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Constant disturbance suppression for nonlinear systems design using singular perturbation theory

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Abstract

A relatively practical method of suppressing the effect of constant disturbances on nonlinear systems is presented. By adding an integrator to a stabilising controller, it is possible to achieve both constant disturbance rejection and zero tracking error. Sufficient conditions for the rejection of a constant input disturbance are given. We give both local and global conditions such that the inclusion of an integrator in the closed loop maintains closed loop stability. The analysis is based on singular perturbation theory. Furthermore, we extend these methods to deal with Multiple-input Multiple-output nonlinear systems. Finally, we implement our method in the control of a simulated helicopter model. The simulation results show that this method achieves satisfactory performance.

1 Introduction

An important objective of control system design is to minimise the effects of external disturbances. For linear systems, the classical method of rejecting a constant disturbance is to include an integrator in the controller. This paper extends this idea to nonlinear systems, using singular perturbation methods to guarantee stability. Paper [5] demonstrates that for disturbance suppression, an output feedback controller must contain an integrator. In this paper we ask whether we can directly add an integrator to an already existing controller to achieve constant disturbance rejection, while still retaining the stability of the system. Often that would be both a simpler and more practical way to deal with the nonlinear constant disturbance suppression problem. This paper not only gives the affirmative answer but also suggests several locations where an integrator with low gain away from DC ($\frac{\epsilon}{s}$ for short) maybe included, in order to deal with the constant input disturbance rejection problem. Furthermore, this method can also be applied to cope with the constant reference tracking problem, even for nonlinear MIMO systems, such as the helicopter system of [1].

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In the next section, we give a description of the problem. In Section 3, we present a proof that an exponentially stabilising nonlinear controller appropriately augmented with a small integrator (a linear transfer function $\frac{\epsilon}{s}$) can yield constant disturbance suppression. In Section 4, we will give both local and global conditions for the existence of a gain of the integrator that is sufficiently small to guarantee stability. Section 5, by using singular perturbation methods, gives a upper bound on a value of the gain that guarantees closed loop stability. Section 6 extends our method to deal with the constant disturbance rejection problem and constant reference tracking problem for Multiple-input Multiple-output(MIMO) systems. Finally in Section 7, we present simulation results obtained by implementing constant disturbance rejection and zero steady state tracking error control for a helicopter model by using this method.

2 Problem Description

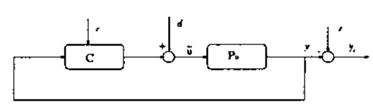

Figure 1: A nonlinear disturbance suppression problem

Firstly, let us consider a nonlinear input disturbance rejection problem as shown in Figure 1. This depicts a nonlinear single-input single output (SISO) system (We will extend our methods to MIMO systems later). It consists of the interconnection of a nonlinear plant P_0 and controller C, forced by a constant command signal r, as well as a constant input disturbance d. Here, y_r is the reference tracking error, and $\tilde{u}$ is the input to the plant. What we are concerned with here is how to design a controller C which possesses the ability to both reject a constant input disturbance d, and to give zero steady state tracking error for a constant reference input r. More precisely, we consider the question of how we might modify a pre-existing controller C_0 not achieving these properties, so that the properties are secured throughout the modification (See Figure 2). In the case of a linear plant, the classical method employed to reject a constant disturbance is to include

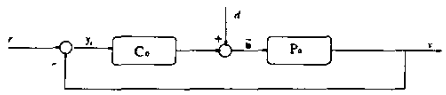

Figure 2: A nonlinear system with an existing stabilising controller C_0

an integrator in the controller. Here, we extend this idea to deal with the nonlinear constant disturbance rejection problem. Consider Figure 3. Suppose that we have already designed a controller C_0 which stabilises the plant P_0 (Later, we shall be precise concerning the type of stability). We then augment the closed loop with the addition of a small gain integrator. The original controller C_0 and small gain integrator $\frac{\epsilon}{s}$ in Case 1 of Figure 3 represents a solution to the problem of designing C in Figure 1. Then, the interconnection is equivalent to a single stable plant P as shown in Figure 3. By stating that the two cases in Figure 3 are equivalent, we mean that if the exogenous input signals d and r in the two cases are equal, then all labelled signals (including the output signals) will also be equal(given suitable matching of initial conditions, or after clear of initial condition effects). Hence, we can focus our attention on the simplified second case. In the second

