PROCEEDINGS OF 1968
JOINT AUTO CONTROL CONFERENCE,
UNIV. OF MICHIGAN, JUNE 1968.

GENERALIZATIONS OF THE "CIRCLE CRITERION" *

by

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ABSTRACT

This paper considers extensions of the "Circle Criterion" to systems consisting of a linear subsystem with an arbitrary number of time-varying or nonlinear memoryless feedback elements.

Stability and relative stability information is given when the linear subsystem is infinite dimensional or finite dimensional, and time-invariant or time-varying. Lyapunov functions are exhibited for the case when the linear subsystem is finite dimensional.

1. INTRODUCTION

Recently, a number of generalizations have been given of the Popov Criterion for nonlinear system stability [1, 2, 3]. This paper gives corresponding and additional generalizations of the related "Circle Criterion" [4, 5, 6] for time-varying and nonlinear systems. Since zero input stability is to be investigated, without loss of generality a system $\underline{S}$ (see Figure 1) is considered consisting of a linear subsystem $\underline{W}$ having "at least one integration in each feedforward path", with an arbitrary number of time-varying memoryless feedback elements.

Stability and relative stability information is given when the subsystem $\underline{W}$ is infinite dimensional, and either time-invariant or time-varying. When the subsystem $\underline{W}$ is time-invariant, the key sufficiency condition for stability is that a certain matrix function be positive real, and it is shown that when $\underline{W}$ is finite dimensional, Lyapunov functions may be constructed using a system theory criterion for positive real matrices [7]. For the case when the subsystem $\underline{W}$ is time-varying, the key sufficiency condition is that a certain matrix function be a covariance; again it is shown that in the finite dimensional case Lyapunov functions may be constructed using recent results in linear optimal control theory [8, 9].

Referring to Figure 1, the outputs $y(t) = [y_1(t), y_2(t), \dots, y_n(t)]'$ of system $\underline{S}$ are both the outputs of the subsystem $\underline{W}$ and the inputs to the time-varying feedback elements. The time-varying gains represented by $K(t) = \text{diag.}\{k_1(t), k_2(t), \dots, k_n(t)\}$ satisfy the condition

* This work was supported by the Australian Research Grants Committee

(S1) $K_1 \leq K(t) \leq K_2$ where $K_1 = \text{diag.}\{k_{11}, k_{12}, \ldots, k_{1n}\}$ and $K_2 = \text{diag.}\{k_{21}, k_{22}, \ldots, k_{2n}\}$ are constant and $K_2 - K_1 > 0$. (Note: The notation $X > Y$ $[X \geq Y]$ is used to indicate that $(X-Y)$ is positive [semi] definite.)

Because the case of time-invariant $\underline{W}$ is more familiar, we shall introduce this case first. To develop a stability criterion, a hypothetical system $\underline{Z}$ is introduced whose transfer function matrix $Z(s)$ is related to the transfer function matrix $W(s)$ of $\underline{W}$ via

$$Z(s) = A(K_2-K_1)^{-1} + AW(s)[I + K_1W(s)]^{-1} \qquad (1)$$

where

(Z1) $A = \text{diag.}\{a_1, a_2, \ldots, a_n\}$ is a constant positive definite matrix.

In (1), the term $W(s)[I + K_1W(s)]^{-1}$ will be recognized as the transfer function matrix of a system with forward part $\underline{W}$ and feedback K_1.

The criterion, to be proved later, establishing stability is

(Z2) $Z(s)$ is positive real for some A satisfying (Z1).

When $\underline{W}$ is time-varying, a hypothetical system $\underline{Z}$ is considered having an impulse response matrix

$$z(t,\tau) = A(t)(K_2-K_1)^{-1}\delta(t-\tau) + A(t)w_1(t,\tau) \qquad (2)$$

where $\delta(t)$ is the delta function, $w_1(t,\tau)$ is the impulse response of the system $\underline{W}$ consisting of the subsystem $\underline{W}$ with feedback K_1, and

(Z1') $A(t) = \text{diag.}\{a_1(t), a_2(t), \ldots, a_n(t)\}$, and is bounded above and below by positive definite diagonal matrices.

