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Additive, Multiplicative and Inverse Multiplicative Robust Stability: Connections with the ν-Gap Metric[1]

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Abstract

Existing Robust Stability Results based on the ν-gap metric can be extended by the introduction of various weighting functions. By appropriatly specialising these weighting functions it is possible to recover earlier robust stability results based on additive, multiplicative and inverse multiplicative uncertainty models. In this way it is demonstrated that some of these classical robustness conditions are a special case of the extended ν-gap Robust Stability tests.

1 Introduction

This work indicates that earlier "classical" robust stability results for additive and multiplicative perturbations, can be recovered by specialising stability robustness results derived using the so-called ν-gap or Vinnicombe distance between plants [8, 9].

Stability *robustness* analysis investigates the closed-loop stability of the interconnection of a particular controller and any member of an entire *set* of plants (as opposed to a particular plant). Earlier last century, stability robustness results focussed on perturbations in the nominal open loop gain or the phase lag that might be introduced by a time delay. These results led to design robustness specifications involving gain margin and phase margin [6]. Much later however, in the early 1980s, additive, multiplicative and inverse multiplicative perturbations of the nominal plant were investigated for their robustness properties [4, 5, 3, 6, 7, 10].

Recently, another type of stability robustness result, one that exploits the so-called ν-gap or Vinnicombe metric distance between plants, has become available [8, 9]. The main contribution of this extended abstract is to advertise results indicating that although these results superficially *appear* to be quite different, in fact, they are not. By incorporating certain limiting weights in the ν-gap robustness results, the earlier results of [3, 4, 5, 6, 7, 10] can be obtained.

Section 2 reviews the various aforementioned approaches to robustness. Sections 3 and 4 show how these results are related. Note that this extended abstract only presents the main results and full proofs are available in [1].

2 Existing Robustness Results

2.1 Additive and Multiplicative Perturbations

The stability robustness of a stable closed-loop comprising a plant and controller can be investigated by analysing whether stability is retained when the plant is perturbed. (It is also possible to investigate controller perturbation.) Many such results are obtained by investigating particular classes of plant variation or uncertainty. Such perturbation classes are typically given names such as additive uncertainty, multiplicative input or output uncertainty or inverse multiplicative output uncertainty: see [3, 4, 5, 6, 7, 10].

More precisely, with a nominal plant P, and W_1, W_2 two given weighting transfer functions (both of which are both stable and stably invertible), one can consider perturbed plant sets of the type $P + W_1\Delta W_{21}$, $P(I + W_1\Delta W_2)$, $(I + W_1\Delta W_2)P$, $P(I + W_1\Delta W_2)^{-1}$ and so on, where the perturbation parameter $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\| < 1$. (Here, as usual, $\mathcal{RH}_\infty$ denotes the set of stable, proper, rational transfer functions.)

Sufficient conditions for robustness are given which involve the weights, W_1 and W_2, and one of the input or output sensitivities (S_i and S_o), or com-

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Perturbed Model Set	Stability Test
$P + W_1\Delta W_2$	$\|\|W_2 C S_o W_1\|\|_\infty \leq 1$
$P(I + W_1\Delta W_2)$	$\|\|W_2 T_i W_1\|\|_\infty \leq 1$
$(I + W_i\Delta W_2)P$	$\|\|W_2 T_o W_1\|\|_\infty \leq 1$
$P(I + W_1\Delta W_2)^{-1}$	$\|\|W_2 S_i W_1\|\|_\infty \leq 1$
$(I + W_1\Delta W_2)^{-1}P$	$\|\|W_2 S_o W_1\|\|_\infty \leq 1$

Table 1: Robustness results for $\Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty < 1$

plementary sensitivities (T_i and T_o). With a plant P and controller C, these sensitivities are defined

$$
\begin{aligned}
S_i &= (I + CP)^{-1}, \\
S_o &= (I + PC)^{-1}, && (1) \\
T_i &= I - S_i = CP(I + CP)^{-1}, \\
T_o &= I - S_o = PC(I + PC)^{-1}. && (2)
\end{aligned}
$$

Suppose that the interconnection of C with P is stable, and that one seeks to ensure that the stability of the interconnection of C with any plant in the perturbed plant set. Table 1, drawn from [10], lists some sufficient conditions. These conditions are also necessary in the case that the perturbations Δ are stable but otherwise unstructured.