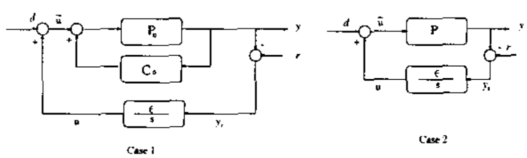

Figure 3: Two Equivalent Cases

case of Figure 3, we suppose that the state equation of the plant P is modelled as follows.

$$P: \left\{ \begin{array}{rcl} \dot{x} & = & f(x, \tilde{u}) \\ y & = & g(x, \tilde{u}). \end{array} \right. \quad (1)$$

If there is no particular declaration in this paper, we suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^l$ are unbiased in the sense that

$$\left\{ \begin{array}{rcl} f(0,0) & = & 0 \\ g(0,0) & = & 0. \end{array} \right. \quad (2)$$

The state equation for the small integrator is expressed as a transfer function block $\frac{\epsilon}{s}$:

$$\frac{\epsilon}{s}: \left\{ \begin{array}{rcl} \dot{\xi} & = & \epsilon y_r \\ u & = & \xi. \end{array} \right. \quad (3)$$

In the above, the reference tracking error y_r is equal to $y - r$. We suppose that the disturbance d and the reference input r are both constant. The following parts of this paper will focus on two key questions. The first question is whether a controller that is augmented with an integrator will reject the constant disturbance. The second question is how to ensure the stability of the closed loop. Another but nevertheless important question is whether constant reference trajectory following occurs, with zero steady state error.

3 Sufficient Conditions for Constant Disturbance Rejection

In [5], it was shown that for input disturbance suppression an output feedback $\mathcal{H}_\infty$ controller must contain an integrator in the controller. In this section, we will still start our discussion from the point of view of an $\mathcal{H}_\infty$ treatment. As in [5], we also extend the constant input disturbance rejection problem to a mixed sensitivity $\mathcal{H}_\infty$ problem (Figure 4). We introduce an integrator into one of the input weights(the disturbance weight), and choose cost variable $z = \tilde{u}$. The input $\hat{w}_1$ gives rise to the input disturbance d. The introduction of the input $\hat{w}_2$ can be interpreted as a way of capturing modelling uncertainty or as a reference input signal. Without an integrator weight function, the introduction of $\hat{w}_2$ is necessary for ensuring that the $\mathcal{H}_\infty$ problem is standard. Here, the input weighting function W_{d2} of $\hat{w}_2$ and the output weighting function W_z are both stable. In order to set up the relationships

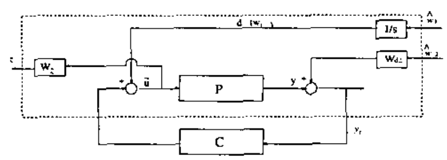

Figure 4: The mixed sensitivity $\mathcal{H}_\infty$ form

between input-output stability [6] and Lyapunov stability for this constant disturbance rejection problem, we present a theorem from [6]. We will later identify the controller C in Figure 4 with the small gain integrator $(\frac{\epsilon}{s})$.

Definition 1 *A system is globally exponentially stable (GES) iff there exists a Lyapunov function $U(x) \leq 0$ such that $\rho_1|x|^2 \leq U(x) \leq \rho_2|x|^2$ and with zero input $\frac{d}{dt}U(x(t)) \leq -\rho_3|x|^2$. where $\rho_i > 0$, $i = 1,2,3$ are suitable scalar constants. If these conditions hold, it follows that there exists some constant $\rho \geq 0$ such that with $x(0) = x_0$, $|x(t)| \leq \rho|x_0|e^{-\rho_3 t/2}$ for all $t \geq 0$.*

Definition 2 *Consider the nonlinear system of the form*

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x, u) \\ y & = & g(x, u). \end{array} \right. \quad (4)$$

The system (4) is said to be "$\mathcal{L}_p$-stable with finite gain" if there exist constants b_p and $\gamma_p < \infty$ such that $u \in \mathcal{L}_p^m \Longrightarrow y \in \mathcal{L}_p^l$ and $\|y\|_p \leq \gamma_p\|u\|_p + b_p$. If $p = 2$, γ_p is said to be the $\mathcal{L}_2$ bound from u to y.