The criterion for establishing stability for the case when the subsystem $\underline{W}$ is time-varying is as follows.

(Z2') $[z(t,\tau) + z'(\tau,t)]$ is a covariance for some $A(t)$ satisfying (Z1').

We note that (Z2') is equivalent to the condition

$$\int_{T_1}^{T_2}\int_{T_1}^{t^+} q'(t)z(t,\tau)q(\tau)d\tau dt \geq 0 \qquad (3)$$

where $q(\cdot)$ is an arbitrary vector function defined over $[T_1, T_2]$ with T_1 and T_2 arbitrary.

<u>All the statements for the time-varying case should be recognized as appropriate generalizations of corresponding statements for the time-invariant case.</u>

The next section considers in detail the development of the stability criteria for the finite-dimensional case and indicates a procedure for the construction of Lyapunov functions. The distributed parameter case is considered in section 3.

2. FINITE DIMENSIONAL SYSTEMS

(i) Subsystem $\underline{W}$ time-invariant

Representing the subsystem $\underline{W}$ by the state space equations (of minimal dimension p)

$$\dot{x} = Fx + Gu; \quad y = H'x \qquad (4)$$

where u is the input vector and is given by $u(t) = -K(t)y(t)$, the following equations may be written for the transfer function matrices $W(s)$ and $Z(s)$:

$$W(s) = H'(sI_p - F)^{-1}G \qquad (5)$$

$$Z(s) = A(K_2-K_1)^{-1} + AH'(sI_p - F + GK_1H')^{-1}G \qquad (6)$$

where I_p is the unit matrix of order p.

The minimality of the dimension of F implies the complete controllability of the pair $[F, G]$ and the complete observability of the pair $[F, H]$; it is straightforward to then verify that $[F-GK_1H',G]$ is completely controllable, and $[F-GK_1H',HA]$ is completely observable. If now $Z(s)$ is positive real, (i.e. (Z2) is satisfied), a system theory criterion for positive real matrices [7] may be applied to the minimal dimension quadruple $\{F-GK_1H', G, HA, A(K_2-K_1)^{-1}\}$ to indicate the existence of a positive definite symmetric P, and matrices L and W_0 such that

$$P(F-GK_1H') + (F'-HK_1G')P = -LL' \qquad (7a)$$

$$PG = HA - LW_0 \qquad (7b)$$

$$W_0'W_0 = 2A(K_2-K_1)^{-1} \qquad (7c)$$

Consider now as a tentative Lyapunov function for the system $\underline{S}$,

$$V(x) = x'Px \qquad (8)$$

On differentiating (8), the appropriate substitutions of (4) and (7) and a rearrangement of terms yields

$$\dot{V}(x) = -[L'x-W_0(K-K_1)y]'[L'x-W_0(K-K_1)y] - 2y'(K-K_1)A(K_2-K_1)^{-1}(K_2-K)y \qquad (9)$$

We observe that conditions (S1), (Z1) and (Z2) ensure that the various terms are such that $V(x)$ is in fact a suitable Lyapunov function.

A relative stability result follows if for some real negative σ_0, $Z(s + \sigma_0)$ rather than $Z(s)$ is positive real. Replacement of $Z(s)$ by $Z(s + \sigma_0)$ in (6) is equivalent to replacing F by $F-\sigma_0 I$; the system theory criterion still applies, with F in (7) replaced by $F-\sigma_0 I$ and one finds with $V(x)$ as in (8); $\dot{V}(x)$ is as in (9) save for the addition of a term $2\sigma_0 x'Px$, or $2\sigma_0 V$. We conclude:

Stability Criterion 1 For the system $\underline{S}$ of Figure 1 with the transfer function matrix $W(s)$ of the subsystem $\underline{W}$ satisfying $W(\infty) = 0$ and the time-varying element gains satisfying (S1), if a matrix A satisfying (Z1) can be found such that for $\sigma_0 = 0$ [for some real constant $\sigma_0 < 0$]

$$Z(s + \sigma_0) = A(K_2-K_1)^{-1} + AW(s+\sigma_0)[I+K_1W(s+\sigma_0)]^{-1} \qquad (10)$$

is positive real, then the system is stable in the sense of Lyapunov [asymptotically stable in the sense of Lyapunov], and possesses a Lyapunov function $V(x)$ of the form $V = x'Px$ with P a positive definite symmetric matrix, [and $\dot{V}/V \leq 2\sigma_0$].