The requirements that W_1, Δ and W_2 each be stable is somewhat restrictive from the perspective of practical utility. The table only captures the possibility that the perturbation is stable, even if the plant P that undergoes an additive perturbation is unstable. Perturbation of the location of an unstable pole in P, for example, is ruled out. The restriction has in fact been removed for some time: see, for example [7].

In the following, we drop the restriction that $\Delta \in \mathcal{RH}_\infty$, and simply require that $\Delta \in \mathcal{L}_\infty$ with $\|\Delta\|_\infty < 1$, that is, the maximum singular value of Δ is less than unity on the $j\omega$-axis. For any transfer function $X(s)$ that has poles as its only singularities in $Re[s] > 0$, let $\eta(X)$ denote number of right-half plane poles of $X(s)$, that is the McMillan degree of the transfer function matrix $X_+(s)$, where $X = X_-(s) + X_+(s)$, and $X_-(s)$ has no singularities in $Re[s] > 0$. For a square, nonsingular X (for which X^{-1} is not necessarily proper), define $\mathcal{Z}(X)$ to be $\eta(X^{-1})$, the count excluding any poles of X^{-1} at $s = \infty$. Similarly we define $\bar{\mathcal{Z}}(X)$ to be $\bar{\eta}(X^{-1})$, the count *including* poles of X^{-1} at $s = \infty$. Then the following results of Table 2, which are somewhat harder to derive than those in

Perturbed Model Set	Stability Test
$P+W_1\Delta W_2$ $\eta(P+W_1\Delta W_2)=\eta(P)$	$\|\|W_2 C(I+PC)^{-1}W_1\|\|_\infty \leq 1$
$(I+W_1\Delta W_2)P$ $\eta[(I+W_1\Delta W_2)P]=\eta(P)$	$\|\|W_2 PC(I+PC)^{-1}W_1\|\|_\infty \leq 1$
$(I+W_1\Delta W_2)^{-1}P$ $(I+W_1\Delta W_2)$ is invertible $\mathcal{Z}[(I+W_1\Delta W_2)^{-1}P]=\mathcal{Z}(P)$	$\|\|W_2(I+PC)^{-1}W_1\|\|_\infty \leq 1$

Table 2: Robustness Results for $\Delta \in \mathcal{L}_\infty, \|\Delta\|_\infty < 1$

Table 1, apply: see also [8]. Note that for brevity we have omitted the cases involving input sensitivities S_i and T_i.

In fact, Table 2 sets out sufficient *and* necessary conditionsfor arbitrary unstructured, $\mathcal{L}_\infty$-norm bounded perturbations Δ. That is, $[P_1, C]$ is stable for all P_1 in the perturbed model set if and only if $[P, C]$ is stable and the corresponding stability test inequality holds.

2.2 The Vinnicombe-Gap and Robustness

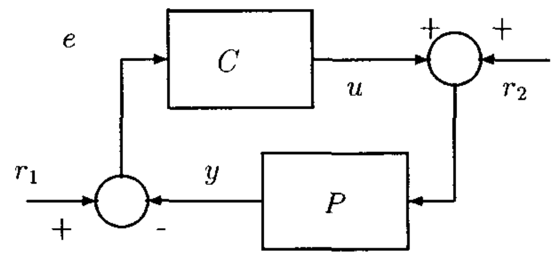

Figure 1: Closed-loop system

Robust stability results are also found in [8, 9]. These results are based on the ν-gap metric (which, incidentally, induces the weakest topology on operator space in which stability is a robust property). We review some of these results in this section. In the feedback system of Figure 1, the transfer function from $[r_1^T \quad r_2^T]^T$ to $[y^T \quad u^T]^T$ is given by

$$
\begin{aligned}
T(P,C) &= \begin{bmatrix} P(I+CP)^{-1}C & P(I+CP)^{-1} \\ (I+CP)^{-1}C & (I+CP)^{-1} \end{bmatrix}, \\
&= \begin{bmatrix} P \\ I \end{bmatrix} (I+CP)^{-1} \begin{bmatrix} C & I \end{bmatrix}.
\end{aligned}
$$

One can define a generalized stability margin by

$$
b_{P,C} = \begin{cases} \|T(P,C)\|_\infty^{-1} & \text{if } (P,C) \text{ is stable,} \\ 0 & \text{otherwise.} \end{cases}
$$

