified central controller $C_{c,bis} = C_c + \Delta C$ using the relation between $\Theta_{bis}$ and the central controller.

**Proposition 5.1** *Consider the same variables as in Proposition 4.1. Then the modified controller $C_{c,bis}$ delivered by the two-block $H_\infty$ problem with weights $W_{1,bis} = W_1 + \Delta W_1$ and $W_2$ can be approximated as follows:*

$$C_{c,bis} = C_c + \Delta C \approx C_c - \overbrace{\frac{det(\Theta)}{\theta_{22}^2} \psi_{12}}^{\Delta C_{appr}} \tag{14}$$

*where $det(A)$ denotes the determinant of the matrix $A$ and $\psi_{ij}$ are the entries of $\Psi$.*

**Proof.** From Proposition 2.1, we have that: $C_{c,bis} = HOM(\Theta_{bis}, 0) = \theta_{12,bis}/\theta_{22,bis}$, where $\theta_{ij,bis}$ are the entries of $\Theta_{bis}$. Using (7), it is easy to find (approximate) expression for $\theta_{12,bis}$ and $\theta_{22,bis}$ i.e.

$$\theta_{12,bis} \approx \theta_{12} - \overbrace{\begin{pmatrix} \theta_{11} & \theta_{12} \end{pmatrix}}^{=\alpha^T} \overbrace{\begin{pmatrix} \psi_{12} \\ -\psi_{22} \end{pmatrix}}^{=\beta} \tag{15}$$

$$\theta_{22,bis} \approx \theta_{22} - \overbrace{\begin{pmatrix} \theta_{21} & \theta_{22} \end{pmatrix}}^{=\gamma^T} \begin{pmatrix} \psi_{12} \\ -\psi_{22} \end{pmatrix} \tag{16}$$

Using (15) and (16), the expression of the modified controller can be approximated as follows: $C_{c,bis} \approx (\theta_{12} - \alpha^T\beta)/(\theta_{22} - \gamma^T\beta)$ and using the first order Taylor expansion around $\beta = 0$ of the ratio between $\theta_{12} - \alpha^T\beta$ and $\theta_{22} - \gamma^T\beta$, we finally obtain (14). ■

**Remark.** From the approximation of the controller $C_{c,bis}$ given in (14), we can deduce that the transfer function $\psi_{12}$ must be strictly proper i.e.

$$\psi_{12}(j\infty) = 0 \tag{17}$$

in order that $\Delta C_{appr}(j\infty) = 0$ which is in accordance with the fact that $C_c(j\infty) = C_{c,bis}(j\infty) = 0$. This property is also in accordance with the definition and properties of the matrix $\Psi$ given in (8)-(9). Since (8) only defines $\Psi$ to within a skew matrix, $\psi_{12}(j\infty)$ can be adjusted

Proposition 5.1 delivers an approximation $\Delta C_{appr}$ of the modification in the central controller due to the deviation $\Delta W_1$. This approximation is a function of the matrix $\Theta$ of the initial two-block problem and of the second entry $\psi_{12}$ of the matrix $\Psi$ defined implicitly in (8). In order to be able to compute $\Delta C_{appr}$, we show in the sequel how to compute $\psi_{12}$ (explicitly) and moreover we analyze the relation between this transfer function $\psi_{12}$ and the deviation $\Delta W_1$.

Let us deduce the following from (8): $\psi_{12} + \psi_{21}^* = (W_1^* \Delta W_1 + W_1 \Delta W_1^*)\theta_{11}^*\theta_{12} \triangleq f(s)$ where $f(s)$ is a known transfer function since it is a function of some entries of the matrix $\Theta$, the weight $W_1$ and the change of weight $\Delta W_1$. Notice also that $f$ is equal to 0 when $\Delta W_1 = 0$. An expression for $\psi_{12}$ can now be computed by partial fraction decomposition of $f(s)$ into its unstable part $\psi_{21}^*$ and its stable part $\psi_{12}$.

