Proceedings of the 38th
Conference on Decision & Control
Phoenix, Arizona USA • December 1999

ON A NONLINEAR GENERALIZATION OF THE ν-GAP METRIC [1]

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Abstract

This paper presents a nonlinear extension of the ν-gap metric introduced in [5]. Indeed, we present an input-output version of the generalized stability margin and the ν-gap metric. The notion of image representation as presented in [4] allows one to use these alternative definitions in a nonlinear context for the derivation of a robust stability theorem.

Keywords: robust stability, nonlinear, metric, generalized stability margin.

1 Introduction

It has long been observed that the closed-loop behaviour of two systems can be very similar (different) even though the norm of the difference between the open-loop systems can be very large (small). It is to deal with this problem that the gap metric and the ν-gap metric have been introduced as a tool for the study of uncertainty in closed-loop systems; see e.g. [2, 5]. Indeed, perturbations which are small in the gap metric or in the ν-gap metric give rise to small closed-loop errors in a feedback loop. The main advantage of the ν-gap metric is that can easily be interpreted as the distance between two frequency responses. Also, it has been shown in [5] that the ν-gap metric provides sharper results in the sense that it is the smallest metric for which certain robust stability results hold. The Vinnicombe metric has also found an increasing use as an assessment tool for the safe update of a controller in adaptive control problems; see [1] for further details.

It is of course of interest to try to extend the above robust stability results to a nonlinear setting. An input-output approach for the generalization of the gap metric can be found in [3]. However, a suitable generalization of the ν-gap metric is of even greater because of its shaper results and its potential usefulness in nonlinear adaptive control problems. In this paper, we propose an input-output version of the measures that are of crucial importance for establishing the robust stability theorem of [5], namely the generalized stability margin and the ν-gap metric. The advantage of these alternative definitions is that they are valid in a nonlinear framework. Indeed, the notion of image representation as presented in [4] is used in conjunction with these alternative definitions to derive a nonlinear robust stability theorem. Note that some of the results presented in this paper are incomplete. For instance, it is not entirely clear yet if the distance used in this paper is the same as the one used in [5] in a linear context.

We now outline the contents of the paper. In section 2, we recall the definitions of the ν-gap metric (Vinnicombe metric) and the stability margin in a linear context and we give some insights on how to rewrite these definitions in an input-output approach. We also recall the notion of image and kernel representations for a nonlinear system and present some results that will be used in the sequel of the paper. Section 3 presents the main robust stability results. In Section 4, we present a nonlinear extension of the Vinnicombe metric that obeys all the properties of a metric. We conclude in Section 5.

[1] The authors wish to acknowledge the funding of the Cooperative Research Centres for Robust and Adaptive Systems by the Australian Commonwealth Government under the Cooperative Research Centres Program, the US Army Research Office, Far East and the Office of Naval Research, Washington.

2 Background material for a nonlinear metric

In this section, we introduce background material and concepts that will be used in the sequel of the paper.

Linear theory

To motivate the later nonlinear system treatment, we will recall a number of results concerning linear systems; see [5].

Notations: $\mathcal{H}_\infty$ denotes the Hardy space of matrix-valued functions analytic in the open-right-half plane. $\mathcal{L}_2$ is the Hilbert space of matrix-valued functions on $\mathbb{R}$, with inner product

$$< f, g > = \int_{-\infty}^{\infty} \text{trace}[f(t)^* g(t)]\, dt.$$

$\mathcal{L}_{2+} = \mathcal{L}_2[0, \infty]$ is the subspace of $\mathcal{L}_2$ with functions zero for $t < 0$. $\mathcal{L}_{2e^+}$ is the Hilbert space of matrix-valued functions f on $\mathbb{R}$ such that $P_T f \in \mathcal{L}_{2+}$ for all $T > 0$ where $P_T u = u$ for $t < T$ and $P_T u = 0$ for $t > T$.

