ON A NONLINEAR GENERALIZATION OF THE $\nu$-GAP METRIC

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Abstract

This paper presents a nonlinear extension of the $\nu$-gap metric introduced in [5]. Indeed, we present an input-output version of the generalized stability margin and the $\nu$-gap metric. The notion of image representation as presented in [4] allows one to use these alternative definitions in a nonlinear context for the derivation of a robust stability theorem.

Keywords: robust stability, nonlinear, metric, generalized stability margin.

1 Introduction

It has long been observed that the closed-loop behaviour of two systems can be very similar (different) even though the norm of the difference between the open-loop systems can be very large (small). It is to deal with this problem that the gap metric and the $\nu$-gap metric have been introduced as a tool for the study of uncertainty in closed-loop systems; see e.g. [2, 5]. Indeed, perturbations which are small in the gap metric or in the $\nu$-gap metric give rise to small closed-loop errors in a feedback loop. The main advantage of the $\nu$-gap metric is that it can easily be interpreted as the distance between two frequency responses. Also, it has been shown in [5] that the $\nu$-gap metric provides sharper results in the sense that it is the smallest metric for which certain robust stability results hold. The Vinnicombe metric has also found an increasing use as an assessment tool for the safe update of a controller in adaptive control problems; see [1] for further details.

It is of course of interest to try to extend the above robust stability results to a nonlinear setting. An input-output approach for the generalization of the gap metric can be found in [3]. However, a suitable generalization of the $\nu$-gap metric is of even greater because of its shaper results and its potential usefulness in nonlinear adaptive control problems. In this paper, we propose an input-output version of the measures that are of crucial importance for establishing the robust stability theorem of [5], namely the generalized stability margin and the $\nu$-gap metric. The advantage of these alternative definitions is that they are valid in a nonlinear framework. Indeed, the notion of image representation as presented in [4] is used in conjunction with these alternative definitions to derive a nonlinear robust stability theorem. Note that some of the results presented in this paper are incomplete. For instance, it is not entirely clear yet if the distance used in this paper is the same as the one used in [5] in a linear context.

We now outline the contents of the paper. In section 2, we recall the definitions of the $\nu$-gap metric (Vinnicombe metric) and the stability margin in a linear context and we give some insights on how to rewrite these definitions in an input-output approach. We also recall the notion of image and kernel representations for a nonlinear system and present some results that will be used in the sequel of the paper. Section 3 presents the main robust stability results. In Section 4, we present a nonlinear extension of the Vinnicombe metric that obeys all the properties of a metric. We conclude in Section 5.

2 Background material for a nonlinear metric

In this section, we introduce background material and concepts that will be used in the sequel of the paper.

Linear theory

To motivate the later nonlinear system treatment, we will recall a number of results concerning linear systems; see [5].

Notations: $\mathcal{H}_\infty$ denotes the Hardy space of matrix-valued functions analytic in the open-right-half plane. $\mathcal{L}_2$ is the Hilbert space of matrix-valued functions on $\mathbb{R}$, with inner product

$$<f, g> = \int_{-\infty}^{\infty} \text{trace}(f(t)^* g(t)) \, dt.$$ 

$\mathcal{L}_2^+ = \mathcal{L}_2[0, \infty]$ is the subspace of $\mathcal{L}_2$ with functions zero for $t < 0$. $\mathcal{L}_2^{u+}$ is the Hilbert space of matrix-valued functions $f$ on $\mathbb{R}$ such that $P_T f \in \mathcal{L}_2^+$ for all $T > 0$ where $P_T u = u$ for $t < T$ and $P_T u = 0$ for $t > T$.

Vinnicombe metric: Throughout this paper, $(N_t, M_t)$ will denote a normalized right coprime factorization (ncf) of $P_t$, and $(\widetilde{N}_t, \widetilde{M}_t)$ a normalized left coprime fac-
torization (nlcf) of $P_i$. We will write

$$G_i := \begin{bmatrix} N_i \\ M_i \end{bmatrix}, \quad \tilde{G}_i := [-\tilde{M}_i \, \tilde{N}_i]. \quad (2.1)$$

Recall that $(N_i, M_i)$ is nrcf of $P_i$ if, and only if, i) $P_i = N_i M_i^{-1}$, ii) $G_i \in \mathcal{H}_\infty$, iii) there exists an $X \in \mathcal{H}_\infty$ such that $XG_i = I$ (coprimeness condition) and iv) $G_i^* G_i = I$ (normalization condition). Similarly, $(\tilde{N}_i, \tilde{M}_i)$ is a nlcf of $P_i$ if, and only if, i) $P_i = \tilde{N}_i \tilde{M}_i^{-1}$, ii) $G_i \in \mathcal{H}_\infty$, iii) there exists an $Y \in \mathcal{H}_\infty$ such that $G_i Y = I$ (coprimeness condition) and iv) $G_i Y G_i^* = I$ (normalization condition).