**Important Comments.** The deviation $\Delta W_1$ will generally be a stable filter having a form similar to: $\Delta W_1 = (\pm K\ s)/((s+\omega_1)(s+\omega_2))$. The modification of the weight due to $\Delta W_1$ is therefore mainly restricted to the frequency band $B_{\Delta W_1} = [\ \omega_1\ \omega_2\ ]$. Indeed, outside this band $B_{\Delta W_1}$, the amplitude of $\Delta W_1$ converges to zero ( i.e. $|\Delta W_1(j\omega)| \to 0$ when $\omega \to 0$ and when $\omega \to \infty$). The representation of such a filter $\Delta W_1$ can be found in Figure 4 (in Section 7). Due to the decomposition into partial fractions, the transfer function $\psi_{12}$ will generally have a different behaviour than $\Delta W_1$. The amplitude of $\psi_{12}$ will indeed only converge to 0 in high frequencies (see (17)). In low frequencies, this amplitude will generally converge to a non-zero constant and, in the band $B_{\Delta W_1} = [\ \omega_1\ \omega_2\ ]$, $\psi_{12}(j\omega)$ will have a non-constant dynamic behaviour. An example of such a transfer function $\psi_{12}$ can be found in Figure 4. Consequently, the region where $\psi_{12}(j\omega)$ will have a significant influence on $\Delta C_{appr}$ (i.e. $B_{\psi_{12}} = [\ 0, \omega_2\ ]$ is much more important than the region $B_{\Delta W_1}$ for $\Delta W_1$. This is in fact a well-known property of the decomposition into partial fractions.

The approximation $\Delta C_{appr}$ (see (14)) can be computed *a-priori*, that is without having first to solve the second two-block $H_\infty$ problem. Indeed, $\Delta C_{appr}$ is only a function of the transfer matrix $\Theta$ (involved in the initial two-block $H_\infty$ problem) and of the transfer function $\psi_{12}$ which can be computed by partial fraction decomposition of $f(s)$.

Due to (14) and the properties of $\psi_{12}$ with respect to $\Delta W_1$ presented in the previous subsection, we can also state that the modification of the central controller due to $\Delta W_1$ will not only be restricted to the band $B_{\Delta W_1}$, but will also persist in low frequencies. In high frequencies, the amplitude $\Delta C_{appr}$ converges to 0 because of (17).

## 6 Consequences for the closed-loop transfer functions

The results presented in the previous section allow one to compute very easily an approximation of the modified central controller $C_{c,bis} = C_c + \Delta C$ due to the modification $\Delta W_1$, and this without having to perform the $H_\infty$ control design problem with the weights $W_1 + \Delta W_1$ and $W_2$ which would have given $C_{c,bis}$ as solution. From the approximation $\Delta C_{appr}$ given in Proposition 5.1, it is then easy to compute the effects of the change in the weights on the obtained closed-loop transfer functions $T_{1,bis}$ and $T_{2,bis}$

Although the influence of a change of the weights will vary depending on the considered model, the (modified) weights, etc. We can nevertheless deduce some general comments which will will be based on the first order Taylor expansion around $\Delta C_{appr} = 0$:

$$\begin{aligned} T_{1,bis} &\approx \frac{C_c + \Delta C_{appr}}{1 + C_c G_{mod}} \\ T_{2,bis} &\approx \frac{1}{1 + C_c G_{mod}} \left(1 - \frac{G_{mod}}{1 + C_c G_{mod}} \Delta C_{appr}\right); \end{aligned} \tag{18}$$

and on the following classical behaviours of these closed-loop transfer functions in low frequencies (LF) and in high frequencies (HF):

$$\begin{aligned} \frac{C(j\omega)}{1 + G_{mod}(j\omega)C(j\omega)} &\approx \frac{1}{G_{mod}(j\omega)} \text{ (LF) and } C(j\omega) \text{ (HF)} \\ \frac{1}{1 + G_{mod}(j\omega)C(j\omega)} &\approx \frac{1}{G_{mod}(j\omega)C(j\omega)} \text{ (LF) and 1 (HF)} \end{aligned} \tag{19}$$

