THERMAL NOISE BEHAVIOR OF A NONLINEAR BRIDGE CIRCUIT

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Abstract – **It is well known that the thermal noise behavior at the terminals of any LTI RLC circuit can be predicted from knowledge of the driving-point impedance and temperature alone. This paper offers the conjecture that similar results hold if the capacitors and inductors are nonlinear. We refine the conjecture by analyzing the behavior of an RLC bridge circuit with the nonlinear inductor and capacitor carefully matched so the terminal behavior reduces to that of a linear resistor R. We show that the terminal noise current is precisely that predicted by the Nyquist-Johnson model for R if the driving voltage is zero or constant, but not if the driving voltage is time-dependent or the inductor and capacitor are time-varying. This paper makes exact calculations using techniques from stochastic differential equations and using reversibility arguments.**

1. INTRODUCTION

Consider the bridge circuit of Figure 1. It is a standard result of linear circuit theory that under the matching condition $L = R^2 C$, the driving-point impedance reduces to R and the natural frequency of the circuit does not appear as a pole [1, 2]. Regardless of the values of the capacitor and inductor, for high frequencies, the capacitor is essentially a short circuit, whereas the inductor is essentially an open circuit; at low frequencies, the opposite occurs. The matching condition ensures that a balance is preserved for intermediate frequencies: the charging of the capacitor is matched by the fluxing of the inductor. In the language of control theory, the state equations become *nonminimal* in the matched case.

1.1. The LTI Case

It is straightforward to verify directly in the LTI case that if a Nyquist-Johnson noise model [3, 4] is associated with each resistor, then the spectrum of the short-circuit terminal current in a matched bridge circuit is also that of a Nyquist-Johnson noise model for a single resistor of value R. The verification can be done by standard frequency-domain techniques or by stochastic calculus [5]. The highpass filtering of the RC branch is precisely balanced by the lowpass filtering of the RL branch, so that the terminal noise spectrum is flat. Of course, both resistors must be at the same temperature. As noted in [2], applying a d.c. voltage to the circuit would result in differential heating of the resistor in the RL branch. If the resistors were not properly connected to thermal reservoirs, one could heat up and become noisier than the other, and the noise spectrum would no longer be flat. This is a trivial nonequilibrium exception to the results of this paper, which assumes uniform, constant temperature.

The result above is a particular example of a general circuit theory result, namely, that a one-port network of LTI passive elements with port admittance $Y(j\omega)$ presents a short-circuit thermal noise current with power spectrum $2kT\,\mathrm{Re}\{Y(j\omega)\}$, where k is Boltzmann's constant and T is the absolute temperature [6]. Physicists regard such results as particular cases of the fluctuation-dissipation theorem [7].

1.2. Generalizations in this Paper

This paper studies one carefully-chosen example, motivated by the question of whether some form of fluctuation-dissipation theorem holds for some class of nonlinear circuits. Our initial formulation appears below as a conjecture for any pair of two-terminal networks, each comprising an interconnection of LTI resistors at a uniform, constant temperature, described by the Nyquist-Johnson model, and possibly also capacitors and inductors that may be nonlinear or time-varying. Two such networks are said to be *zero-state deterministically equivalent* if every applied terminal voltage waveform $v(t), t \geq 0$, produces the same current response $i(t)$ from both networks, provided all capacitor voltages and inductor currents are initially zero and all noise sources in the resistor models are set to zero. (In the LTI case this just means the two input admittances are identical.)

> *Preliminary Fluctuation-Dissipation Conjecture for Networks:*
> No two zero-state deterministically equivalent networks can be distinguished by their terminal noise current responses to any applied voltage waveform.

The conjecture just hypothesizes that the deterministic terminal behavior uniquely determines the noise current response for all voltage drives, independent of the details of the network. The conjecture is true in the LTI case. (Closely related formulations for the current-driven and multiport cases [6] also hold true for LTI networks, but we ignore them here for simplicity.) However, an examination of the bridge circuit will show that this conjecture is wrong in other cases and must be revised.

This paper considers only the Nyquist-Johnson model for noise in a linear resistor. That model does not assume any knowledge of

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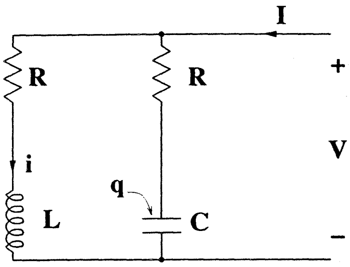

Figure 1: Linear noise-free bridge circuit is matched and has input impedance R if $L = R^2C$.

the deterministic current flow mechanism. The results of this paper disprove the existence of such "black-box" noise models for systems with internal nonlinearities when the nonlinearities are in the lossless subsections.