The formula for the $\nu$-gap metric defined in [5] in a Linear Time Invariant (LTI) framework is

$$\delta_\nu(P_1, P_2) := \begin{cases} \|G_2 G_1\|_\infty \quad \text{if} \det(G_2^* G_1(j\omega)) \neq 0, \forall \omega \\
\quad \quad \text{and} \; \det(G_2^* G_1) = 0 \quad (2.2) \quad \text{wno det}(G_2^* G_1) = 0 \end{cases}$$

where wno($g$) denotes the winding number about the origin of $g(s)$, as $s$ follows the standard Nyquist D-contour.

**Stability margin**: This paper is concerned with the closed-loop system of Figure 2.1 and its relation to the robust stability and robust performance of feedback systems. Define

$$T(P_0, C) = \begin{bmatrix} P_0 \\ -C P_0 \end{bmatrix} (I - C P_0)^{-1} [-C \, I]. \quad (2.3)$$

In [5] and in a linear context, a generalized stability margin was defined as follows

$$b_{P_0, C} := \begin{cases} \|T(P_0, C)\|_\infty, & \text{if} \; [P_0, C] \; \text{is stable} \\
0, & \text{otherwise}. \quad (2.4) \end{cases}$$

Throughout this paper, $(N_e, M_e)$ will denote a normalized right coprime factorization (nrcf) of $C$, and $(\tilde{N}_e, \tilde{M}_e)$ a normalized left coprime factorization (nlcf) of $C$. We will write

$$K := \begin{bmatrix} M_e \\ N_e \end{bmatrix}, \quad \tilde{K} := [-\tilde{N}_e \, \tilde{M}_e]. \quad (2.5)$$

As noted in [5] if $[P_0, C]$ is stable, then

$$b_{P_0, C} = \inf_{\omega} e(\tilde{K} G_0(j\omega)). \quad (2.6)$$

Note that $[P_0, C]$ is stable if, and only if, $\det(\tilde{K} G_0(j\omega)) \neq 0, \forall \omega$ and wno $\det(\tilde{K} G_0) = 0$.

**Insights in the linear context**

In this subsection, we show that, without taking into account winding number conditions, one can rewrite both the Vinnicombe distance and the stability margin using an input-output approach. It will become clear in the sequel of the paper how to take into account the winding number conditions.

**Vinnicombe metric**: Using the fact that $[G_2, \tilde{G}_2]$ is unitary, one shows that

$$G_1^* \left[ G_2 \, \tilde{G}_2 \right] \left[ \frac{G_2}{G_2^*} \right] G_1 = G_1^* \left[ \tilde{G}_2 \, G_2 + G_2 \tilde{G}_2 \right] G_1 = I.$$

It now follows that

$$\sigma^2(\tilde{G}_2 G_1)(j\omega) + \sigma^2(\tilde{G}_2^* G_1)(j\omega) = 1, \forall \omega. \quad (2.7)$$

So we can think of working with

$$\sqrt{1 - \sigma^2(\tilde{G}_2 G_1)(j\omega)}$$

instead of $\tilde{\sigma}(\tilde{G}_2 G_1)(j\omega)$ as suggested by (2.8). Indeed, one has

$$\|G_2 G_1\|_\infty = \sup_\omega \sigma(\tilde{G}_2 G_1) = \frac{1}{\sqrt{1 - \sigma^2(\tilde{G}_2 G_1)(j\omega)}}$$

where

$$\inf_\omega \sigma(\tilde{G}_2 G_1)(j\omega) = \inf_\|u\|_1 \|G_2 G_1 u\|$$

The last equality follows from the normalization assumption on $G_1$ and $G_2$. Here $u$ and $v(u)$ are assumed to be in $L_2^+$. Note that it is only the shape of $v(u)$ that affects the expressions above, i.e. without loss of generality one can impose the additional constraint