For the case when only one time-varying element exists in the system $\underline{S}$ (i.e. $n=1$), the above criterion reduces to the "Circle Criterion" which has graphical interpretations on the complex plane [4, 10] and parameter plane [11].

(ii) Subsystem $\underline{W}$ time-varying

Representing the system $\underline{S}$ by the state space equations (of minimal dimension p)

$$\dot{x} = (F-GKH')x; \quad y = H'x \qquad (11)$$

where F, G and H' may be time-varying, the following assumption is made.

(S2) F, G and H are bounded.

The impulse response matrix $w(t,\tau)$ of the subsystem $\underline{W}$ is given by

$$w(t,\tau) = H'(t)\phi(t,\tau)G(\tau)\xi 1(t-\tau) \qquad (12)$$

where

$$\dot{\phi}(t,\tau) = F\phi(t,\tau); \quad \phi(\tau,\tau) = I_n \qquad (13)$$

and $1(t)$ is the unit step function at time t. This means that the impulse response $z(t,\tau)$ of the hypothetical system $\underline{Z}$ may be written as

$$z(t,\tau) = \tfrac{1}{2}R_z\delta(t-\tau) + H_z'(t)\phi_z(t,\tau)G(\tau)1(t-\tau) \qquad (14)$$

where

$$\dot{\phi}_z(t,\tau) = (F-GK_1H')\phi_z(t,\tau); \quad \phi_z(\tau,\tau) = I_n$$
$$R_z = 2A(K_2-K_1)^{-1}; \qquad H_z = HA \qquad (15)$$

Further restrictions are now made. It is assumed that <u>either</u>

(S3) $[F, H']$ is uniformly completely observable

and

(Z4) $[z(t,\tau) + z'(\tau,t) - 2\eta I_n\delta(t-\tau)]$ is a covariance for some positive η over $[t_0,\infty)$.

<u>or</u>

(S4) A selection of F, G and H be made over an interval $[t_0-T_0,t_0]$ such that $[z(t,\tau) + z'(\tau,t)]$ is a covariance or $[t_0-T_0,\infty)$ and that $[F, G]$ is uniformly completely controllable on $[t_0,\infty)$.

A further condition is now stated for convenience

(S5) $(F-GK_1H')$ is asymptotically stable.

With (S1) and (S2) satisfied and $A(\cdot)$ chosen such that (Z1)', (S3) and <u>either</u> (Z4) and (S5) <u>or</u> (S4) are satisfied, then a recent control theory result [9] may be used to define a matrix function $\bar{P}(t)$ for $t_0 \leq t < \infty$ such that

$$0 < \alpha_1 I_n \leq \bar{P}(t) < \alpha_2 I_n < \infty; \quad \bar{P} = \bar{P}' \qquad (16)$$

for some positive α_1 and α_2, where the time derivative of $\bar{P}(t)$ is given as

$$-\dot{\bar{P}} = \bar{P}(F-GR_z^{-1}H_z') + (F'-H_zR_z^{-1}G')\bar{P} + \bar{P}GR_z^{-1}G'\bar{P} + H_zR_z^{-1}H_z' \qquad (17)$$

for all t. The value of $\bar{P}(t)$ satisfying (17) for all t is given as

$$\bar{P}(t) = \lim_{t_1\to\infty} \Pi(t,t_1) \qquad (18)$$

where $\Pi(t,t_1)$ is the solution of

$$-\dot{\Pi} = \Pi(F-GR_z^{-1}H_z') + (F'-H_zR_z^{-1}G')\Pi + \Pi GR_z^{-1}G'\Pi + H_zR_z^{-1}H_z' \qquad (19a)$$

