Parametrization of internally stabilizing controllers via differentially coprime kernel representations

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Abstract In this paper, we use the notion of a differentially coprime kernel representation to parametrize the set of all stabilizing controllers using a so called Youla parameter and to unify understanding of some stability concepts for nonlinear systems. By utilizing the differential kernel representation concept, we are able to convert a closed-loop identification problem into one of open-loop identification. The idea of a differential kernel representation allows us also to clarify the relationship between three different notions of internal stability available in the literature. The results in the paper thus provide new insights to the stability of nonlinear feedback systems.

1 Introduction

In this paper, we present a parametrization of all internally stabilizing controllers using kernel representations. It is applicable to the identification of nonlinear systems operating under a nonlinear feedback. The parametrization of nonlinear stabilizing controllers were tackled by many researchers and left coprime factorizations approach have derived satisfactory results so far [7]. Moreover some applications have been obtained based on such parametrizations [1, 8]. Recently kernel representations were introduced as a generalization of left factorizations in [10]. Although there is no essential difference for linear systems, for nonlinear systems it was shown in [11] that state-space realizations of kernel representations are often computable under mild assumptions whereas those of left factorizations are hard or impossible to obtain.

Results on the parametrization of stabilizing controllers based on kernel representations are available in [10, 9, 6, 5]. However, the parametrization based on kernel representations adopts a different definition of the stability of feedback systems than the usual one and we have problems in utilizing it to some applications, e.g. a closed-loop identification [2, 8]. To overcome such problems, we employ a special kernel representation, i.e. we assume that 1. the direct feedthrough part is a constant, and that 2. it has a differential coprimeness property. The property 1 is also employed in [6, 8] to render the parametrized feedback system well-posed. Kernel representations which obey these two assumptions also connect the stability of feedback systems as employed in the kernel representation approach to the one used in the usual sense. Furthermore a framework for nonlinear closed-loop identification is exhibited as an application of the parametrization. All proofs are omitted for the reason of space. See [4, 3] for the proofs and detail.

2 Preliminary background

\[
\Sigma^0 : L_{m_2}^0 \to L_{n_2}^0 \text{ denotes an operator with an initial state } x(0) = x_0 \in X^0 \subset \mathbb{R}^n \text{ which is a mapping from } L_{m_2}^0 \text{ to } L_{n_2}^0, \text{ where } X^0 \ni 0 \text{ is a connected subset of } \mathbb{R}^n. \Sigma \text{ is used as shorthand for } \Sigma_{m_2}^0. \text{ We employ the differential operator } \partial(\cdot) \text{ borrowed from } [8].
\]

\[
\partial \Sigma^0_{m_2}(\cdot) := \Sigma^0_0 (\cdot + u) - \Sigma^0_0 (\cdot) \tag{1}
\]

The operator acts on \( u \), and is parametrized by \( u \); in case \( \Sigma^0_0 \) is a linear operator, the differential operator is independent of \( u \) and identical with \( \Sigma^0_0 \).

The smoothing concept is a powerful tool for establishing the well-posedness property of interconnected systems [12].

Definition 1 Consider an operator \( \Sigma^0 : L_{m_2}^0 \to L_{n_2}^0 \). It is said to be smoothing if it is weakly Lipschitz and for every \( T > 0, \gamma > 0 \) and \( x^0 \in X^0 \) there exists \( t_1 = t_1(T, \gamma, x^0) \in (0, T) \) such that

\[
\| \tau_{T+t_1}(\Sigma^0_0 \circ \tau_{T+t_1} - \Sigma^0_0 \circ \tau_T) \|_L \leq \gamma \tag{2}
\]

holds for \( \forall t \in [0, T - t_1] \). Here \( \tau_T \) is the truncation operator.

In this paper, it is also assumed that any operator has a finite dimensional state-space realization. Please refer to [4, 3, 6] for definitions of other terminologies.

2.1 Kernel representations

This subsection introduces kernel representations [10] as generalization of left factorizations. A kernel representation of a causal operator \( \Sigma^0 : L_{m_2}^0 \to L_{n_2}^0 \) is a causal operator \( K^2_0 : L_{m_2}^0 \times L_{n_2}^0 \to L_{n_2}^0 \) such that

\[
y = \Sigma^0_0 (u) \Leftrightarrow K^2_0 (u, y) = 0 \tag{3}
\]

holds for \( \forall x^0 \in X^0 \) and \( \forall y \in L_{n_2}^0 \) and \( y \in L_{n_2}^0 \).