Vinnicombe metric: Throughout this paper, (N_i, M_i) will denote a normalized right coprime factorization (nrcf) of P_i, and $(\tilde{N}_i, \tilde{M}_i)$ a normalized left coprime factorization (nlcf) of P_i. We will write

$$G_i := \begin{bmatrix} N_i \\ M_i \end{bmatrix}, \quad \tilde{G}_i := [-\tilde{M}_i \ \ \tilde{N}_i]. \tag{2.1}$$

Recall that (N_i, M_i) is nrcf of P_i if, and only if, i) $P_i = N_i M_i^{-1}$, ii) $G_i \in \mathcal{H}_\infty$, iii) there exists an $X \in \mathcal{H}_\infty$ such that $XG_i = I$ (coprimeness condition) and iv) $G_i^* G_i = I$ (normalization condition). Similarly, $(\tilde{N}_i, \tilde{M}_i)$ is a nlcf of P_i if, and only if, i) $P_i = \tilde{M}_i^{-1}\tilde{N}_i$, ii) $\tilde{G}_i \in \mathcal{H}_\infty$, iii) there exists an $Y \in \mathcal{H}_\infty$ such that $\tilde{G}_i Y = I$ (coprimeness condition) and iv) $\tilde{G}_i \tilde{G}_i^* = I$ (normalization condition).

The formula for the ν-gap metric defined in [5] in a Linear Time Invariant (LTI) framework is

$$\delta_\nu(P_1, P_2) := \begin{cases} \|\tilde{G}_2 G_1\|_\infty & \text{if } \det(G_2^* G_1)(j\omega) \neq 0, \ \forall\omega \\ & \text{and wno } \det(G_2^* G_1) = 0 \\ 1 & \text{otherwise,} \end{cases} \tag{2.2}$$

where wno(g) denotes the winding number about the origin of $g(s)$, as s follows the standard Nyquist D-contour.

Stability margin: This paper is concerned with the closed-loop system of Figure 2.1 and its relation to the robust stability and robust performance of feedback systems. Define

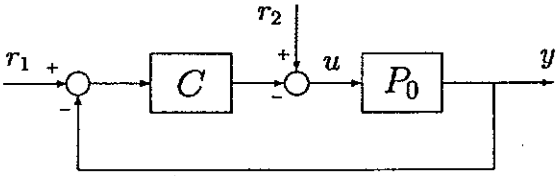

Figure 2.1: Closed-loop system $[P_0, C]$

$$T(P_0, C) = \begin{bmatrix} P_0 \\ I \end{bmatrix} (I - CP_0)^{-1}[-C \ I]. \tag{2.3}$$

In [5] and in a linear context, a generalized stability margin was defined as follows

$$b_{P_0,C} := \begin{cases} \|T(P_0, C)\|_\infty^{-1}, & \text{if } [P_0, C] \text{ is stable} \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Throughout this paper, (N_c, M_c) will denote a normalized right coprime factorization (nrcf) of C, and $(\tilde{N}_c, \tilde{M}_c)$ a normalized left coprime factorization (nlcf) of C. We will write

$$K := \begin{bmatrix} M_c \\ N_c \end{bmatrix}, \quad \tilde{K} := [-\tilde{N}_c \ \ \tilde{M}_c]. \tag{2.5}$$

As noted in [5] if $[P_0, C]$ is stable, then

$$b_{P_0,C} = \inf_\omega \underline{\sigma}[\tilde{K}G_0](j\omega). \tag{2.6}$$

Note that $[P_0, C]$ is stable if, and only if, $\det(\tilde{K}G_0(j\omega)) \neq 0, \forall\omega$ and wno $\det(\tilde{K}G_0) = 0$.

Insights in the linear context

In this subsection, we show that, without taking into account winding number conditions, one can rewrite both the Vinnicombe distance and the stability margin using an input-output approach. It will become clear it the sequel of the paper how to take into account the winding number conditions.