We now describe how to obtain a distance measure betwen two plants P_1 and P_2 with real rational transfer functions and the same input and output dimensions. In the sequel, for a real rational $X(s)$, we define $X(s)^* = X^T(-s)$. For real rational $\Phi(s)$ with $\Phi(s) = \Phi(s)^*$ and $\Phi(j\omega) \geq 0$ for all ω, the notation $\Phi(s)^{\frac{1}{2}}$ denotes a (possibly but not necessarily stable minimum-phase) spectral factor of $\Phi(s)$. Set

$$\kappa[P_1(j\omega), P_2(j\omega)] = \lim_{\hat{\omega}\to\omega} \bar{\sigma}\{[I + P_2(j\hat{\omega})P_2(j\hat{\omega})^*]^{-\frac{1}{2}} \times [P_2(j\hat{\omega}) - P_1(j\hat{\omega})][I + P_1(j\hat{\omega})^* P_1(j\hat{\omega})]^{-\frac{1}{2}}\}. \quad (3)$$

(In cases where $j\omega$ is a pole of either P_1 or P_2 the limiting process assures that the above quantity is well-defined). The ν-gap distance between P_1 and P_2 is further defined by

$$\delta_\nu(P_1, P_2) = \sup_\omega \kappa[P_1(j\omega), P_2(j\omega)]$$

provided that

$$\text{wno}[\det(I + P_2^* P_1)] + \eta(P_1) - \bar{\eta}(P_2) = 0. \quad (4)$$

Here, $\text{wno}[\det(I + P_2^* P_1)]$ is the number of encirclements of the origin made by $\det[I + P_2^*(s)P_1(s)]$ as s follows the standard Nyquist contour, indented into the right half plane around any imaginary axis pole or zero. As before, $\eta(P_1)$ is the number of poles of P_1 in the *open* right half plane $Re[s] > 0$, while $\bar{\eta}(P_2)$ is the number of poles of P_2 in the *closed* right half plane $Re[s] \geq 0$. In the event that the winding number condition (4) does not hold, then one sets

$$\delta_\nu(P_1, P_2) = 1.$$

There are two robust stability results to record. Firstly, suppose that (P_1, C) is stable. Then for any $\beta < b_{P_1,C}$, the set $\{P_2 : \delta_\nu(P_1, P_2) \leq \beta\}$ is stabilized by C, while if $\beta > b_{P_1,C}$, there exists at least one P_2 in the set which is not stabilized by C.

Second, suppose again that (P_1, C) is stable. Suppose also that for all ω, there holds

$$\kappa[P_1(j\omega), P_2(j\omega)]\bar{\sigma}[T(P_1(j\omega), C(j\omega))] < 1. \quad (5)$$

Then $[P_2, C]$ is stable if and only if the winding number condition (4) holds.

There is of course a connection between these results. Notice that if $\delta_\nu(P_1, P_2) \leq \beta < b_{P_1,C}$, then

$$\begin{aligned} \kappa[P_1(j\omega), P_2(j\omega)] &\leq \delta_\nu(P_1, P_2) < \\ b_{P_1,C} &= \frac{1}{\sup_\omega \bar{\sigma}\{T[P_1(j\omega), C(j\omega)]\}}, \\ &\leq \frac{1}{\bar{\sigma}\{T[P_1(j\omega), C(j\omega)]\}}, \end{aligned}$$

that is, equation (5) holds.

The winding number condition (4) is a rather complicated condition involving poles and zeros. (Notice that for a nonsingular X, $\text{wno}[\det(X)] = \mathcal{Z}(X) - \eta(X)$). As such, it corresponds to (but generalizes) the pole or zero counting conditions of Table 2.

For future use, let us note that from (3), it follows easily that for invertible P_1 and P_2,

$$\kappa[P_1(j\omega), P_2(\omega)] = \kappa[P_1^{-1}(j\omega), P_2^{-1}(j\omega)].$$

Also note that the winding number condition (4) is easily seen to be equivalent to

$$\text{wno}[\det(I + P_2^{-*} P_1^{-1})] + \mathcal{Z}(P_1) - \bar{\mathcal{Z}}(P_2) = 0.$$

3 Extending ν-Gap Robustness Results

We aim to demonstrate that a connection exists between the ν-gap robustness results, and the earlier "classical" results in Table 2. The older results are are actually consequences of "limiting" versions of the ν-gap result.