The system (4) is said to be "$\mathcal{L}_p$-stable without bias" if there exists a constant $\gamma_p < \infty$ such that $x(0) = 0$, $u \in \mathcal{L}_p^m \Longrightarrow y \in \mathcal{L}_p^l$ and $\|y\|_p \leq \gamma_p \|u\|_p$.

The system (4) is "small signal $\mathcal{L}_p$-stable without bias" if there exist constants $r_p > 0$ and $\gamma_p < \infty$ such that $x(0) = 0$, $u \in \mathcal{L}_p^m$ with $\|u\|_p \leq r_p \Longrightarrow y \in \mathcal{L}_p^l$ and $\|y\|_p \leq \gamma_p \|u\|_p$.

Theorem 3 *Consider the system described by equation (4). Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^l$ are unbiased, which ensures that $x = 0$ is an equilibrium of the* **unforced** *system*

$$\dot{x} \;=\; f(x, 0). \tag{5}$$

Suppose that $x = 0$ is an exponentially stable equilibrium of (5), and that f is C^1. Suppose also that f and g are locally Lipschitz continuous at $(0,0)$, that is, suppose there exist finite constants k_f, k_g, r such that

$$\|f(x,u) - f(z,v)\|_2 \leq k_f[\|x - z\|_2 + \|u - v\|_2], \tag{6}$$

$$\|g(x,u) - g(z,v)\|_2 \leq k_g[\|x - z\|_2 + \|u - v\|_2], \tag{7}$$

for all $(x,u)(z,v) \in B_r$. Here, B_r is the open ball of the radius r, that is, $B_r = \{x \,:\, \|x - x_0\| < r\}$. Then the system (4) is small signal $\mathcal{L}_p$-stable without bias for each $p \in [1, \infty)$. If $x = 0$ is a globally exponentially stable equilibrium, and (6) and (7) hold with B_r replaced by $\mathbb{R}^{(m+n)}$, then the system (4) is $\mathcal{L}_p$-stable without bias for each $p \in [1, \infty)$. Furthermore, there exists a Lyapunov function $U(x) \geq 0$ which satisfies the requirements of exponential stability of Definition 1, and the gain γ_p is related to the constants ρ_i defining the properties of $U(x)$ by

$$\|y\|_p \leq k_g[(\rho_3 k_f / 4{\rho_1}^2 {\rho_2}^2) + 1]\|u\|_p.$$

Proof: See [6]. Other proofs are omitted to save space. ■

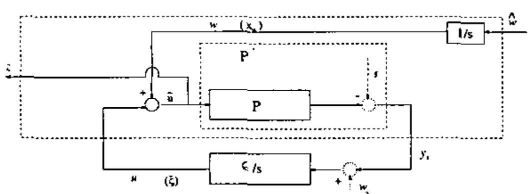

Figure 5: The simplified mixed sensitivity $\mathcal{H}_\infty$ form

Now, let us consider the mixed sensitivity $\mathcal{H}_\infty$ problem depicted by Figure 4. One design goal of $\mathcal{H}_\infty$ methods is to ensure that a finite $\mathcal{L}_2$ gain $\gamma_{\hat{w}z}$ exists from input $[\hat{w}_1 \; \hat{w}_2]^T$ to output z, in other words to ensure that the system is "$\mathcal{L}_2$-stable with finite gain" (see Definition 2.). In this section, in order to emphasise the problem of constant input disturbance rejection as opposed to reference tracking and to simplify our discussion, we will not consider the input $\hat{w}_2$, that is, we set $\hat{w}_2 = 0$. We also assume that the weight function W_z is unity. Because the weighting functions W_{d2} and W_z are both stable, we can use Theorem 3 to see that these simplifications will not influence the existence of $\gamma_{\hat{w}z}$ and our further discussion. We set the controller C in Figure 4 to be $\frac{\epsilon}{s}$. The system is then as depicted in Figure 5.

Theorem 4 *Consider the system depicted in Figure 5. The plant P and $\frac{\epsilon}{s}$ blocks are respectively described by equations (1),(2) and (3). Suppose that $(0,0)$ is an exponentially stable equilibrium of the* **unforced closed loop** $(P, \frac{\epsilon}{s})$. *Further, assume that f is C^1, and that f, g are locally Lipschitz continuous at $(0,0)$ with Lipschitz constants k_f and k_g to the Euclidean norm $\|.\|_2$(See Definition 1 and Theorem 3.).*

Then the system depicted in Figure 5 is small signal $\mathcal{L}_2$ stable without bias from $\hat{w}$ to z.