**Comments on $T_{1,bis}$.** In low frequencies, according to (19), the difference between $T_1$ and $T_{1,bis}$ will be negligible and this even though $\Delta C_{appr}(j\omega) \neq 0$ at those frequencies. In high frequencies, according to (19), $|T_{1,bis}(j\omega)| < |T_1(j\omega)|$ if $|C_{c,bis}(j\omega)| < |C_c(j\omega)|$, and vice versa. Moreover, since both controllers are strictly proper, $T_1(j\omega)$ and $T_{1,bis}(j\omega)$ generally converge to zero when $\omega \to \infty$. According to (18), we will have that $|T_{1,bis}(j\omega)| < |T_1(j\omega)|$ if $|C_{c,bis}(j\omega)| < |C_c(j\omega)|$ (and vice versa) around the peak of $T_1$.

**Comments on $T_{2,bis}$.** Since $\Delta C_{appr}(j\omega) \neq 0$ in low frequencies, we will have a difference between $T_2$ and $T_{2,bis}$ at those frequencies. According to (19), this difference will be such that $|T_{2,bis}(j\omega)| < |T_2(j\omega)|$ if $|C_{c,bis}(j\omega)| > |C_c(j\omega)|$, and vice versa. In high frequencies, according to (19), $T_{2,bis}(j\omega) \approx T_2(j\omega) \approx 1$. A general comment about the difference between $T_{2,bis}$ and $T_2$ around the resonance peak is not obvious.