Our starting point is the simple observation that (P, C) is stable if and only if, with V_1, V_2, V_1^{-1} and V_2^{-1} all stable, $(V_1 P V_2, V_2^{-1} C V_1^{-1})$ is also stable. Consequently, the following simple variant on the robust stability condition is associated with (5).

Lemma 3.1 *Let (P_1, C) be stable. Suppose that for some V_1, V_2 with V_1, V_2, V_1^{-1} and V_2^{-1} each stable, there holds for all ω*

$$\kappa[V_1 P_1 V_2(j\omega), V_1 P_2 V_2(j\omega)] \ldots \bar{\sigma}[T(V_1 P_1 V_2(j\omega), V_2^{-1} C V_1^{-1}(j\omega)] < 1. \quad (6)$$

Then (P_2, C) is stable if and only if the following winding number condition holds:

$$\text{wno}[\det(I + V_2^* P_2^* V_1^* V_1 P_1 V_2)] + \eta(P_1) - \bar{\eta}(P_2) = 0. \quad (7)$$

The lemma suggests the possibility, given a plant pair P_1 and P_2 and a controller C, of choosing weights V_1 and V_2 in order to secure (6), and then checking that (7) indeed holds. This is roughly what we shall do to recover the traditional robust stability results from the δ_ν results, in particular the results associated with (5).

By way of motivation for the choice of weights, let us observe what happens in the scalar case. Use lower case symbols p_1, p_2 and c for this special case, and let v denote the multiplicative combination of V_1 and V_2. Then at each ω,

$$\begin{aligned}\kappa(vp_1, vp_2) &= \frac{|v(p_1 - p_2)|}{\sqrt{1+|vp_1|^2}\sqrt{1+|vp_2|^2}},\\ \bar{\sigma}[T(vp_1, v^{-1}c)] &= \frac{\sqrt{1+|vp_1|^2}\sqrt{1+|v^{-1}c|^2}}{|1+p_1c|}\end{aligned}$$

and

$$\kappa(vp_1, vp_2)\bar{\sigma}\left[T(vp_1, v^{-1}c)\right] = \frac{|p_1-p_2|}{|1+p_1c|}\left(\frac{|v|^2+|c|^2}{1+|vp_2|^2}\right)^{\frac{1}{2}}.$$

The function

$$g(x) = \frac{x+|c|^2}{1+x|p_2|^2}$$

has a monotonic derivative on $x \geq 0$. Consequently

$$\arg\min_{x\geq 0} g(x) = \begin{cases} 0 & \text{if } |c|^2 < |p_2|^{-2} \\ \infty & \text{if } |c|^2 > |p_2|^{-2} \\ [0,\infty) & \text{if } |c|^2 = |p_2|^{-2} \end{cases}$$

with $\min_{x\geq 0} g(x) = \min(|c|^2, |p_2|^{-2})$.

In the scalar case then, the left side of (6) is minimized by letting $|v|$ tend to either zero or infinity, according to whether $|p_2c| < 1$ or $|p_2c| > 1$. A choice of v which would be optimal in the sense of minimizing the left side of (6) might then have $|v|$ very small over some frequencies and very large over others. Such a choice would be likely to make the investigation of (7) rather difficult.

4 Connecting "Classical" and ν-gap Robustness Conditions

In this section we will demonstrate that by taking $|v|$ either vanishingly small or arbitrarily large at *all* frequencies, we can recover the earlier robustness results follow on as a consequence. Let us therefore focus on equations (6) and (7) in these two circumstances.

4.1 Vanishingly Small Weights

Although we have briefly looked at scalar plants to motivate the analysis, we now investigate the more general multivariable case. Suppose that we set $V_2 = \epsilon W_2^{-1}$ and $V_1 = \epsilon W_1^{-1}$, where W_1 and W_2 together with their inverses are stable and ϵ is a positive scalar (which will become very small below). Then

$$T[V_1P_1V_2(j\omega), V_2^{-1}CV_1^{-1}(j\omega)] = \begin{bmatrix} W_1^{-1}P_1(I+CP_1)^{-1}CW_1 & \epsilon^2 W_1^{-1}P_1(I+CP_1)^{-1}W_2^{-1} \\ \epsilon^{-2}W_2(I+CP_1)^{-1}CW_1 & W_2(I+CP_1)^{-1}W_2^{-1}\end{bmatrix}.$$