A kernel representation \( K^2_0 : L_{m_2}^0 \times L_{n_2}^0 \to L_{n_2}^0 \) is said to be well-defined if there exists the causal pseudo-inverse operator \( (K^2_0)^\# : L_{m_2}^0 \times L_{n_2}^0 \to L_{n_2}^0 \) such that

\[
y = (K^2_0)^\# (u, y) \Leftrightarrow K^2_0 (u, y) = y \tag{4}
\]

holds for \( \forall x^0 \in X^0 \) and \( \forall u \in L_{n_2}^0 \) and \( y \in L_{n_2}^0 \).
Definition 2  A bounded kernel representation $R^{0}_{G}$ : $L^{m+p}_{2e} \to L^{p}_{2e}$ is said to be \textit{coprime} if there exists a bounded operator $X^{0} : L^{p}_{2e} \to L^{m+p}_{2e}$ such that
\[ R^{0}_{G} \circ X^{0} = \text{Id} \] \hspace{1cm} (4)
holds for $\forall x^{0} \in X^{0}$. It is said to be \textit{uniformly differentially coprime} if it is globally Lipschitz and there exists a set of bounded operators $X^{0}_{(w)} : L^{p}_{2e} \to L^{m+p}_{2e}$ which are parametrized by the signal $w$ and have a finite gain uniformly over $w \in L^{m+p}_{2e}$ such that
\[ \partial R^{0}_{G(w)} \circ X^{0}_{(w)} = \text{Id} \] \hspace{1cm} (5)
holds for $\forall w \in L^{m+p}_{2e}$. Here $X^{0}_{(w)}(v)$ is causally dependent on $w$ and $v$.

2.2 Internal stability

Figure 1: The feedback system $\{G, K\}$ with additive disturbances

We now consider the feedback system depicted in Figure 1. Such a feedback system that interconnects $G^{0} : L^{m}_{2e} \to L^{p}_{2e}$ and $K^{0} : L^{p}_{2e} \to L^{m}_{2e}$ is denoted by $\{G^{0}, K^{0}\}$ or just $\{G, K\}$. We use the following condensed notations if no confusion arises: $w := (u, y)$, $z_{G} := (z_{K}, z_{G})$ and $e_{12} := (e_{1}, e_{2})$. The stability of the feedback system $\{G, K\}$ with additive disturbances as in Figure 1 is considered. Let us define a new operator $E^{(x^{0}_{G}, x^{0}_{K})} : L^{m+p}_{2e} \to L^{m+p}_{2e}$ which is a mapping from the external additive signal $(e_{1}, e_{2})$ to $(u, y)$ in Figure 1.

Definition 3  A feedback system $\{G, K\}$ is said to be \textit{well-posed} if the operator $E^{(x^{0}_{G}, x^{0}_{K})}$ exists and is weakly Lipschitz. It is said to be \textit{internally stable} if $E^{(x^{0}_{G}, x^{0}_{K})}$ is bounded.

Figure 2: Null and strong internal stability of $\{G, K\}$

We state two other stability concepts of feedback systems based on kernel representations. The stability of feedback systems as shown in Figure 2 is discussed here, where $R_{G} : (u, y) \mapsto z_{G}$ and $R_{K} : (y, u) \mapsto z_{K}$ denote the kernel representations of the components $G$ and $K$ respectively. Employing $R_{G}$ and $R_{K}$, a kernel representation of the operator $E_{(G, K)}$ can be defined by

\[ R^{0}_{E_{(G, K)}}(e_{12}, w) := \begin{pmatrix} R^{0}_{K}(w + e_{12}) \\ R^{0}_{G}(w) \end{pmatrix} = z_{G} \]

It is easy to see that
\[ w = E_{(G, K)}(e_{12}) \Leftrightarrow R_{E_{(G, K)}}(e_{12}, w) = 0 \]
which is the definition of the kernel representation (3). Further the kernel representation of the feedback system in the case $e_{12} = 0$ can also be defined by
\[ R^{0}_{w_{E_{(G, K)}}}(w) := R^{0}_{E_{(G, K)}}(0, w). \]

Definition 4  A feedback system $\{G, K\}$ with a weakly Lipschitz kernel representation $R_{(G, K)}$ is said to be \textit{null well-posed} if the operator $R_{(G, K)}$ exists and is weakly Lipschitz, and is said to be \textit{null internally stable} if $R_{(G, K)}$ is bounded. It is said to be \textit{strongly well-posed} if the operator $R_{(G, K)}$ is well-defined and $R_{(G, K)}$ is weakly Lipschitz, and is said to be \textit{strongly internally stable} if $R_{(G, K)}$ is bounded.