$$\|G_2 v(u)\| = \|G_1 u\|. \quad (2.12)$$

**Stability margin**: Using the fact that $[K \, \tilde{K}^*]$ is unitary, one shows that

$$G_0^* \left[ K \, \tilde{K} \right] \left[ K^* \, \tilde{K} \right] G_0 = G_0^* \left[ K^* \, \tilde{K} + K K^* \right] G_0 = I.$$
It now follows that
\[ g^2[\kappa G_0](j\omega) + \vartheta^2[K^*G_0](j\omega) = 1, \quad \forall \omega. \] (2.13)
So we can think of working with
\[ \sqrt{1 - \vartheta^2[K^*G_0](j\omega)} \]
instead of \( g[K^*G_0](j\omega) \) as suggested by (2.6). Indeed, one has
\[ \inf_{\omega} g[K^*G_0](j\omega) = \inf_{\omega} \sqrt{1 - \vartheta^2[K^*G_0](j\omega)} = \sqrt{1 - \sup_{\omega} \vartheta^2[K^*G_0](j\omega)} \]
where
\[ \sup_{\omega} \vartheta[K^*G_0](j\omega) \]

\[ = \sup_{||v||=1} \|K^*G_0 u\| \]
\[ = \sup_{||v||=1} \left\{ \sup_{||v||=1} ||v(u)|| \right\} \]
\[ = \sup_{u} \left\{ \sup_{||v||=1} ||v(u)|| \right\} \]
\[ = \sup_{v} \left\{ \sup_{||v||=1} ||v(u)|| \right\} \]
\[ = \sup_{u} \left\{ \sup_{||v||=1} ||v(u)|| \right\} \]

(2.14)

(2.15)

(2.16)

(2.17)

[In the linear case, there is no difference between (2.15)-(2.16).] Here \( u \) and \( v(u) \) are assumed to be in \( L_{\infty} \). Note that it is again only the shape of \( v(u) \) that affects the expressions above, i.e. without loss of generality one can impose the additional constraint
\[ \|K^*u\| = \|G_0 u\|. \] (2.18)

Kernel and image representations of a nonlinear system

A nonlinear generalization of the results in [5] hinges on the generalization of normalized left and right coprime representations of a system to a nonlinear setting. This generalization is, respectively, provided by the notion of stable kernel and image representation of a nonlinear system; see e.g. [4].

For simplicity, let us consider affine systems of the type
\[ P := \left\{ \begin{array}{l} \dot{x} = f(x) + g(x)u, \quad x \in X, u \in \mathbb{R}^m \\ y = h(x) \end{array} \right. \] (2.19)

Consider the following two steady-state Hamilton Jacobi equations
\[ V_\omega(x)f(x) - \frac{1}{2} V_\omega(x)g(x)g^T(x)V_\omega(x) = 0, \quad \frac{1}{2} h^T(x)h(x) = 0, \] (2.20)

\[ W_\omega(x)g(x) + \frac{1}{2} W_\omega(x)g(x)g^T(x)W_\omega(x) = 0, \quad \frac{1}{2} h^T(x)h(x) = 0. \] (2.21)

Stable kernel representation: Suppose there exists a stabilizing solution \( W \geq 0 \) to (2.21). Assume there exists a \( n \times p \) matrix \( k(x) \) satisfying
\[ W_\omega(x)k(x) = h^T(x). \] (2.22)

Then the system
\[ \dot{z}_1 = \tilde{G}[u^T \ y^T]^T \]
\[ \tilde{G} := \left\{ \begin{array}{l} \dot{x} = [f(x) - k(x)h(x)] + g(x)u + k(x)y \\ z_1 = y - h(x) \end{array} \right. \] (2.23)

is a stable kernel representation of \( P \). The construction above specialized to the linear case gives rise to a normalized left coprime factorization. In the nonlinear (and linear) case, one can define a second output
\[ \tilde{z}_1 = u - g(x)^T z(x) \] (2.25)

with the property that for all \( u, y \) and \( z(0) = 0 \) there holds
\[ \int_0^\infty (u^T u + y^T y) dt = \int_0^\infty (\tilde{z}_1^T \tilde{z}_1 + \tilde{z}_1^T \tilde{z}_1) dt. \]

These facts justify our regarding the nonlinear (2.24) as a normalized kernel representation of \( P \). Also, \( G_i \) defined in (2.1) is a normalized kernel representation of \( P_i \) defined in Section 2.