At each ω, the maximum singular value is dominated by the 2–1 block for ϵ sufficiently small:

$$\begin{aligned}&\bar{\sigma}\{T[\epsilon^2 W_1^{-1}P_1W_2^{-1}, \epsilon^{-2}W_2CW_1]\} \simeq \\ &\epsilon^{-2}\bar{\sigma}[W_2(I+CP_1)^{-1}CW_1].\end{aligned} \tag{8}$$

Also, for all ω:

$$\begin{aligned}&\kappa[\epsilon^2W_1^{-1}P_1W_2^{-1}, \epsilon^2W_1^{-1}P_2W_2^{-1}] = \\ &\bar{\sigma}\left\{[I+\epsilon^4W_1^{-1}P_2W_2^{-1}W_2^{-*}P_2^*W_1^{-*}]^{-\frac{1}{2}}\right.\\ &\times\epsilon^2W_1^{-1}(P_2-P_1)W_2^{-1}\\ &\left.\times[I+\epsilon^4W_2^{-*}P_1^{-*}W_1^{-*}W_1^{-1}P_1W_2^{-1}]^{-\frac{1}{2}}\right\}\end{aligned}$$

It can be shown (although the calculations are somewhat tedious), that if $j\omega_0$ is a pole of $P_1 - P_2$, then κ is not continuous at $\epsilon = 0$. If $P_1 - P_2$ has no $j\omega$-axis poles (a sufficient condition being that neither P_1 nor P_2 has any $j\omega$-axis poles), it can be verified quite easily that for very small ϵ,

$$\begin{aligned}&\kappa[\epsilon^2W_1^{-1}P_1W_2^{-1}, \epsilon^2W_1^{-1}P_2W_2^{-1}] \simeq \\ &\epsilon^2\bar{\sigma}[W_1^{-1}(P_2-P_1)W_2^{-1}](j\omega)\end{aligned}$$

Hence, given that $P_1 - P_2$ possesses no $j\omega$-axis poles, then with the substitutions $V_1 = \epsilon W_1^{-1}$ and $V_2 = \epsilon W_2^{-1}$ and with $0 < \epsilon << 1$, the inequality (6) of Lemma 3.1, becomes,

$$\begin{aligned}&\bar{\sigma}[W_1^{-1}(P_2-P_1)W_2^{-1}](j\omega)\\ &\quad\ldots\times\bar{\sigma}[W_2(I+CP_1)^{-1}CW_1(j\omega)]\\ &\quad< 1\end{aligned} \tag{9}$$

Next, we investigate the winding number part of the stability condition. (We shall continue to demand that $P_1 - P_2$ has no $j\omega$-axis poles and that ϵ is suitably small.) To this end, we shall use the following lemma.

Lemma 4.1 *Let $\Delta \in \mathcal{RL}_\infty$, that is with Δ rational, and with the same dimensions as some prescribed proper real rational Φ. Then for scalar $\epsilon < 1$ such that $\|\epsilon^2\Delta\|_\infty < 1$, there holds*

$$\text{wno}[\det(I + \epsilon^4\Phi^*\Phi + \epsilon^4\Delta^*\Phi)] = \bar{\eta}(\Phi) - \eta(\Phi). \quad (10)$$

Proof: See [1]. ∎

Set $P_2 - P_1 = W_1\Delta W_2$, so that because $P_2 - P_1$ has no imaginary axis poles it follows that $\Delta \in \mathcal{L}_\infty$. Also, set $\Phi = W_1^{-1}P_1W_2^{-1}$. Condition (7) becomes

$$\text{wno}[\det(I + \epsilon^4\Phi^*\Phi + \epsilon^4\Delta^*\Phi)] + \eta(P_1) - \bar{\eta}(P_2) = 0.$$

Using Lemma 4.1, and noting the equalities $\bar{\eta}(\Phi) = \bar{\eta}(P_1)$ and $\eta(\Phi) = \eta(P_1)$, we see that (7) is equivalent to $\bar{\eta}(P_1) = \bar{\eta}(P_2)$. Since $P_2 - P_1 \in \mathcal{L}_\infty$, this is also equivalent to

$$\eta(P_1) \;\; = \;\; \eta(P_2). \quad (11)$$

In summary, we have established that if $P_2 - P_1$ has no $j\omega$-axis poles, and if (9) and (11) hold, then (6) and (7) hold with $V_i = \epsilon W_i^{-1}$ and ϵ suitably small for some stable and stably invertible W_1 and W_2. Accordingly the stability of (P_1, C) implies the stability of (P_2, C).