$$\Pi(t_1,t_1) = 0 \qquad (19b)$$

Consider now as a tentative Lyapunov function for the system $\underline{S}$

$$\bar{V}(x,t) = x'\bar{P}(t)x \qquad (20)$$

Differentiating (20), the appropriate substituting of (11), (15) and (17) and a rearrangement of terms yields

$$\dot{\bar{V}} = -[R_z^{-1}(H_z'-G'\bar{P})x - (K-K_1)y]'R_z[R_z^{-1}(H_z'-G'\bar{P})x - (K-K_1)y] - 2y'(K-K_1)A(K_2-K_1)^{-1}(K_2-K)y \qquad (21)$$

We observe that conditions (S1), (S2), (Z1′) and (Z2′) ensure that (16) is satisfied and that the various terms in (20) and (21) are bounded with the appropriate sign. Thus $\bar{V}(x,t)$ of (20) is in fact a Lyapunov function.

For relative stability, it is assumed that $z(t,\tau)$ in (Z2′) is no longer defined via (14) and (15), but by (14) and (15) with F replaced by $F-\sigma_0 I$ for some negative real σ_0. It is also assumed that (S3) holds with F replaced by $F-\sigma_0 I$. Then $\bar{V}(x,t)$ is defined as $x'\bar{P}(t)x$, with $\bar{P}$ being found from the sequence of equations (17) through (19b) with F replaced by $F-\sigma_0 I$. In computing $\dot{V}$, it is found that an additional term $2\sigma_0 x'Px$ appears in the right side of (21). We conclude:

> <u>Stability Criterion 2</u> For the system $\underline{S}$ of Figure 1 with state space equations (11) satisfying (S1) and (S2), if for some $A(\cdot)$ conditions (Z1′), (S3) (see (14) and (15) for $z(t,\tau)$) and <u>either</u> (Z4) and (S5) <u>or</u> (S4) are satisfied [with F replaced by $(F-\sigma_0 I_n)$ for some *nonpositive real* constant σ_0 in (S5), (14) and (15)], then the system is [asymptotically] stable in the *sense of Lyapunov and possesses a* Lyapunov function $\bar{V}(x,t)$ of the form $\bar{V}(x,t) = x'\bar{P}(t)x$, with $\bar{P}(\cdot)$ suitably constrained [and $\dot{V}/V \le 2\sigma_0$].

We note that in the case when F, G, H and A are constant, then $\bar{P} = P$ and the above criterion corresponds to Stability Criterion 1. In fact a convenient method to construct P for the time-invariant case is to use (18) and (19).

3. DISTRIBUTED PARAMETER SYSTEMS

In this section, we shall explicitly prove a general relative stability result for time-varying systems. Appropriate specializations of the result will yield results for time-invariant systems, and ordinary stability results.

It proves convenient to introduce a variation of (S1):

> (S1)′ $K_1 + \varepsilon I \le K(t) \le K_2 - \varepsilon I$ for some positive .

Stability for a distributed parameter system can only refer to properties of the system outputs, and accordingly we shall examine boundedness and square integrability properties of the output. The following generalizations of (S2) and (S5) are required, with $\underline{W}_1$ denoting $\underline{W}$ with feedback K_1:

> (S2)′ For some real constant $\sigma_0 < 0$ and for any initial condition the zero input response $y_0(t)$ of $\underline{W}$ is such that $\exp(-\sigma_0 t)y_0(t)$ is bounded and square integrable

> (S5)′ The impulse response matrix $w_1(t,\tau)$ of $\underline{W}_1$ satisfies $||w_1(t,\tau)|| < \alpha_3 \exp[-\alpha_4(t-\tau)]$, *for all* $t \ge \tau$, and for some positive α_3 and $\alpha_4 + \sigma_0$, with σ_0 as in (S2)′.