Vinnicombe metric: Using the fact that $[G_2 \ \ \tilde{G}_2^*]$ is unitary, one shows that

$$G_1^* [G_2 \ \ \tilde{G}_2^*] \begin{bmatrix} G_2^* \\ \tilde{G}_2 \end{bmatrix} G_1 = G_1^* [\tilde{G}_2^* \tilde{G}_2 + G_2 G_2^*] G_1 = I.$$

It now follows that

$$\overline{\sigma}^2[\tilde{G}_2 G_1](j\omega) + \underline{\sigma}^2[G_2^* G_1](j\omega) = 1, \quad \forall\omega. \tag{2.7}$$

So we can think of working with

$$\sqrt{1 - \underline{\sigma}^2[G_2^* G_1](j\omega)}$$

instead of $\overline{\sigma}[\tilde{G}_2 G_1](j\omega)$ as suggested by (2.8). Indeed, one has

$$\begin{aligned} \|\tilde{G}_2 G_1\|_\infty &= \sup_\omega \left[\overline{\sigma}[\tilde{G}_2 G_1](j\omega)\right] \\ &= \sup_\omega \left[\sqrt{1 - \underline{\sigma}^2[G_2^* G_1](j\omega)}\right] \\ &= \sqrt{1 - \inf_\omega[\underline{\sigma}^2[G_2^* G_1](j\omega)]} \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} &\inf_\omega[\underline{\sigma}[G_2^* G_1](j\omega)] && (2.9) \\ &= \inf_{\|u\|=1} \|G_2^* G_1 u\| \\ &= \inf_{\|u\|=1} \left\{ \sup_{\|v(u)\|=1} |v(u)^* G_2^* G_1 u| \right\} \\ &= \inf_{\|u\|=1} \left\{ \sup_{\|v(u)\|=1} |\langle G_2 v(u), G_1 u\rangle| \right\} \\ &= \inf_u \left\{ \sup_{v(u)} \frac{|\langle G_2 v(u), G_1 u\rangle|}{\|v(u)\| \, \|u\|} \right\} && (2.10) \\ &= \inf_u \left\{ \sup_{v(u)} \frac{|\langle G_2 v(u), G_1 u\rangle|}{\|G_2 v(u)\| \, \|G_1 u\|} \right\}. && (2.11) \end{aligned}$$

The last equality follows from the normalization assumption on G_1 and G_2. Here u and $v(u)$ are assumed to be in $\mathcal{L}_{2+}$. Note that it is only the shape of $v(u)$ that affects the expressions above, i.e. without loss of generality one can impose the additional constraint

$$\|G_2 v(u)\| = \|G_1 u\|. \tag{2.12}$$

Stability margin: Using the fact that $[K \ \ \tilde{K}^*]$ is unitary, one shows that

$$G_0^* [K \ \ \tilde{K}^*] \begin{bmatrix} K^* \\ \tilde{K} \end{bmatrix} G_0 = G_0^* [\tilde{K}^* \tilde{K} + KK^*] G_0 = I.$$

It now follows that

$$\underline{\sigma}^2[\tilde{K}G_0](j\omega) + \bar{\sigma}^2[K^*G_0](j\omega) = 1, \quad \forall\omega. \tag{2.13}$$

So we can think of working with

$$\sqrt{1 - \bar{\sigma}^2[K^*G_0](j\omega)}$$

instead of $\underline{\sigma}[\tilde{K}G_0](j\omega)$ as suggested by (2.6). Indeed, one has

$$\begin{aligned} \inf_\omega \underline{\sigma}[\tilde{K}G_0](j\omega) &= \inf_\omega \sqrt{1 - \bar{\sigma}^2[K^*G_0](j\omega)} \\ &= \sqrt{1 - \sup_\omega \bar{\sigma}^2[K^*G_0](j\omega)} \end{aligned}$$