The connection between this result and the first entry on Table 2 should now be clear. In (9), we are modelling the difference between P_2 and P_1 by $P_2 = P_1 + W_1\Delta W_2$ where $\Delta \in \mathcal{L}_\infty$ and P_1 and P_2 have the same right half plane pole count, as in Table 2. However, in (9), we are simply requiring the frequency by frequency condition that

$$\bar{\sigma}(\Delta)\bar{\sigma}\left[W_2C(I + P_1C)^{-1}W_1\right] \;\; < \;\; 1 \quad (12)$$

Table 2 requires the apparently stronger conditions that $\|\Delta\|_\infty < 1$ and $\left\|W_2C(I + P_1C)^{-1}W_1\right\|_\infty \leq 1$. However, because of the freedom in the choice of the pertubation parameter, the conditions in Table 2 are actually equivalent to the condition that equation (12) holds. It is certainly easy to see that equation (12) is implied by the conditions $\|\Delta\|_\infty < 1$ and $\left\|W_2C(I + P_1C)^{-1}W_1\right\|_\infty \leq 1$.

Let us emphasise that we are describing a robust stability result. We are not advocating a design method where the closed loop transfer function $T[V_1PV_2(j\omega), V_2^{-1}CV_1^{-1}(j\omega)]$ is implemented. Such a design strategy would not be prudent, since $\bar{\sigma}(T)$ goes to infinity as ϵ goes to zero.

It is possible to make connections with the second entry of Table 2 using similar ideas. Let us suppose that $\Delta \in \mathcal{L}_\infty$, and temporarily suppose that P_1 has no imaginary axis pole. We consider

$$P_2 \;\; = \;\; (I + W_1\Delta W_2)P_1$$

and suppose that (with all $\Delta \in \mathcal{L}_\infty$ with $\|\Delta\|_\infty \leq 1$) there holds

$$\eta(P_2) \;\; = \;\; \eta(P_1). \quad (13)$$

Define $\bar{W}_{2\epsilon}$ to be the stable and stably invertible spectral factor of $\epsilon I + (W_2P_1)^*(W_2P_1)$ for small ϵ. Select

$$\begin{aligned} V_1 &= \epsilon W_1^{-1} \\ V_2 &= \epsilon \bar{W}_{2\epsilon}^{-1} \end{aligned}$$

Using similar methods to the previous case (see [1] for details) we have that for small ϵ

$$\kappa[V_1P_1V_2(j\omega), V_1P_2V_2(j\omega)] \simeq \epsilon^2\bar{\sigma}(\Delta)$$

and equation (6) becomes

$$\bar{\sigma}(\Delta)\bar{\sigma}[W_2P_1(I + CP_1)^{-1}CW_1] \;\; < \;\; 1. \quad (14)$$

The winding number condition is equivalent to (13), and so stability of (P_1, C) will imply stability of (P_2, C) if both (13) and (14) hold.

This result can also be extended to allow P_1 to possess imaginary axis poles so that (13) and (14) yields a robust stability results also for the case: again see [1].

4.2 Arbitrarily Large Weights

By suitably specialising the Vinnicombe form of the robustness conditions it is also possible to derive the third result of Table 2. We restrict attention to square nonsingular P for simplicity of exposition. (Nonsingular rectangular P may be dealt with by a "squaring-up" operation that will result in a square, nonsingular P.) In the formulas for $T[V_1P_1V_2(j\omega), V_2^{-1}CV_1^{-1}(j\omega)]$, we shall choose,

$$\begin{aligned} V_1 &= MW_2, \\ V_2 &= M\bar{W}_1, \end{aligned}$$

where M is a scalar which we will allow to become arbitrarily large, and $\bar{W}_1$ is a minimum phase and stable spectral factor such that (under the temporary assumption that P_1 has no extended $j\omega$-axis pole or zero)

$$\bar{W}_1^{-*}\bar{W}_1^{-1} = (W_1^{-1}P_1)^*(W_1^{-1}P_1).$$

For each ω, the maximum singular value of T is determined by the 1–2 block. For very large M

$$\begin{aligned}&\bar{\sigma}\left\{T[M^2 W_2 P_1 \bar{W}_1, M^{-2}\bar{W}_1^{-1}CW_2^{-1}]\right\} \simeq \\ &M^2\bar{\sigma}[W_2(I+P_1C)^{-1}P_1\bar{W}_1] = \\ &M^2\bar{\sigma}[W_2(I+P_1C)^{-1}W_1].\end{aligned} \tag{15}$$