If $(0,0)$ is a globally exponentially stable equilibrium, and f, g are globally Lipschitz continuous at $(0,0)$, then the system is $\mathcal{L}_2$ stable without bias.

Note: Consider the plant P' in Figure 5. If we have a nonzero constant reference input r, we can consider the original plant P and reference input r to be equivalent to a new plant P' with an equilibrium point (x_e, ξ_e), where $g(x_e, \xi_e) = r$. Sufficient conditions for stability in this situation are that the conditions of Theorem 4 are satisfied for the new equilibrium point.

4 Guaranteeing Stability with Integrator Augmentation

We have established that a controller augmented with an integrator will reject a constant input disturbance provided that the combination is stabilisation. We are now concerned with the problem of how to design such a controller so as to ensure the stability of the closed loop $(P, \frac{\epsilon}{s})$. In this section, using singular perturbation theory, we will investigate both local and global conditions for the existence of a small scalar ϵ^* such that when $0 < \epsilon < \epsilon^*$ the closed loop $(P, \frac{\epsilon}{s})$ is stable.

Consider the set up of Figure 3 described by equations (1) and (3). If we set the constant input signal r and d to zero in order to analyse the Lyapunov stability of the unforced closed loop $(P, \frac{\epsilon}{s})$, then the state equation for the closed loop $(P, \frac{\epsilon}{s})$ can be expressed as:

$$(P, \frac{\epsilon}{s}) : \begin{cases} \dot{x} &= f(x, \xi) \\ \dot{\xi} &= \epsilon g(x, \xi). \end{cases} \tag{8}$$

In order to use the singular perturbation method, we first transform equation (8) to its standard singular perturbation form [2]. Let $\tau = \epsilon(t - t_0)$, so that $\tau = 0$ at $t = t_0$. That leads to $\frac{d\tau}{dt} = \epsilon$. Then, we have

$$\begin{cases} \epsilon\frac{d}{d\tau}x &= f(x,\xi) \\ \frac{d}{d\tau}\xi &= g(x,\xi). \end{cases} \tag{9}$$

It should be noticed that x is a vector; on the other hand, with a SISO problem, ξ is a scalar. In order to be consistent with standard singular perturbation notation, we will for the moment use the notation $\dot{x}$ to denote the derivative on the *slow* time scale τ when we analyse singular perturbation models.

Theorem 5 *(Global conditions for the existence of ϵ^*) Consider the second case depicted in Figure 3 described by equation (9) which satisfies the requirement of equation (2), and suppose that the following assumptions are satisfied:*

(i) The equation $0 = f(x,\xi)$ obtained by setting $\epsilon = 0$ in equation (9) implicitly defines a unique C^2 function $x = h(\xi)$.

(ii) For fixed $\xi \in R$, the equilibrium $x_e = h(\xi)$ of the subsystem $\dot{x} = f(x,\xi)$ is Globally Asymptotically Stable (GAS) [3] and Locally Exponentially Stable (LES).

(iii) The equilibrium $\xi = 0$ of the reduced model (slow time scale) $\dot{\xi} = g(h(\xi),\xi)$ is GAS and LES (See Definition 1). A sufficient condition is that $g(h(\xi),\xi)\xi < 0$ (when $\xi \neq 0$) and $g(h(\xi),\xi)\xi \leq -\rho|\xi|^2$ for ξ in a neighbourhood of $\xi = 0$.

Then there exists $\epsilon^* > 0$, such that for all $0 \leq \epsilon \leq \epsilon^*$, the equilibrium $(x,\xi) = (0,0)$ is GAS. Furthermore if the conditions in (ii) and (iii) involve GES instead of GAS, then the equilibrium $(x,\xi) = (0,0)$ is GES.