2. NONLINEAR, NOISE-FREE CASE

Consider the circuit of Figure 2, but with the current noise sources turned off. Of course, $R > 0$. In addition, we constrain the nonlinear elements as follows: The mappings $h : \phi \to i_L$ and $f : q \to v_C$ obey

(i) $h(0) = 0, f(0) = 0$

(ii) h and f are continuously differentiable functions, and for all values of the arguments and some fixed $\epsilon > 0$, there holds

$$\frac{dh}{d\phi} \geq \epsilon > 0 \quad \text{and} \quad \frac{df}{dq} \geq \epsilon > 0.$$

These constraints, drawn essentially from [8, 9], ensure that the circuit is passive (for positive resistors), and that $(q, \phi) = (0, 0)$ is a globally asymptotically stable equilibrium point for $V = 0$.

As noted in Section 1, in the linear case the condition $L = R^2C$ ensures that the bridge appears as a simple linear resistor at its terminals. The generalization of this condition is found in the following condition relating the two nonlinearities.

2.1. Matching Condition for the Nonlinear Bridge

Consider the circuit of Figure 2, with the mappings constrained as above and the noise sources turned off. Suppose the circuit is in the zero state at $t = 0$ and is excited by a voltage $V(t)$ for $t > 0$. Then for all $V(t)$ there holds $V(t) = R\,I(t)$ for all $t \geq 0$, if and only if

$$f(q) = R\,h(Rq) \tag{1}$$

for all values of q.

Remark: Since $f'(q) = 1/C(q)$ is the reciprocal of the incremental capacitance and $h'(\phi) = 1/L(\phi)$ is the reciprocal of the incremental inductance, then Eq. (1) implies $L(\phi) = R^2C(q)|_{q=\phi/R}$, a local version of the linear matching condition $L = R^2C$.

Remark: The above result is almost certainly not novel. However, we are unaware of a reference.

Reference [10] presents this result as a theorem and gives a complete proof.

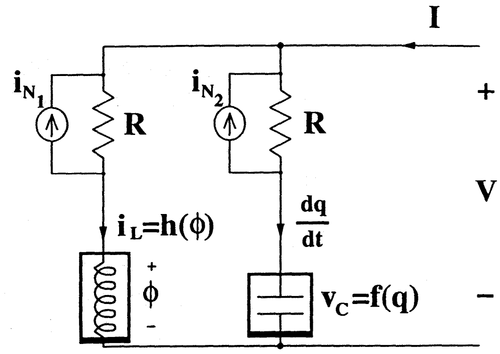

Figure 2: Nonlinear bridge circuit with Nyquist-Johnson noise sources.

3. NONLINEAR, NOISY CASE: SUCCESSFUL RESULTS

For this section, a Norton-form Nyquist-Johnson noise model is associated with each resistor in the circuit, as in Figure 2. We would like to show that the terminal current noise of the matched bridge is the same as that for a single linear resistor, when V is constant and the circuit is in steady-state. To first order, this result is clear. Recall that the incremental capacitance and inductance satisfy $L(\phi) = R^2\,C(q)|_{q=\phi/R}$. A linearization about the noise-free equilibrium operating point (q, ϕ) for a d.c. applied voltage of a nonlinear matched circuit will yield a matched linear circuit. By superposition, the noise current for the linearized circuit is unaffected by the applied voltage. The point of this section is to show that this equivalence holds *exactly*, even for high temperatures or strong nonlinearities for which the noise could drive the circuit out of the valid region of linearization.

The circuit is described by stochastic differential equations (SDE's):

$$\frac{dq}{dt} = \frac{V - f(q)}{R} - i_{N_2} \tag{2}$$

$$\frac{d\phi}{dt} = V - R\,h(\phi) - R\,i_{N_1}, \tag{3}$$

where i_{N_1} and i_{N_2} are independent Gaussian white noise processes with power spectral density $2kT/R$. The port current is

$$I = h(\phi) + \frac{dq}{dt} = h(\phi) + \frac{V}{R} - \frac{f(q)}{R} - i_{N_2}. \tag{4}$$

3.1. The $I(t) - V(t)$ relation in the presence of noise

Before proceeding to study the noise power spectrum, we show that the nonlinear inductor and capacitor cannot "rectify" the noise, even with a time-varying $V(t)$. Rectification would cause incorrect "average" behavior, or first-order statistics of the circuit, such that it would be pointless to study the second-order statistic of the power spectral density.