## 7 Illustration

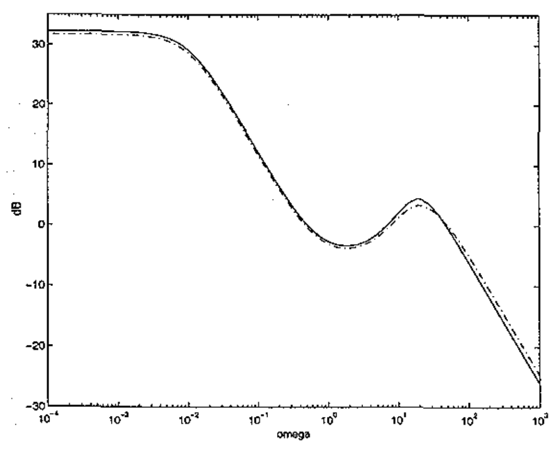


**Figure 1:** $|C_c(j\omega)|$ (solid) and $|C_{c,bis}(j\omega)|$ (dashdot)

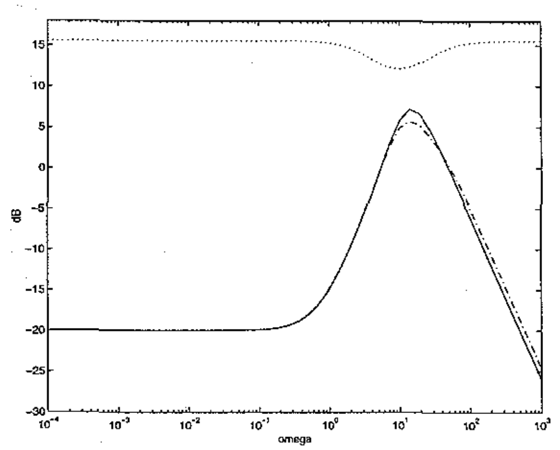


**Figure 2:** $|T_1(j\omega)|$ *(solid) and* $|T_{1,bis}(j\omega)|$ *(dashdot) and* $|W_{1,bis}(j\omega)^{-1}|$ (dotted)

In this section, we will illustrate the results presented in this paper. We will consider the following system: $G_{mod} = 10/((s-1)(0.2s+1))$. In the first $H_\infty$ control design problem, we will as usual only constrain the sensitivity function and choose the constraint $W_1$ on the other transfer function as a small constant. The chosen weights are: $W_1(s) = 1/6$ and $W_2(s) = (0.1s+1)/(0.003(100s+1))$. The $H_\infty$ problem is solved with these elements and we obtain the following central controller: $C_c = (50.2564(s+0.69)(s+5))/((s+0.01)(s^2+30.85s+423.3))$. Figures 2 and 3 represent the amplitude of the closed-loop transfer functions $T_1$ and $T_2$ achieved by this controller $C_c$ with the system $G_{mod}$. Now in a second step, we want to decrease the resonance peak of $T_1(G_{mod}, C_c)$. So, we choose the following new weight $W_{1,bis}$: $W_{1,bis} = W_1 + \Delta W_1 = W_1 + ((1.8967s)/((s+17.78)(s+5.623)))$ that can be considered as a small deviation of $W_1$.

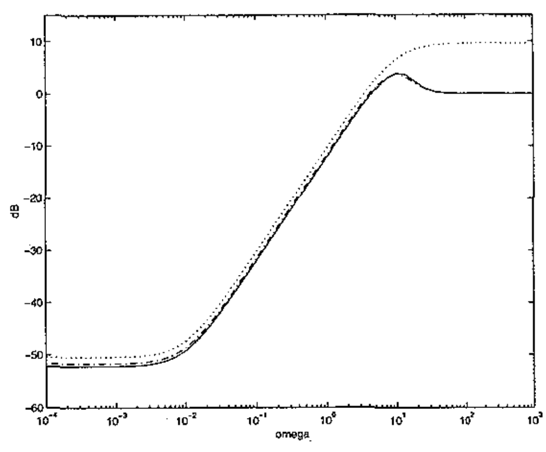


**Figure 3:** $|T_2(j\omega)|$ *(solid) and* $|T_{2,bis}(j\omega)|$ *(dashdot) and* $|W_2(j\omega)^{-1}|$ (dotted)

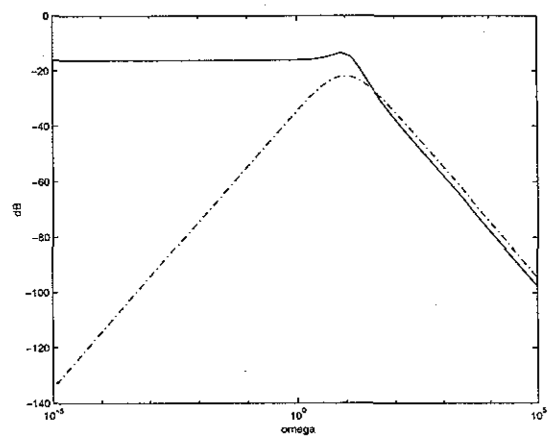


**Figure 4:** $|\psi_{12}(j\omega)|$ (solid) and $|\Delta W_1(j\omega)|$ (dashdot) at each frequency

The $H_\infty$ problem is solved with this new weight and we obtain the following central controller: $C_{c,bis} = (58.9691(s+17.78)(s+5.623)(s+5)(s+0.6782))/((s+27.12)(s+5.628)(s+0.01)(s^2+32.49s+339.1))$. The controller $C_{c,bis}$ is represented in Figure 1 and the new closed-loop transfer functions $T_{1,bis}$ and $T_{2,bis}$ in Figure 2 and 3, respectively. In these three last figures, the modified transfer functions are compared to the corresponding transfer function in the initial two-block problem. Now, we will show that the results presented in this paper would have allowed us to predict the modification caused by $\Delta W_1$ without having to perform the second $H_\infty$ design problem. For this purpose, let us first compute the transfer function $\psi_{12}$ that is necessary to approximate the change in the central controller according to (14). This function can be computed using the procedure presented in Section 5 and is represented in Figure 4. In this figure, we notice that $\psi_{12}$ converges to a non-zero constant in low frequencies as opposed to