Assuming the strong detectability of kernel representations, we can derive the parametrization of all strongly (and null) internally stabilizing controllers. We do not repeat the results here for the reason of space. See [6] for detail. However, there are problems in applications of these results. For example, we need the parametrization of all plants which are internally stabilized by a given controller for closed-loop identification, because we cannot check whether the given controller strongly (or null) internally stabilizes the plant or not. Therefore we will investigate the parametrization based on \textit{internal stability} in the sequel.

3 Equivalence of internal stabilities

In this section, the relationship between the three different well-posedness and internal stability definitions of feedback systems are discussed. Firstly, we can prove the equivalence between null well-posedness and strong well-posedness under the assumption that an operator $\Sigma : u \mapsto y$ has a construction
\[ \Sigma(u) = \Sigma^{\text{smth}}(u) + \Sigma^{\text{const}} \cdot u \]
which implies that its state-space realization has a constant direct feedthrough.

Lemma 1  Consider a well-posed feedback system $\{G, K\}$ with kernel representations $R_{G} : L^{m+p}_{2e} \to L^{p}_{2e}$ and $R_{K} : L^{m+p}_{2e} \to L^{m}_{2e}$. Suppose either $R_{G}$ or $R_{K}$ has a construction as in (7). Then the feedback system $\{G, K\}$ with the kernel representation $R_{(G, K)}$ is strongly well-posed if and only if it is null well-posed.

Furthermore, we can show the equivalence between null internal stability and strong internal stability when one of the kernel representations is globally Lipschitz.
Theorem 1 Consider a well-posed feedback system \( \{G, K\} \) with kernel representations \( R_G : L^{m+1}_{2e} \to L^p_{2e} \) of \( G \) and \( R_K : L^{m+1}_{2e} \to L^p_{2e} \) of \( K \). Suppose either \( R_G \) or \( R_K \) is globally Lipschitz and has a construction as in (7). Then the feedback system \( \{G, K\} \) with the kernel representation \( R_{\{G,K\}} \) is strongly internally stable if and only if it is null internally stable.

If both \( R_G \) and \( R_K \) have constructions as in (7), then all three well-posedness notions coincide. The equivalence between well-posedness and null well-posedness does not hold in general. We can show the relation between these three well-posedness notions by employing a special kernel representation.

Lemma 2 Consider a well-posed feedback system \( \{G, K\} \) with a kernel representation \( R_K : L^{m+1}_{2e} \to L^p_{2e} \) of \( K \) which has a construction as in (7). Then there exists a kernel representation \( R_G : L^{m+1}_{2e} \to L^p_{2e} \) of \( G \) such that the system \( \{G, K\} \) with \( R_{\{G,K\}} \) is null well-posed.

Lemma 2 only proves that the null well-posedness of the feedback system with a certain kernel representation \( R_G \) of \( G \). The class of all kernel representations of \( G \) such that the feedback system is null well-posed is given by \( R_G \circ R_{\{G,K\}} \) with any well-defined kernel representation \( R_G \) of a zero operator [10] (where \( R_{\{G,K\}} \) is invertible). Lemmas 1 and 2 imply that if either of the kernel representations \( R_G \) or \( R_K \) has a construction as in (7) then the three well-posedness notions are equivalent in some sense. Furthermore, we investigate the property of differential coprimeness of the kernel representation \( R_{\{G,K\}} \).

Lemma 3 Consider an invertible globally Lipschitz kernel representation \( R_G : L^p_{2e} \to L^p_{2e} \). Then \( R_G^{-1} \) is globally Lipschitz if and only if \( R_G \) is uniformly differentially coprime.

Theorem 2 Consider a feedback system \( \{G, K\} \) with kernel representations \( R_G : L^{m+1}_{2e} \to L^p_{2e} \) of \( G \) and \( R_K : L^{m+1}_{2e} \to L^p_{2e} \) of \( K \). Suppose \( R_{\{G,K\}} \) is uniformly differentially coprime, and either \( R_G \) or \( R_K \) has a construction as in (7). Then the feedback system \( \{G, K\} \) with the kernel representation \( R_{\{G,K\}} \) is strongly internally stable.

Uniform differential coprimeness of \( R_{\{G,K\}} \) implies strong internal stability of the feedback system \( \{G, K\} \) provided either \( R_G \) or \( R_K \) has a construction as in (7). This property will be used in the parametrization of all internally stabilizing controllers in the next section.