Taking expectations on both sides of Eq. (4),

$$E\{I\} = E\{h(\phi)\} + \frac{V(t)}{R} - \frac{E\{f(q)\}}{R} - 0. \tag{5}$$

In order to compute the expectations of $f(q)$ and $h(\phi)$, we need to know something about the probability densities ρ for q and ϕ. The Fokker-Planck equations [11, 12] for the evolutions of these densities are, for Eqs. (2) and (3), respectively,

$$\frac{\partial \rho_q}{\partial t} = -\frac{\partial}{\partial q}\left[\frac{V(t)-f(q)}{R}\rho_q\right] + \frac{kT}{R}\frac{\partial^2 \rho_q}{\partial q^2} \quad (6)$$

$$\frac{\partial \rho_\phi}{\partial t} = -\frac{\partial}{\partial \phi}\left[\Big(V(t) - R\,h(\phi)\Big)\rho_\phi\right] + kTR\,\frac{\partial^2 \rho_\phi}{\partial \phi^2}. \quad (7)$$

Using the matching condition (1), these two equations become identical up to a scaling. The reader can verify that a density $\rho_\phi(\phi, t)$ satisfies Eq. (7) if and only if the scaled version

$$\rho_q(q,t) = R\,\rho_\phi(R\,q,t) \quad (8)$$

satisfies Eq. (6). The densities corresponding to zero initial conditions (delta functions) also satisfy Eq. (8) at $t = 0$. Thus, the solutions of Eqs. (6) and (7) satisfy Eq. (8) for all time, and it follows by direct calculation that

$$E\{f(q(t))\} = R\,E\{h(\phi(t))\}, \quad t \geq 0. \quad (9)$$

Substituting Eq. (9) into Eq. (5) shows immediately that

$$E\{I(t)\} = \frac{V(t)}{R}. \quad (10)$$

More details are given in [5].

3.2. Terminal noise current power spectrum

Consider the circuit of Figure 2, described by Eqs. (2) to (4), with the nonlinear functions $f(\cdot)$ and $h(\cdot)$ obeying the constraints of Section 2 and the the matching condition (1). Now, let $V(t) \equiv V$ be constant, and assume the circuit is in steady-state at $t = 0$. Denote by R_{nn} the autocorrelation of the terminal noise current $n(t) = I(t) - V/R$. Then for $t, \tau > 0$,

$$R_{nn}(t-\tau) = \frac{2kT}{R}\,\delta(t-\tau). \quad (11)$$

The further result that $\int_0^t n(s)\,ds$ is a scaled Wiener process, *i.e.*, that $n(t)$ is "Gaussian white noise," is proven in [10].

Remark: Of course, the same autocorrelation may be obtained for arbitrary $V(t)$ if bridge circuit is LTI; this follows directly from superposition.

The autocorrelation of $n(t)$ is, from Eq. (4),

$$R_{nn}(t,\tau) = E\left\{\left[h(\phi(t)) - \frac{f(q(t))}{R} - i_{N_2}(t)\right] \times \left[h(\phi(\tau)) - \frac{f(q(\tau))}{R} - i_{N_2}(\tau)\right]\right\}.$$

By independence and zero mean of the driving currents i_{N_1} and i_{N_2} and through the above-mentioned similarity of the Fokker-Planck equations (6) and (7), the autocorrelation can be simplified to

$$R_{nn}(t,\tau) = 2E\left\{\frac{f(q(t))}{R}\,\frac{f(q(\tau))}{R}\right\} - 2E\left\{\frac{f(q(t))}{R}\right\}E\left\{\frac{f(q(\tau))}{R}\right\} + E\left\{\frac{f(q(t))}{R}\,i_{N_2}(\tau)\right\} + E\left\{i_{N_2}(t)\,\frac{f(q(\tau))}{R}\right\} + E\Big\{i_{N_2}(t)\,i_{N_2}(\tau)\Big\}. \quad (12)$$

The key to reducing this equation further (we want to eliminate all but the last line) is the following step. If we multiply both sides of the differential equation (2) for $q(t)$ by $f(q(\tau))$ and take expectations, we obtain

$$\frac{d}{dt}E\Big\{f(q(\tau))\,q(t)\Big\} = \frac{V}{R}\,E\Big\{f(q(\tau))\Big\} - \frac{1}{R}E\Big\{f(q(t))\,f(q(\tau))\Big\} - E\Big\{i_{N_2}(t)\,f(q(\tau))\Big\}. \quad (13)$$

Call this $F(t,\tau)$. The dummy time indices t and τ may be interchanged, corresponding to writing the SDE in τ and multiplying through by $f(q(t))$, to get $F(\tau, t)$.