We can show, using the invertibility of the P_i and results from Section 2 that

$$\kappa[V_1P_1V_2, V_1P_2V_2] = \kappa[V_2^{-1}P_1^{-1}V_1^{-1}, V_2^{-1}P_2^{-1}V_1^{-1}].$$

and hence for very large M, (see [1] for details)

$$\kappa[V_1P_1V_2, V_1P_2V_2] \simeq M^{-2}\bar{\sigma}(\Delta). \tag{16}$$

Equation (6) then becomes

$$\bar{\sigma}(\Delta)\bar{\sigma}[W_2(I+P_1C)^{-1}W_1] < 1.$$

The winding number condition (Section 2) is,

$$\begin{aligned}&\text{wno}[\det\left(I + M^{-4}W_2^{-*}P_2^{-*}\bar{W}_1^{-*}\bar{W}_1^{-1}P_1^{-1}W_2^{-1}\right)] \\ &+\eta(P_1^{-1}) - \bar{\eta}\left(\bar{W}_1^{-1}P_2^{-1}W_2^{-1}\right) = 0.\end{aligned}$$

It can also be shown that for very large M, condition (17) becomes

$$\mathcal{Z}(P_1) = \mathcal{Z}(P_2) = \mathcal{Z}\left[(I + W_1\Delta W_2)^{-1}P_1\right].$$

These results can be extended to the case where P_1 has poles or zeros on the $j\omega$-axis by perturbation arguments similar to those introduced previously.

5 Summary

The main contribution of this extended abstract is to show that earlier "classical" robust stability results of can be obtained by using specialized weighting functions in the more recent ν-gap robust stability theory [8, 9]. The pole or zero counts preservation conditions in the earlier results are specialisations (and thus restrictions) of winding number conditions required for the application of the ν-gap robustness results. In the course of demonstrating the results, we showed how the introduction of specialized weights into the generalized sensitivity matrix, can isolate particular input or output sensitivity or complementary sensitivity functions. Of course, similar analysis could be performed for any of those sensitivity functions. For scalar P and C, as has been observed in [2], the choice $|V_1V_2| = \sqrt{|C|/|P|}$ yields at each frequency a minimum for $\bar{\sigma}(T)$, of $|(I+CP)^{-1}| + |P(I+CP)^{-1}C|$. These observations suggest the possibility of obtaining results in addition to those for robust stability.

References

[1] B. D. O. Anderson and T. S. Brinsmead. *Classical and ν-gap metric robust stability results.* Technical Report, RSISE, Australian National University, Acton 0200, Australia, March 2001.

[2] K. J. Åstrom. Model uncertainty and robust control. In P. Albertos and A. Sala, editors, *Lecture Notes on Iterative Identification and Control Design.* Universidad Politechnice de Valencia, 2000.

[3] J. Doyle, B. Francis, and A. Tannenbaum. *Feedback Control Theory.* Macmillan Publishing Company, New York, 1992.

[4] J. C. Doyle, , J. Wall, and G. Stein. Performance and robustness analysis for structured uncertainty. *Proc. 21st IEEE Conf. on Decision and Control*, pages 629–636, 1982.

[5] J. C. Doyle and G. Stein. Multivariable feedback design: Concepts for a classicalmodern synthesis. *IEEE Trans. Auto. Control*, AC-26(1):4–16, February 1981.

[6] M. Morari and E. Zafiriou. *Robust Process Control.* Prentice Hall, NJ, 1989.

[7] M. Vidyasagar. *Control System Synthesis: A Factorization Approach.* MIT Press, Cambridge, Massachusetts, 1985.

[8] G. Vinnicombe. Frequency domain uncertainty and the graph topology. *IEEE Trans. Auto. Control*, 38(9):1371–1383, September 1993.

[9] G. Vinnicombe. *Uncertainty and Feedback: $\mathcal{H}_\infty$ Uncertainty and the ν-gap metric.* Imperial College Press, London, 1999. to appear.

[10] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, NJ, 1996.