4 Parametrization

This section discusses the parametrization of all internally stabilizing controllers, in contrast to the results in [6] which provide the parametrization of all strongly (and null) internally stabilizing ones.

Theorem 3 Consider a null internally stable feedback system \( \{G, K\} \) with bounded kernel representations \( R_G : L^{m+1}_{2e} \to L^p_{2e} \) and \( R_K : L^{m+1}_{2e} \to L^p_{2e} \). Suppose \( R_K \) is uniformly differentially coprime with a weakly Lipschitz operator \( X(w) \) in (5), completely controllable and has a construction as in (7). Then the parametrization of all weakly Lipschitz plants which are internally stabilized by \( K \) is given by \( G_S \) defined by

\[
R_{G_S} := R_S \circ R_{\{G,K\}} : L^{m+1}_{2e} \to L^p_{2e}
\]  

with any weakly Lipschitz bounded operator \( S \).

Differential coprimeness plays an important role to connect the different definitions of internal stability. Theorem 3 is useful in its application to closed-loop identification which is the topic of the following section. Notice that Theorem 3 can be easily made applicable by choosing the stabilizing controller \( K \) to satisfy the differential coprimeness assumption. Further the simultaneous parametrization of plant and controller pairs can be obtained in a similar manner [4].

5 Closed-loop identification

This section demonstrates how the parametrization in Theorem 3 is utilized for closed-loop identification. The configuration as shown in Figure 3 is considered. \( G \) is the plant to be identified and we use the (shorthand) notations \( r_{12} := (r_1, r_2) \) and \( r_3 \) for the known reference signals, \( d_{12} := (d_1, d_2) \) for the unknown disturbance signals and \( w := (u, y) \) for the measurable loop signals. \( G_0 \) is a nominal model and \( K \) is a stabilizing controller for both \( G \) and \( G_0 \) with \( R_{G_0} \) and \( R_K \) their kernel representations. Conventionally, \( r_3 \) would be taken to be zero, and \( R_{G_0}^{-1} \) would be replaced by \( K \). It suits us to consider the more general arrangement. We make some assumptions pertinent to the application of Theorem 3.

Assumptions

(A1) \( R_K \) is completely controllable.
(A2) \( R_K \) is differentially coprime.
(A3) \( R_K \) has a construction as in (7).
(A4) The feedback system \( \{G_0, K\} \) with the kernel representations \( R_{G_0} \) and \( R_K \) is null internally stable.
(A5) The feedback system \( \{G, K\} \) is internally stable.

Suppose the assumptions (A1)-(A5) are satisfied. Then \( G \) can be depicted as Figure 4 with a bounded weakly Lipschitz operator \( S \) by Theorem 3. The objective of this section is to identify the operator \( S \) using the reference input signal \( r_1, r_2 \) and \( r_3 \) and the measurable loop signal \( w \). It is shown in [4] that the mapping
Theorem 4. Consider the closed-loop identification as in Figure 4, and suppose (A1)-(A5) hold. Then, (i) we can choose an arbitrary open-loop input \( \delta z_k \in L_{2e} \) by choosing an appropriate \( r_1 \in L_{2e} \) when \( r_2, r_3, d_1, d_2 \in L_{2e} \) are prescribed, (ii) we can choose an arbitrary open-loop input \( \delta z_k \in L_{2e} \) by choosing appropriate \( r_1, r_2 \in L_{2e} \) when \( r_3, d_1, d_2 \in L_{2e} \) are prescribed.

Theorem 4 implies that we can choose the open-loop input signal \( \delta z_k \) arbitrarily. We now obtain an open-loop identification problem. However this is not a standard open-loop identification problem because both the input and output signals are contaminated by noise and the unknown additive signals have correlation with the reference signal \( \delta z_k \). To overcome such problems, we assume a high SNR (signal-to-noise ratio) as in [2, 8]. More precisely, we make the following assumptions.

Assumptions

(A6) \[ ||d_1|| \leq ||w||. \]

(A7) \[ ||d_2|| \leq \frac{||R_G(w)||}{2} ||\delta z_k||. \]

(A6) allows us to regard the operators \( \partial R_G(w) \cdot (\cdot) \) and \( \partial R_K(w) \cdot (\cdot) \) as being modellable by time-varying linear systems when they are acting on \( d_1 \). These time-varying systems result from the linearization around the trajectory produced by the input \( w \). Further, employing the differential operator of \( S \) we obtain

\[ \delta z = S(\delta z_K) + (\partial S(\delta z_K) \circ \partial R_K(w) - \partial R_G(w)) \cdot (d_1). \]