Note: If we consider the more general case that the equilibrium point ξ is not zero but fixed at $\xi = \xi_e$ by the influence of a constant reference input r, we require a slight adjustment to Condition (iii). In particular, we require that the equilibrium ξ_e of the reduced model (slow time scale) is GAS and LES (i.e. we should have that $[g(h(\xi)) - g(h(\xi_e))](\xi - \xi_e) \leq 0$, and $[g(h(\xi)) - g(h(\xi_e))](\xi - \xi_e) \leq \rho|\xi - \xi_e|^2$ is valid for ξ in a neighbourhood of $\xi = \xi_e$.). This will be satisfied for all ξ_e if $\frac{\partial g(h(\xi),\xi)}{\partial \xi} < -\rho < 0$, that is, if the "incremental DC gain" of the nonlinear plant is uniformly bounded away from zero.

Theorem 6 *(Local conditions for the existence of ϵ^*) Consider the second case in Figure 3 described by equation (9), and suppose that $f(.,.)$ and $g(.,.)$ are continuously differentiable with $f(0,0) = 0$ and $g(0,0) = 0$. Define:*

$$A_{11} = \frac{\partial g}{\partial \xi}|_{(x,\xi)=(0,0)},\ A_{12} = \frac{\partial g}{\partial x}|_{(x,\xi)=(0,0)},$$

$$A_{21} = \frac{\partial f}{\partial \xi}|_{(x,\xi)=(0,0)},\ A_{22} = \frac{\partial f}{\partial x}|_{(x,\xi)=(0,0)},$$

and suppose that A_{22} is nonsingular. Suppose further that the equation $0 = f(x,\xi)$ obtained by setting $\epsilon = 0$ in equation (9) has a unique C^2 solution $x = h(\xi)$, and that $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ is nonzero. Then if all eigenvalues of A_{22} and of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ have negative real parts, there exists $\epsilon^* > 0$, such that for all $0 \leq \epsilon \leq \epsilon^*$, the equilibrium $(x_e = 0, \xi_e = 0)$ is an asymptotically stable equilibrium point.

5 An Integrator Gain Bound

In the last section, we gave sufficient conditions for the existence of a bound on the integrator gain that will guarantee closed loop stability. Here, we will give an explicit expression for such an ϵ^*, based on singular perturbation theory.

Theorem 7 *(An integrator gain bound) Consider the second case in Figure 3 described by equation (9) which satisfies the requirement of equation (2), and suppose that the following conditions are satisfied:*

(i) There exists a function h such that $x = h(\xi)$ is the unique root of $0 = f(x,\xi)$ in $(x,\xi) \in B_x \times B_\xi$ (Here, B_x and B_ξ are some open balls on x and ξ space respectively).

(ii) There exists a Lyapunov function $W(x,\xi)$ such that for all $(x,\xi) \in B_x \times B_\xi$:

a. $W(x,\xi) > 0$ for all $x \neq h(\xi)$ and $W(h(\xi),\xi) = 0$.

b. There exists some $\alpha_2 > 0$, such that $\frac{\partial W}{\partial x}f(x,\xi) \leq -\alpha_2[\phi(x - h(\xi))]^2$.

c. There exists some γ and β_2 such that $\frac{\partial W}{\partial \xi}g(x,\xi) \leq \gamma[\phi(x - h(\xi))]^2 + \beta_2\psi(\xi)\phi(x - h(\xi))$.

In the above, $\psi(.)$ and $\phi(.)$ are scalar functions of vector arguments which vanish only when their arguments are zero, e.g. $\psi(\xi) = 0$ iff $\xi = 0$.

(iii) There exists a Lyapunov function $V(\xi)$ such that:

d. $\frac{\partial V}{\partial \xi}g(h(\xi),\xi) \leq -\alpha_1\psi^2(\xi)$, for some $\alpha_1 > 0$.

e. There exist some β_1 such that $\frac{\partial V}{\partial \xi}[g(x,\xi) - g(h(\xi),\xi)] \leq \beta_1\psi(\xi)\phi(x - h(\xi))$.

Then, when $0 < \epsilon < \epsilon^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \beta_1\beta_2}$, there exists a Lyapunov function for the closed loop system $(P, \frac{\epsilon}{s})$ of the form:

$W_\gamma(x,\xi) = (1-d)V(\xi) + dW(x,\xi)$, where d is allowed to be any fixed value in the range $(0,1)$.

Furthermore, the origin is an asymptotically stable equilibrium of $(P, \frac{\epsilon}{s})$.