where

$$\begin{aligned} &\sup_\omega \bar{\sigma}[K^*G_0](j\omega) && (2.14) \\ &= \sup_{\|u\|=1} \|K^*G_0\, u\| \\ &= \sup_{\|u\|=1} \left\{ \sup_{\|v(u)\|=1} |v(u)^* K^* G_0\, u| \right\} \\ &= \sup_u \left\{ \sup_{v(u)} \frac{|\langle Kv(u), G_0\, u\rangle|}{\|v(u)\|\,\|u\|} \right\}, && (2.15) \\ &= \sup_v \left\{ \sup_{u(v)} \frac{|\langle Kv, G_0\, u(v)\rangle|}{\|v\|\,\|u(v)\|} \right\}, && (2.16) \\ &= \sup_u \left\{ \sup_{v(u)} \frac{|\langle Kv(u), G_0\, u\rangle|}{\|Kv(u)\|\,\|G_0\, u\|} \right\}. && (2.17) \end{aligned}$$

[In the linear case, there is no difference between (2.15)-(2.16).] Here u and $v(u)$ are assumed to be in $\mathcal{L}_{2+}$. Note that it is again only the shape of $v(u)$ that affects the expressions above, i.e. without loss of generality one can impose the additional constraint

$$\|Kv(u)\| = \|G_0\, u\|. \tag{2.18}$$

Kernel and image representations of a nonlinear system

A nonlinear generalization of the results in [5] hinges on the generalization of normalized left and right coprime representations of a system to a nonlinear setting. This generalization is, respectively, provided by the notion of stable kernel and image representation of a nonlinear system; see e.g. [4].

For simplicity, let us consider affine systems of the type

$$P := \left\{ \begin{array}{rcll} \dot{x} &=& f(x) + g(x)\,u, & x \in \mathcal{X}, u \in \mathbb{R}^m \\ y &=& h(x), & y \in \mathbb{R}^p \end{array} \right. \tag{2.19}$$

Consider the following two steady-state Hamilton Jacobi equations

$$\begin{aligned} V_x(x)f(x) - \frac{1}{2}V_x(x)g(x)g^T(x)V_x^T(x) & \\ +\frac{1}{2}h^T(x)h(x) &= 0, && (2.20) \\ W_x(x)f(x) + \frac{1}{2}W_x(x)g(x)g^T(x)W_x^T(x) & \\ -\frac{1}{2}h^T(x)h(x) &= 0. && (2.21) \end{aligned}$$

Stable kernel representation: Suppose there exists a stabilizing solution $W \geq 0$ to (2.21). Assume there exists a $n \times p$ matrix $k(x)$ satisfying

$$W_x(x)k(x) = h^T(x). \tag{2.22}$$

Then the system

$$z_l = \tilde{G}\,[u^T \;\; y^T]^T \tag{2.23}$$

$$\tilde{G} := \left\{ \begin{array}{rcl} \dot{x} &=& [f(x) - k(x)h(x)] + g(x)\,u + k(x)\,y \\ z_l &=& y - h(x) \end{array} \right. \tag{2.24}$$

is a stable kernel representation of P. The construction above specialized to the linear case gives rise to a normalized left coprime factorization. In the nonlinear (and linear) case, one can define a second output

$$\bar{z}_l = u - g(x)^T W_x(x) \tag{2.25}$$

with the property that for all u, y and $x(0) = 0$ there holds

$$\int_0^\infty (u^T u + y^T y)\, dt = \int_0^\infty (z_l^T z_l + \bar{z}_l^T \bar{z}_l)\, dt.$$

These facts justify our regarding the nonlinear (2.24) as a normalized kernel representation of P. Also, $\tilde{G}_i$ defined in (2.1) is a normalized kernel representation of P_i defined in Section 2.