Of course, σ_0 in (S2)′ and (S5)′ measures the relative stability of $\underline{W}_1$. Thus in (S2)′, boundedness of $\exp(-\sigma_0 t)y_0(t)$ means that $y_0(t)$ must decay at least as fast as $\exp(\sigma_0 t)$. Also, taking $\sigma_0 = 0$ corresponds to considering ordinary stability rather than asymptotic stability.

In precise terms, $y(t)$ for an arbitrary initial state will be shown to have the properties $\exp(-\sigma_0 t)y(t)$ is bounded and square integrable, given that

> (Z2″) $\exp(-\sigma_0 t)z(t,\tau)\exp(\sigma_0\tau) + \exp(\sigma_0 t)z'(\tau,t)\exp(-\sigma_0\tau)$ is a covariance, where $z(t,\tau)$ is given by (2), and $A(\cdot)$ satisfies (Z1′).

Let the initial state of system $\underline{W}_1$ (at time t_0) be such that its zero input response is $y_0(t)$; *then the response of system* $\underline{S}$

$$y(t) = y_0(t) - \int_{t_0}^{t} w_1(t,\tau)[K(t)-K_1]y(\tau)d\tau \tag{22}$$

This equation, together with (2), gives the following:

$$\int_{t_0}^{t_1}\int_{t}^{t+} y'(t)[K(t)-K_1]\exp(-\sigma_0 t)$$
$$\times [\exp(-\sigma_0\tau)z(t,\tau)\exp(\sigma_0\tau)]$$
$$\times \exp(-\sigma_0\tau)[K(\tau)-K_1]y(\tau)d\tau dt$$
$$= \int_{t_0}^{t_1} y'(t)[K(t)-K_1]\exp(-2\sigma_0 t)A(t)$$
$$\times \{(K_2-K_1)^{-1}[K(t)-K_1] - I\}y(t)dt$$
$$+ \int_{t_0}^{t_1} y'(t)[K(t)-K_1]$$
$$\times \exp(-2\sigma_0 t)A(t)y_0(t)dt \qquad (23)$$

For convenience this equation is written $I_1 = I_2 + I_3$ where each I_j corresponds to the appropriate integral in (23). The application of (S1) and (Z1') to I_2 yields

$$I_2 \le -\xi_2\int_{t_0}^{t_1} y'(t)\exp(-2\sigma_0 t)y(t)dt \qquad (24)$$

where ξ_2 is a positive constant independent of t_1. The integral I_3 may be examined using the Cauchy-Schwarz inequality which yields

$$|I_3| \le (\int_{t_0}^{t_1} y'(t)[K(t)-K_1]^2$$
$$\times \exp(-2\sigma_0 t)A^2(t)y(t)dt)^{1/2}$$
$$(\int_{t_0}^{t_1} y_0(t)\exp(-2\sigma_0 t)y_0(t))^{1/2} \qquad (25)$$
$$\le \xi_3(\int_{t_0}^{t_1} y'(t)\exp(-2\sigma_0 t)y(t)dt)^{1/2} \qquad (26)$$

where ξ_3 is a positive constant independent of t_1, existing by (S1)', (S2)' and (Z1'). Now observe that if $\exp(-\sigma_0 t)y(t)$ is not square integrable, (24) and (26) imply that $I_2 + I_3$ diverges to $-\infty$. But I_1 is guaranteed nonnegative by condition (Z2''). [In (3), replace $z(t,\tau)$ by $\exp(-\sigma_0 t)z(t,\tau)\exp(\sigma_0\tau)$ and $q(t)$ by $\exp(-\sigma_0 t)[K(t)-K_1]y(t)$]. Hence $\exp(-\sigma_0 t)y(t)$ is square integrable.