6 MIMO Systems

So far in the development, we have concentrated our attention on SISO systems. In this section, we will extend our results to MIMO systems as well. In the following, we will give some sufficient conditions to guarantee local stability for a MIMO system augmented with low gain integrators. As for the SISO analysis, we neglect the constant input signals r and d when we analyse the stability.

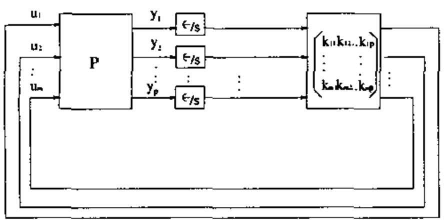

Figure 6: MIMO system augmented with small gain integrators

Consider Figure 6. Let P be a MIMO system with m inputs and p outputs (here, $m \geq p$) described by equation (4). We assume that $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^p$ are unbiased. The state equation of the augmented system can be described as below.

$$\begin{cases} \dot{x} &= f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) \\ \dot{\xi} &= \epsilon g(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i). \end{cases} \quad (10)$$

Again, we change equation (10) to its standard singular perturbation form.

$$\begin{cases} \epsilon\dot{x} &= f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) \\ \dot{\xi} &= g(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i). \end{cases} \quad (11)$$

Where the dot means the derivative with respect to τ.

Lemma 8 *Define* A_i*, the* $i \times i$ *(upper left) sub-matrix of* A *as*

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} \end{bmatrix}. \quad (12)$$

For any nonsingular matrix $A_0 \in \mathbb{R}^{n\times n}$*, it is possible to reorder the columns of* A_0 *to ensure the reordered matrix* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *satisfies the property that for each* $i = 1, 2, ...n$, $det(A_i) \neq 0$.

Lemma 9 *If all the roots of the equation* $s^n + a_1 s^{n-1} + ... + a_0 = 0$ *have negative real parts, then there exists an* ϵ^* *such that when* $0 < \epsilon < \epsilon^*$ *all the roots of the equation* $s^{n+1} + a_1 s^n + ... + a_0 s + \epsilon = 0$ *have negative real parts.*

Theorem 10 *For a nonsingular square matrix* $A \in \mathbb{R}^{n\times n}$ *with* A_i *defined as in (12) and* $det(A_i) \neq 0$ *for each* $i = 1, 2, ...n$ *there exists a diagonal matrix* $K \in \mathbb{R}^{n\times n}$ *such that all the eigenvalues of the matrix* AK *have negative real parts*

Theorem 11 *Consider the system described by equations (4) and (10) and illustrated in Figure 6. Assume that* $x = 0$ *is an asymptotically stable equilibrium for the plant* P*, and that* $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ *are continuously differentiable with* $f(0,0) = 0$ *and* $g(0,0) = 0$*. We assume that*

(i) The equation

$$f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) = 0$$

obtained by setting $\epsilon = 0$ *in equation (11) has a unique* C^2 *solution* $x = h(\xi)$,

(ii) The matrix $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ *is nonsingular.*

Then, there exists ϵ^* *and a constant matrix* $K = (k_{ij})_{m\times p}$ *(see Figure 6) such that* $(x = 0, \xi = 0)$ *is an asymptotically stable equilibrium whenever* $0 < \epsilon < \epsilon^*$*. Further more, when* $m = p$*, the constant square matrix* K *can be diagonal.*

7 Controller Design for a Nonlinear Helicopter Model

In this section, we implement our constant input disturbance rejection method on the simulated control of a helicopter model provided by [1]. In [1], an output tracking controller is designed based on approximate linearisation. However, this method does not completely suppress constant input disturbances. In order to deal with this shortcoming, we augmented an approximate linearised output tracking controller, by including an extra integrator block. The helicopter model appears as below in both [1] and [4].