Stable image representation: Suppose there exists a stabilizing solution $V \geq 0$ to (2.20). Then the system

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} N \\ M \end{bmatrix} z_r = G\, z_r \tag{2.26}$$

$$G := \left\{ \begin{array}{rcl} \dot{x} &=& [f(x) - g(x)g^T(x)V_x^T(x)] + g(x)\, z_r \\ y &=& h(x) \\ u &=& z_r - g^T(x)V_x^T(x) \end{array} \right. \tag{2.27}$$

is a stable image representation of P. It can also be established that with $x(0) = 0$,

$$\int_0^\infty (u^T u + y^T y)\, dt = \int_0^\infty z_r^T z_r\, dt.$$

This establishes that the factorization (2.27) is a nonlinear equivalent of a normalized right coprime factorization, i.e. with unity gain from z_r to $[u^T \;\; y^T]^T$ for all z_r. It follows that G_i as defined in (2.1) is a particular instance of an image representation as defined in (2.27).

An obvious choice for a nonlinear extension of the Vinnicombe metric would be to use $\|\tilde{G}_2 G_1\|_\infty$ with $\tilde{G}_2$ and G_1, respectively, computed using (2.24) and (2.27). However, this approach makes it difficult to prove metric properties. The authors are still pursuing this option.

Background material
Here we prove a key lemma that will be used in the sequel to prove a robust stability result and to show that that a metric obeys the triangle inequality. The proofs of the main lemma and its corollary have been ommitted but can be obtained from the authors.

Lemma 2.1 *Let $a, b, c \in \mathbb{R}^n$ with $\|a\| = \|b\| = \|c\| = 1$. Then*

$$
\begin{aligned}
| < a, b > | \;\; &\geq \;\; | < a, c >< b, c > | \\
&\quad - \sqrt{1 - < a, c >^2}\sqrt{1 - < b, c >^2}
\end{aligned}
\tag{2.28}
$$

Intuition for Lemma 2.1: Let a, b and c be 3 points *on the surface of the sphere in $\mathbb{R}^3$; refer to Figure 2.2.* Let θ_{ab} be the length of the great circle arc joining a and b.

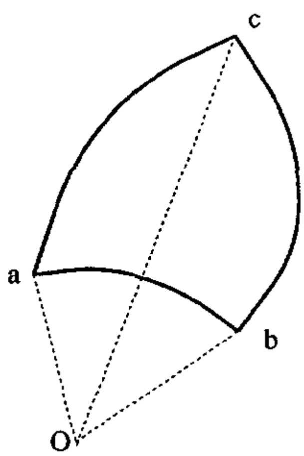

Figure 2.2: Illustration of Lemma 2.1

Then, since $< a, b > = \cos\theta_{ab}$, the inequality says (neglecting signs)

$$
\begin{aligned}
\cos\theta_{ab} &\geq \cos\theta_{ac}\cos\theta_{bc} - \sin\theta_{ac}\sin\theta_{bc} \quad \text{or} \\
\cos\theta_{ab} &\geq \cos(\theta_{ac} + \theta_{bc}) \quad \text{or} \\
\theta_{ab} &\leq \theta_{ac} + \theta_{bc}
\end{aligned}
$$

This expresses the fact that the shortest distance between two points on the surface of a sphere is the great circle distance. Note that equality is obtained in (2.28) if, and only if O, a, b and c are coplanar.

Corollary 2.1 *Let $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ be in $\mathcal{L}_{2+}$ with $\|\alpha\| = \|\beta\| = \|\gamma\| = 1$. Then*

$$
\begin{aligned}
| < \alpha, \beta > | \;\; &\geq \;\; | < \alpha, \gamma >< \beta, \gamma > | \\
&\quad - \sqrt{1 - < \alpha, \gamma >^2}\sqrt{1 - < \beta, \gamma >^2}
\end{aligned}
\tag{2.29}
$$

3 Robust stability theorems

In this section, we present the main robust stability theorem which in turn will provide an extension of the Vinnicombe metric to a nonlinear setting. We consider nonlinear systems and controllers P_i, P_j and C of the type (2.19) with their respective image representations G_i, G_j and K computed using (2.27). Note that for K the last two equations of (2.27) have to be swapped.