Using the square integrability of $\exp(-\sigma_0 t)y(t)$, together with (S5)' and (22) it is not difficult to show that $\exp(-\sigma_0 t)y(t)$ is also bounded. From (22)

$$||\exp(-\sigma_0 t)y(t)|| \le ||\exp(-\sigma_0 t)y_0(t)||$$
$$+ \exp(-\sigma_0 t)$$
$$\int_{t_0}^{t} ||w_1(t,\tau)||\;||[K(\tau)-K_1]y(\tau)||d\tau \qquad (27)$$

Using (S5),

$$||\exp(-\sigma_0 t)y(t)|| \le ||\exp(-\sigma_0 t)y_0(t)||$$
$$+ \alpha_3\int_{t_0}^{t} \exp[-(\alpha_4+\sigma_0)t + \alpha_4\tau]$$
$$\times ||[K(\tau)-K_1]y(\tau)||d\tau \qquad (28)$$
$$= ||\exp(-\sigma_0 t)y_0(t)||$$
$$+ (K_2-K_1-\varepsilon I)\alpha_3\int_{t_0}^{t} \exp[-(\alpha_4+\sigma_0)(t-\tau)]$$
$$\times ||\exp(-\sigma_0\tau)y(\tau)||d\tau \qquad (29)$$

The Cauchy-Schwarz inequality may be applied to the integral in (29), and it is found that the square integrability of $\exp(-\sigma_0 t)y(t)$ and the latter part of *condition* (S5)' (*guaranteeing the* positivity of $\alpha_4 + \sigma_0$) imply that the integral is uniformly bounded for all t.

The above results have thus established the following criterion.

Stability Criterion 3 Systems S satisfying (S1)′, (S2)′ and (S5)′ have an output $y(t)$ such that $\exp(-\sigma_0 t)y(t)$ is bounded and square integrable on $[t_0,\infty)$ for any initial conditions and initial time t_0 provided (Z2′′) is satisfied.

An important special case viz. the case when the subsystem W is time-invariant is considered in the following criterion.

Stability Criterion 4 Systems S satisfying (S1)′, with the subsystem W time-invariant and satisfying (S2)′ and (S5)′ have an output $y(t)$ such that $\exp(-\sigma_0 t)y(t)$ is bounded and square integrable on $[t_0,\infty)$ for any initial conditions and initial time t_0, provided $Z(s + \sigma_0)$ [see (1)] is positive real for some A satisfying (Z1).

4. CONCLUSIONS

It is interesting to consider how the preceding results arise naturally from versions of the circle criterion applicable to single-input, single-output, time-invariant and finite-dimensional systems.

The first generalization is to multi-loop systems, and it will be recognized that the criterion for positive real matrices of [7] is fundamental in carrying out the generalization; it will also be recognized that this criterion does not distinguish greatly between positive real functions and positive real matrices. Further, the subsequent extension to relative stability is a natural one.

Turning then to time-varying systems, it is evident that the criterion for positive real matrices used for time-invariant systems is, as such, inapplicable. But optimal control does indicate a related criterion, which actually reduces to the positive real criterion in stationary cases; this related criterion yields the stability results. Relative stability extensions are again straightforward.

Guided by results for finite-dimensional systems, it becomes possible to suggest the covariance condition (Z2′′) as being the appropriate stability criterion for distributed parameter systems. [Note that, with $z(t,\tau)$ time-invariant, $\exp(-\sigma_0 t)z(t,\tau)\exp(\sigma_0\tau)$ is $\exp[-\sigma_0(t-\tau)]z(t-\tau)$, and the Laplace transform of $\exp(-\sigma_0 t)z(t)$ is $Z(s + \sigma_0)$]. The remaining conditions of section 3 are imposed to convert the Lyapunov stability problem to a stability problem that is more meaningful for distributed parameter systems.

5. REFERENCES

[1] J. B. Moore and B.D.O. Anderson, "A Generalization of the Popov Criterion," Journal of the Franklin Institute, to appear.

[2] J. B. Moore and B.D.O. Anderson, "Stability Criterion for Time-Varying Systems containing Memoryless Nonlinearities", Technical Report EE-6705, Department of Electrical Engineering, University of Newcastle, New South Wales, 2308, Australia, July 1967.

[3] J. B. Moore, "Stability of Linear Dynamical Systems with Memoryless Nonlinearities", International Journal of Control, to appear.