$$\begin{bmatrix} \dot{P} \\ \dot{v}^p \\ \dot{R} \\ \dot{\omega}^b \\ \dot{T}_M \\ \dot{T}_T \\ \dot{a}_{1s} \\ \dot{b}_{1s} \end{bmatrix} = \begin{bmatrix} v^p \\ \frac{1}{m} R f^b \\ R\hat{\omega}^b \\ \mathcal{J}^{-1}(\tau^b - \omega^b \times \mathcal{J}\omega^b) \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}, \quad (13)$$

$$y = [p_x \; p_y \; p_z \; \phi \; \theta \; \psi]^T, \quad (14)$$

where $P \in \mathbb{R}^3$ and $v^p \in \mathbb{R}^3$ are the position and velocity vectors of centre of the mass in spatial coordinates, $R \in SO(3)$ is the rotation matrix of the body axes relative to the partial axes, $w^b = [w_x^b, w_y^b, w_z^b]^T \in \mathbb{R}^3$ is the body angular velocity vector, $m > 0$ is the body mass, $\mathcal{J} \in \mathbb{R}^{3\times3}$ is the inertial matrix and $f^b, \tau^b \in \mathbb{R}^3$ are the body force and torque. The body forces and torques generated by the main rotor are controlled by T_M,a_{1s} and b_{1s}, in which a_{1s} and b_{1s} are respectively the longitudinal and lateral tilt of the tip path plane of the main rotor with respect to the shaft. The tail rotor is considered as the source of pure lateral force and anti-torque, which are controlled by T_T. As in [1], we also assume that all the states are measurable. In order to present the helicopter system in an input-affine form, we define $w = [w_1\ w_2\ w_3\ w_4]^T$, which are the derivatives of T_M, T_T, a_{1s} and b_{1s} as auxiliary inputs to the system. Here the state $x \in R^{16}$, the inputs $w \in R^4$, the output $y \in R^6$.

The system equations of (13) and (14) have four control inputs so the maximum number of outputs for possibly applying an input-output linearisation procedure is four. We choose the outputs p_x, p_y, p_z, ψ as in [1]. Approximate linearisation is implemented by neglecting the coupling terms, a procedure which is presented very clearly in [1].

We define reference tracking error signals as $y_{r_{p_i}} = r_{p_i} - y_{p_i}$ and $y_{r_\psi} = r_\psi - y_\psi$. Here, $i = x, y, z$, while r_{p_i} and r_ψ are the reference inputs for position and yaw angle respectively. We augment the approximate input-output linearisation controller with the small gain integrators of the reference tracking error of position $y_{r_{p_x}}, y_{r_{p_y}}$ and $y_{r_{p_z}}$. According to Theorem 11, it is possible to retain the stability of the augmented system while acquiring constant input disturbance rejection and zero steady state tracking error.

Figures 7 illustrate some simulation results, where the mass of the helicopter has changed 10 percent from the nominal case. Although a 10 percent change in mass is, strictly speaking, a change in the plant rather than a constant input disturbance, it has a similar effect of altering the (constant) control input required to achieve equilibrium. We consider such a "disturbance" in order to make comparison with the results reported in [1]. The augmented controller is still able to track the reference input without steady state errors. In contrast, the approximate input-output linearisation controller without the integrator augmentation does not reject such disturbances.

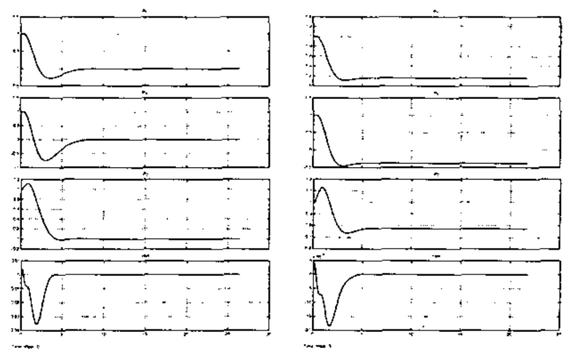

Figure 7: Left: The Output of Augmented System Under 10 percent Change of Mass. Right: The Output of System without the Augmentation Under 10 percent Change of Mass.

8 Conclusion

In this paper we have addressed the problem of achieving constant input disturbance rejection and constant reference tracking, for nonlinear systems. A relatively intuitive solution to this problem has been proposed: we simply augment an existing controller (which stabilises the nonlinear system) with (an) appropriately located integrator(s), with appropriately small gain. We can use singular perturbation theory to guarantee that, even with the addition of such an integrator, closed loop stability will be retained. Our simulation results on a nonlinear helicopter model indicate that satisfactory performance can be achieved in some circumstances, and that the proposed method is a simple but effective way to achieve the suppression of exogenous signals even for MIMO nonlinear systems.

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