Let us define

$$
\begin{aligned}
(e_{P_0,C})_T &= \sup_u \left\{ \sup_{v(u)} \frac{|\langle Kv(u), G_0\, u\rangle_T|}{\|Kv(u)\|_T \, \|G_0\, u\|_T} \right\} \\
&\qquad \text{with } \|Kv(u)\|_T = \|G_0\, u\|_T,
\end{aligned}
\tag{3.1}
$$

$$
(b_{P_0,C})_T = \sqrt{1 - (e_{P_0,C})_T^2} \tag{3.2}
$$

and

$$
\begin{aligned}
(e_{01})_T &= \inf_w \left\{ \sup_{z(w)} \frac{|\langle G_0\, z(w), G_1\, w\rangle_T|}{\|G_0\, z(w)\|_T \, \|G_1\, w\|_T} \right\} \\
&\qquad \text{with } \|G_0\, z(w)\|_T = \|G_1\, w\|_T,
\end{aligned}
\tag{3.3}
$$

$$
(d_{01})_T = \sqrt{1 - (e_{01})_T^2}. \tag{3.4}
$$

Here u, $v(u)$, w and $z(w)$ are assumed to be in $\mathcal{L}_2[0, T]$ with $T \in \,]0, \infty]$. We now show the following theorem.

Theorem 3.1 *For any P_0 and C, the feedback loop $[P_0, C]$ is stable if, and only if,*

$$
\begin{aligned}
&(e_{P_0,C})_T < 1 - \epsilon, \quad \forall T \quad \text{or, equivalently,} \\
&(b_{P_0,C})_T > \epsilon, \quad \forall T.
\end{aligned}
$$

for some arbitrarily small $\epsilon > 0$.

An outline for a proof of this theorem can be found in Appendix A. The complete proof can be obtained from the authors.

Let us define

$$
e_{P_0,C} = \max_T (e_{P_0,C})_T, \tag{3.5}
$$

$$
b_{P_0,C} = \sqrt{1 - e_{P_0,C}^2}. \tag{3.6}
$$

Corollary 3.1 *For any P_0 and C, the feedback loop $[P_0, C]$ is stable if, and only if,*

$$
\begin{aligned}
&e_{P_0,C} < 1, \quad \text{or, equivalently,} \\
&b_{P_0,C} > 0.
\end{aligned}
$$

This corollary shows that (3.6) is a nonlinear generalization of the stability margin defined in [5]. In fact, one can show the following result.

Corollary 3.2 *Consider linear P_0 and C. Then the alternative definition*

$$
b_{P_0,C} = \sqrt{1 - e_{P_0,C}^2} \tag{3.7}
$$

with

$$
e_{P_0,C} = \sup_{u \in \mathcal{L}_{2e+}} \left\{ \sup_{v(u) \in \mathcal{L}_{2e+}} \frac{|\langle Kv(u), G_0\, u\rangle|}{\|Kv(u)\| \, \|G_0\, u\|} \right\} \tag{3.8}
$$

is equivalent to that provided in (2.4), i.e. it takes care of the winding number condition and it is valid whether, or not, $[P_0, C]$ is stable.

An outline for proof of this corollary can be found in Appendix A. The complete proof can be obtained from the authors.

Remark: Note that (3.8) is a short notation for

$$\lim_{T\to\infty} \sup_{u\in\mathcal{L}_2[0,T]} \left\{ \sup_{v(u)\in\mathcal{L}_2[0,T]} \frac{|\langle Kv(u), G_0\, u\rangle_T|}{\|Kv(u)\|_T\,\|G_0\, u\|_T} \right\}.$$

We now present the main robust stability theorem.