[4] J. J. Bongiorno, "An Extension of the Nyquist-Barkhausen Stability Criterion to Linear Lumped Parameter Systems with Time-Varying Elements," IEEE Transactions on Automatic Control, Vol. AC-8, No. 2, April 1963, pp. 166-172.

[5] K. S. Narendra and R. M. Goldwyn, "A Geometrical Criterion for the Stability of Certain Nonlinear Nonautonomous Systems," IEEE Transactions on Circuit Theory, Vol. CT-11, No. 3, September 1964, pp. 406-408.

[6] R. W. Brockett and H. B. Lee, "Frequency Domain Instability Criteria for Time-Varying and Nonlinear Systems," Proceedings of the IEEE, Vol. 55, No. 5, May 1967, pp. 604-619.

[7] B.D.O. Anderson, "A System Theory Criterion for Positive Real Matrices," SIAM Journal of Control, Vol. 5, No. 2, May 1967, pp. 171-182.

[8] J. B. Moore and B.D.O. Anderson, "Extension of Quadratic Minimization Theory, I: Finite Time Results," to be published.

[9] B.D.O. Anderson and J. B. Moore, "Extensions of Quadratic Minimization Theory, II: Infinite Time Results," to be published.

[10] J. B. Moore, "A Circle Criterion Generalization for 'Relative Stability' ", IEEE Transactions on Automatic Control, February 1968.

[11] J. B. Moore, "Circle Criteria in the Parameter Plane," Proceedings IEE, to appear.

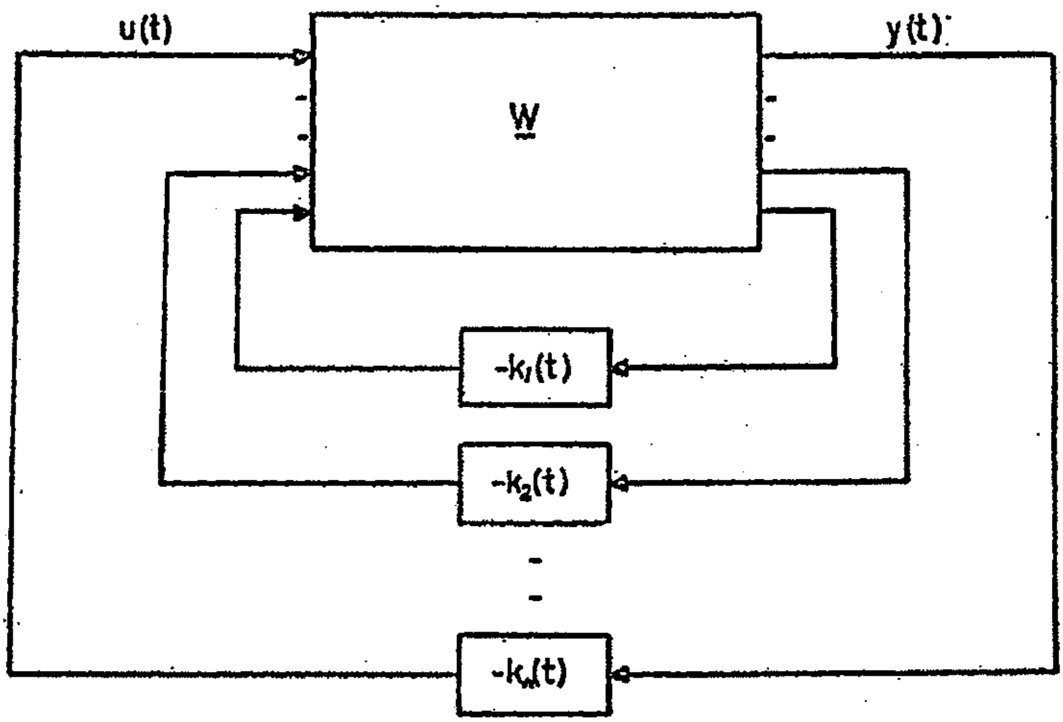

FIGURE 1. SYSTEM S