Since V is constant and the system is initially at steady-state, it remains in steady-state for $t \geq 0$, *i.e.*, $q(t)$ and $\phi(t)$ are *stationary* random processes. Taking expectations of both sides of the differential equation (2),

$$E\left\{\frac{dq}{dt}\right\} = 0 = E\left\{\frac{V - f(q(t))}{R}\right\} + E\Big\{i_{N_2}(t)\Big\},$$

so that

$$V = E\big\{f(q(t))\big\}.$$

Since $q(t)$ is stationary, $F(t,\tau) = F(t-\tau)$ depends only on the difference $(t-\tau)$. Further, a consequence of Eq. (2) and the constraints on the nonlinear constitutive relations of Sect. 2 is that $q(t)$ is a *reversible* process [13], *i.e.*, for all t_1 and t_2,

$$\Pr[\alpha \leq q(t_1) \leq \alpha + d\alpha, \beta \leq q(t_2) \leq \beta + d\beta] = \Pr[\beta \leq q(t_1) \leq \beta + d\beta, \alpha \leq q(t_2) \leq \alpha + d\alpha].$$

As a consequence of reversibility, F is an even function, $F(t-\tau) = F(\tau - t)$, and hence $F'(\cdot)$ must be odd. After several steps of algebra using all these pieces, the autocorrelation reduces to

$$R_{nn}(t,\tau) = E\Big\{i_{N_2}(t)\,i_{N_2}(\tau)\Big\} = \frac{2kT}{R}\,\delta(t-\tau).$$

It is perhaps somewhat surprising that this analysis holds exactly. There are two noise sources driving nonlinear elements, so one might expect a nonlinear "mixing" under which the two drives interact to produce a colored noise spectrum, but this does not happen in this circuit.

4. FAILURES OF THE CONJECTURE

As mentioned in the introduction, there are some situations in which the noise current of the matched bridge circuit is not statistically equivalent to the noise of a single linear resistor. Even if the circuit is kept at constant temperature, the conjecture fails for a time-varying circuit. This failure casts doubts on the hopes of establishing the general nonlinear nonequilibrium result for a time-varying driving voltage.

Suppose the energy storage elements in Figure 2 are linear, but time varying. This will provide the first nontrivial failure of the fluctuation-dissipation hypothesis in the Introduction; it is sufficient to consider the short-circuit (undriven) behavior. The circuit differential equations are

$$\frac{d\phi}{dt} = -\frac{R\,\phi(t)}{L(t)} - R\,i_{N_1}(t) \tag{14}$$

$$\frac{dq}{dt} = -\frac{q(t)}{R\,C(t)} - i_{N_2}(t), \tag{15}$$

and we assume $E\{q(0)\} = E\{\phi(0)\} = 0$ so that $q(t)$ and $\phi(t)$ are zero mean. The port current is

$$I(t) = \frac{\phi(t)}{L(t)} - \frac{q(t)}{R\,C(t)} - i_{N_2}(t). \tag{16}$$

The corresponding matching condition is of course

$$L(t) = R^2\, C(t). \tag{17}$$

The differential equation for $q(t)$ can be solved explicitly in terms of sample paths of the noise process $i_{N_2}(t)$, and this solution used to calculate the terms of the autocorrelation. Most of these terms vanish because the variables are uncorrelated as argued previously but now also zero-mean, or by causality in that $i_{N_2}(\tau)$ cannot affect $q(t)$ for $\tau > t$. The final result of the calculations is that in order for the autocorrelation to be the correct strength delta-function, the the following necessary condition must hold:

$$0 \stackrel{?}{=} -\frac{dC(t)}{dt}\, kT\, \exp\left[2\int_0^t \frac{ds}{RC(s)}\right].$$

Thus, the time-varying bridge does not have stationary current noise at the terminals as required by the Nyquist-Johnson model, except in the trivial case that C is a constant. If the system starts at equilibrium, *i.e.*, $E\{q^2(0)\} = kTC$, then this condition is sufficient as well as necessary. Of course, if C is a constant, then the bridge is simply the standard linear, time-invariant circuit, for which the result was already known.

Remark: For a driving voltage $V(t)$ significantly larger than the noise, one could solve the deterministic system and then compute an approximation for the noise behavior by linearization about this time-varying solution. This approximation would behave like the time-varying linear system described above. Since the second-order statistics for that system are incorrect, we believe that the second-order statistics for the nonlinear system driven by a time-varying voltage will not match the statistics of a single linear resistor driven by that same voltage.

5. CONCLUSION

The negative results in Section 4 show that our original fluctuation-dissipation conjecture is not correct as stated and must be limited to exclude time-varying networks and nonlinear networks with time-varying inputs. Is the modified form below correct? This remains an open question in the field, and some of the ideas in [13] may be of assistance.

> *Modified Fluctuation-Dissipation Conjecture for Circuits*
> No two zero-state deterministically equivalent *time-invariant* networks can be distinguished by the terminal noise currents *at any d.c. voltage input* when the networks are in statistical steady-state.

The assumptions in Sections 1.2 and 2 remain in effect, including LTI Nyquist-Johnson resistors and nonlinear inductors and capacitors. Additional assumptions may be required to guarantee reversibility of the charge or flux random processes.

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