Theorem 3.2 *Let P_0, P_1 and C be such that*

- $[P_0, C]$ *is stable and*
- *the following equivalent statements hold:*
  $(e_{01})_T > (e_{P_0,C})_T \quad \forall T$,
  $(d_{01})_T < (b_{P_0,C})_T \quad \forall T$,
  $\arccos(e_{01})_T < \arccos(e_{P_0,C})_T \quad \forall T$,
  $\arcsin(d_{01})_T < \arcsin(b_{P_0,C})_T \quad \forall T$.

Then the feedback loop $[P_1, C]$ is stable.

The proof of the main theorem has been omitted but can be obtained from the authors. This theorem gives a pointwise (in time) robust stability theorem and our result is analogous to the pointwise (in frequency) robust stability theorem derived in [5].

Let us define

$$e_{01} = \min_T (e_{01})_T, \tag{3.9}$$

$$d_{01} = \sqrt{1-e_{01}^2}. \tag{3.10}$$

The next result is a trivial consequence of Theorem 3.2.

Corollary 3.3 *Let P_0, P_1 and C be such that*

- $[P_0, C]$ *is stable and*
- *the following equivalent statements hold:*
  $e_{01} > e_{P_0,C}$,
  $d_{01} < b_{P_0,C}$,
  $\arccos e_{01} < \arccos e_{P_0,C}$,
  $\arcsin d_{01} < \arcsin b_{P_0,C}$.

Then the feedback loop $[P_1, C]$ is stable.

We show in the next section that one can construct a reciprocal measure that obeys the triangle inequality.

4 Properties of the metric

In this section we will actually work with

$$\theta_{21} = \arcsin d_{21} = \arccos e_{21} \tag{4.1}$$

with $0 \le \theta_{21} \le \frac{\pi}{2}$ with e_{21} and d_{21}, respectively, defined in (3.9) and (3.10). We show that θ_{21} satisfies the triangle inequality. Note that θ_{21} is not a reciprocal measure in general, i.e. $\theta_{21} \neq \theta_{12}$ in case G_1 and G_2 are nonlinear. However, it shown subsequently that can one can construct a reciprocal measure that still satisfies the triangle inequality using θ_{21}. We also show that this symmetric measure satisfies a robust stability theorem of the type presented in the previous section.

We now show the triangle inequality property for the distance defined in (4.1). The proof of the next theorem has been omitted but can be obtained from the authors.

Theorem 4.1 *For any P_1, P_2 and P_3 of the type* (2.19) *with stable image representation defined as in* (2.27), *one has*

$$\theta_{21} \le \theta_{23} + \theta_{31} \tag{4.2}$$

with θ_{ij} defined in (4.1).

We now introduce a symmetric distance and we show the metric property by showing that this distance still obeys the triangle inequality property. Let us define the symmetric distance

$$\begin{aligned} \phi_{12} &= \max\{\theta_{21}, \theta_{12}\} \\ &\le \max\{\theta_{23}+\theta_{31}, \theta_{13}+\theta_{32}\} \\ &\le \max\{\max\{\theta_{23},\theta_{32}\}+\theta_{31}, \\ &\qquad \max\{\theta_{13},\theta_{31}\}+\theta_{32}\} \\ &\le \max\{\max\{\theta_{23},\theta_{32}\}+\max\{\theta_{13},\theta_{31}\}, \\ &\qquad \max\{\theta_{13},\theta_{31}\}+\max\{\theta_{23},\theta_{32}\}\} \\ &= \max\{\theta_{13},\theta_{31}\}+\max\{\theta_{23},\theta_{32}\} \\ &= \phi_{13}+\phi_{32} \end{aligned} \tag{4.3}$$

which shows that (4.3) satisfies the triangle inequality. Note that it follows immediately from the definitions that $\phi_{21} = 0$ if, and only if, $P_1 = P_2$, i.e. the distance defined in (4.3) is a **metric**.

Note that the following robust stability result holds:

Corollary 4.1 *Let P_0, P_1 and C be such that*

- $[P_0, C]$ *is stable and*
- $\phi_{01} < \arcsin b_{P_0,C}$.

Then the feedback loop $[P_1, C]$ is stable.