Stable image representation: Suppose there exists a stabilizing solution \( V \geq 0 \) to (2.20). Then the system
\[ \left[ \begin{array}{l} \dot{y} \\ \dot{u} \end{array} \right] = \left[ \begin{array}{l} N \\ M \end{array} \right] z_r = G z_r \] (2.26)

\[ G := \left\{ \begin{array}{l} \dot{x} = [f(x) - g(x)g^T(x)V_\omega(x)] + g(x)u \\ \dot{y} = h(x) \\ \dot{u} = z_r - g^T(x)V_\omega^T(x) \end{array} \right. \] (2.27)

is a stable image representation of \( P \). It can also be established that with \( z(0) = 0 \),
\[ \int_0^\infty (u^T u + y^T y) dt = \int_0^\infty z_r^T z_r dt. \]

This establishes that the factorization (2.27) is a nonlinear equivalent of a normalized right coprime factorization, i.e. with unity gain from \( z_r \) to \( [u^T \ y^T]^T \) for all \( z_r \). It follows that \( G_i \) as defined in (2.1) is a particular instance of an image representation as defined in (2.27).

An obvious choice for a nonlinear extension of the Vinnicombe metric would be to use \( ||G_2 G_1||_{\infty} \) with \( G_2 \) and \( G_1 \), respectively, computed using (2.24) and (2.27). However, this approach makes it difficult to prove metric properties. The authors are still pursuing this option.
Background material
Here we prove a key lemma that will be used in the sequel to prove a robust stability result, and to show that that a metric obeys the triangle inequality. The proofs of the main lemma and its corollary have been omitted but can be obtained from the authors.

Lemma 2.1 Let \( a, b, c \in \mathbb{R}^n \) with \( \|a\| = \|b\| = \|c\| = 1 \). Then
\[
\langle a, b \rangle \geq \langle a, c \rangle < \langle b, c \rangle \quad (2.28)
\]
Intuition for Lemma 2.1: Let \( a, b \) and \( c \) be 3 points on the surface of the sphere in \( \mathbb{R}^3 \); refer to Figure 2.2.

Let \( \theta_{ab} \) be the length of the great circle arc joining \( a \) and \( b \). Then, since \( \langle a, b \rangle = \cos \theta_{ab} \), the inequality says (neglecting signs)
\[
\cos \theta_{ab} \geq \cos \theta_{ac} \cos \theta_{bc} - \sin \theta_{ac} \sin \theta_{bc} \quad \text{or}
\cos \theta_{ab} \geq \cos(\theta_{ac} + \theta_{bc}) \quad \text{or}
\theta_{ab} \leq \theta_{ac} + \theta_{bc}
\]
This expresses the fact that the shortest distance between two points on the surface of a sphere is the great circle distance. Note that equality is obtained in (2.28) if, and only if, \( O, a, b \) and \( c \) are coplanar.

Corollary 2.1 Let \( \alpha(t), \beta(t) \) and \( \gamma(t) \) be in \( L_2^+ \) with \( \|\alpha\| = \|\beta\| = \|\gamma\| = 1 \). Then
\[
\langle \alpha, \beta \rangle \geq \langle \alpha, \gamma \rangle \rangle \langle \beta, \gamma \rangle \rangle \quad (2.29)
\]

3 Robust stability theorems
In this section, we present the main robust stability theorem which in turn will provide an extension of the Vinicombe metric to a nonlinear setting. We consider nonlinear systems and controllers \( P_i, P_j \) and \( C \) of the type (2.19) with their respective image representations \( G_i, G_j \) and \( K \) computed using (2.27). Note that for \( K \) the last two equations of (2.27) have to be swapped.

Let us define
\[
(e_{P_0,C})_T = \sup_u \left\{ \frac{(Kv(u), G_0 u)_T}{\|Kv(u)\|_T \|G_0 u\|_T} \right\} \quad (3.1)
\]
with \( \|Kv(u)\|_T = \|G_0 u\|_T \),
\[
(b_{P_0,C})_T = \sqrt{1 - (e_{P_0,C})^2_T} \quad (3.2)
\]
and
\[
(e_{\theta_{P_0,C}})_T = \inf_w \left\{ \frac{(G_0 z(w), G_1 w)_T}{\|G_0 z(w)\|_T \|G_1 w\|_T} \right\} \quad (3.3)
\]
with \( \|G_0 z(w)\|_T = \|G_1 w\|_T \),
\[
(d_{\theta_{P_0,C}})_T = \sqrt{1 - (e_{\theta_{P_0,C}})_T^2} \quad (3.4)
\]
Here \( u, v(u), w \) and \( z(w) \) are assumed to be in \( L_2[0, T] \) with \( T \in [0, \infty] \). We now show the following theorem.