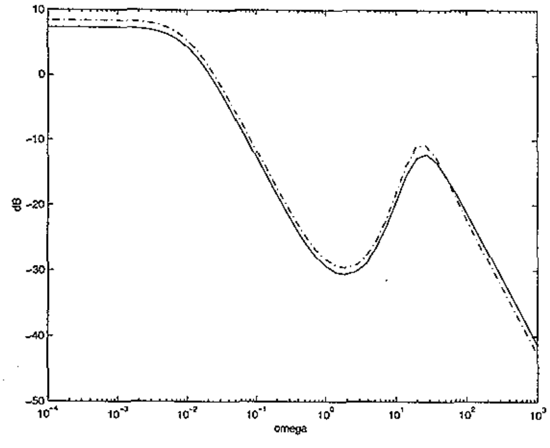


**Figure 5:** $|C_{c,bis}(j\omega) - C_c(j\omega)|$ (solid) and $|\Delta C_{appr}(j\omega)|$ (dashdot) at each frequency

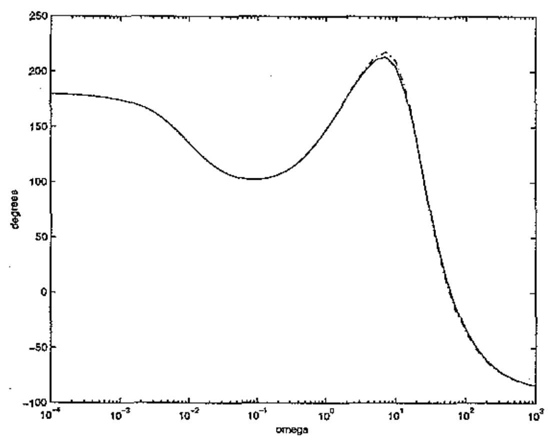


**Figure 6:** $arg(C_{c,bis}(j\omega) - C_c(j\omega))$ (solid) and $arg(\Delta C_{appr}(j\omega))$ (dashdot) at each frequency

$\Delta W_1$ (which converges to 0). This is in accordance with our comments at the end of Section 5. From this function $\psi_{12}$ (and the matrix $\Theta$ of the first two-block problem), we can now compute $\Delta C_{appr}$. This last quantity is compared with the actual difference between $C_c$ and $C_{c,bis}$ in Figures 5 and 6. We observe that our $\Delta C_{appr}$ is a very good approximation of the actual difference between the two successive controllers. Moreover, we also observe that the change in the controller due to $\Delta W_1$ is not only located in the band where $\Delta W_1$ has a significant amplitude, but also persists at low frequencies. The change $\Delta C_{appr}$ converges to 0 when $\omega \rightarrow \infty$ as $\Delta W_1$ does. In Figure 2 and 3, we observe behaviours that are also in accordance with our comments of Section 6. Indeed, at low frequencies, we have $|T_{1,bis}(j\omega)| \approx |T_1(j\omega)|$ and $|T_{2,bis}(j\omega)| > |T_2(j\omega)|$ since $|C_{c,bis}(j\omega)| < |C_c(j\omega)|$. At high frequencies, we have $T_{2,bis}(j\omega) \approx T_2(j\omega) \approx 1$ and $|T_{1,bis}(j\omega)| > |T_1(j\omega)|$ since $|C_{c,bis}(j\omega)| > |C_c(j\omega)|$ at those high frequencies, while both converging to 0. Moreover, we see that the resonance peak of $T_{1,bis}$ has a smaller amplitude than the one of $T_1$. This could have been predicted from the fact that $|C_{c,bis}(j\omega)| < |C_c(j\omega)|$ around the frequency of this peak.

## 8 Conclusions

In this paper, we have analyzed the influence of a small weight modification in a classical two-block $H_\infty$ control design problem on the obtained central controller and on the obtained closed-loop transfer functions. The first contribution has been to give a frequency domain approximation of the change in the central controller induced by the (small) weight modification after a first $H_\infty$ control design step. This approximation is a function of the weight modification frequency response and of the variables involved in the initial control design problem, and is therefore computable before performing the new control design step (i.e. the one with the modified weights). We show also that the modification in the central controller persists outside the frequency band where the weight modification is mainly located. A last contribution is to analyze the influence of this modification of the central controller on the modified closed-loop transfer functions.

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