The proof of this corollary is trivial from Corollary 3.3.

5 Conclusions

In this paper we have presented a nonlinear extension of the ν-gap metric which was originally presented for linear systems in [5]. We have shown that the robust stability results presented in [5] can be extended to a nonlinear framework through the use of alternative input-output definitions for the ν-gap metric and the generalized stability margin and the notion of an image representation of a nonlinear system.

Future research topics include an alternative nonlinear measure based on $\tilde{G}_2 G_1$ and a comparison of the nonlinear ν-gap presented in this paper with the nonlinear gap metric presented in [3].

A Proofs of Section 3

In this section, we give an outline of some of the proofs of Section 3.

Proof of Theorem 3.1
The result is proved using the next lemma. The proof of the lemma is omitted but can be obtained from the authors.

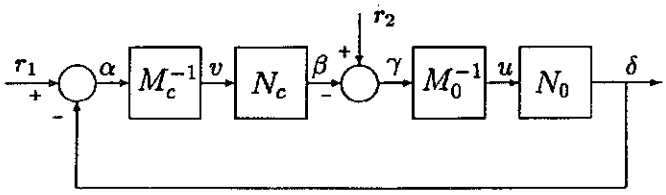

Figure A.1: Closed-loop system $[P_0, C]$

Lemma A.1 *The closed-loop system $[P_0, C]$ depicted in Figure A.1 be unstable if, and only if, for arbitrarily small $\epsilon_1, \epsilon_2 > 0$, there exists T, v, $u \in \mathcal{L}_2[0,T]$ such that*

$$\frac{\langle Kv(u), G_0\, u\rangle_T}{\|Kv(u)\|_T\, \|G_0\, u\|_T} \leq \epsilon_1 - 1 \qquad \text{(A.1)}$$

with

$$\frac{\|G_0\, u\|_T}{\|Kv(u)\|_T} = 1 + O(\epsilon_2). \qquad \text{(A.2)}$$

Proof of Corollary 3.2
We need to show that

$$\sup_{u\in\mathcal{L}_{2+}} \left\{ \sup_{v(u)\in\mathcal{L}_{2+}} \frac{|\langle Kv(u), G_0\, u\rangle|}{\|Kv(u)\|\, \|G_0\, u\|} \right\}$$
$$= \sup_{u\in\mathcal{L}_{2e+}} \left\{ \sup_{v(u)\in\mathcal{L}_{2e+}} \frac{|\langle Kv(u), G_0\, u\rangle|}{\|Kv(u)\|\, \|G_0\, u\|} \right\}.$$

We only need to show (A.3) when $[P_0, C]$ depicted in Figure A.1 is stable. Indeed, it is trivial to see from Theorem 3.1 that when $[P_0, C]$ is unstable, one has

$$\sup_{u\in\mathcal{L}_{2e+}} \left\{ \sup_{v(u)\in\mathcal{L}_{2e+}} \frac{|\langle Kv(u), G_0\, u\rangle|}{\|Kv(u)\|\, \|G_0\, u\|} \right\} = 1.$$

Also, the converse of (A.3) is trivial.

Lemma A.2 *Consider linear P_0 and C. Let the closed-loop system $[P_0, C]$ depicted in Figure A.1 be stable. Then*

$$\sup_{u\in\mathcal{L}_{2+}} \left\{ \sup_{v(u)\in\mathcal{L}_{2+}} \frac{|\langle Kv(u), G_0\, u\rangle|}{\|Kv(u)\|\, \|G_0\, u\|} \right\}$$
$$\geq \sup_{u\in\mathcal{L}_{2e+}} \left\{ \sup_{v(u)\in\mathcal{L}_{2e+}} \frac{|\langle Kv(u), G_0\, u\rangle|}{\|Kv(u)\|\, \|G_0\, u\|} \right\}. \qquad \text{(A.3)}$$

The proof of the lemma is omitted but can be obtained from the authors.

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