Theorem 3.1 For any \( P_0 \) and \( C \), the feedback loop \([P_0, C]\) is stable if, and only if,
\[
(e_{P_0,C})_T < 1 - \epsilon, \quad \forall T \quad \text{or, equivalently,}
(b_{P_0,C})_T > \epsilon, \quad \forall T
\]
for some arbitrarily small \( \epsilon > 0 \).

An outline for a proof of this theorem can be found in Appendix A. The complete proof can be obtained from the authors.

Let us define
\[
e_{P_0,C} = \max_T (e_{P_0,C})_T, \quad (3.5)
\]
\[
b_{P_0,C} = \sqrt{1 - \epsilon^2_{P_0,C}}. \quad (3.6)
\]

Corollary 3.1 For any \( P_0 \) and \( C \), the feedback loop \([P_0, C]\) is stable if, and only if,
\[
e_{P_0,C} < 1, \quad \text{or, equivalently,}
b_{P_0,C} > 0.
\]

This corollary shows that (3.6) is a nonlinear generalization of the stability margin defined in [5]. In fact, one can show the following result.

Corollary 3.2 Consider linear \( P_0 \) and \( C \). Then the alternative definition
\[
b_{P_0,C} = \sqrt{1 - \epsilon^2_{P_0,C}} \quad (3.7)
\]
with
\[
e_{P_0,C} = \sup_{u \in L_2^+} \left\{ \sup_{v(u) \in L_2^+} \frac{(Kv(u), G_0 u)_T}{\|Kv(u)\|_T \|G_0 u\|_T} \right\} \quad (3.8)
\]
is equivalent to that provided in (2.4), i.e. it takes care of the winding number condition and it is valid whether, or not, \([P_0, C]\) is stable.
An outline for proof of this corollary can be found in Appendix A. The complete proof can be obtained from the authors.

Remark: Note that (3.8) is a short notation for
\[
\lim_{T \to \infty} \sup_{u \in C_2[0,T]} \{ \sup_{v \in C_2[0,T]} \frac{|(Ku(u), G_0 u)|_T^2}{\|Ku(u)\|_T \|G_0 u\|_T} \}.
\]

We now present the main robust stability theorem.

**Theorem 3.2** Let \( P_0, P_1 \) and \( C \) be such that
- \([P_0, C]\) is stable and
- the following equivalent statements hold:
  - \((e_{01})_T > (e_{P_0,C})_T \quad \forall T,\)
  - \((d_{01})_T < (b_{P_0,C})_T \quad \forall T,\)
  - \(\arccos(e_{01})_T < \arccos(e_{P_0,C})_T \quad \forall T,\)
  - \(\arcsin(d_{01})_T < \arcsin(b_{P_0,C})_T \quad \forall T.
\]

Then the feedback loop \([P_1, C]\) is stable.

The proof of the main theorem has been omitted but can be obtained from the authors. This theorem gives a pointwise (in time) robust stability theorem and our result is analogous to the pointwise (in frequency) robust stability theorem derived in [5].

Let us define
\[
e_{01} = \min_T (e_{01})_T, \quad (3.9)
\]
\[
d_{01} = \sqrt{1 - e_{01}^2}, \quad (3.10)
\]

The next result is a trivial consequence of Theorem 3.2.

**Corollary 3.3** Let \( P_0, P_1 \) and \( C \) be such that
- \([P_0, C]\) is stable and
- the following equivalent statements hold:
  - \(e_{01} > e_{P_0,C},\)
  - \(d_{01} < b_{P_0,C},\)
  - \(\arccos(e_{01}) < \arccos(e_{P_0,C}),\)
  - \(\arcsin(d_{01}) < \arcsin(b_{P_0,C}).\)

Then the feedback loop \([P_1, C]\) is stable.

We show in the next section that one can construct a reciprocal measure that obeys the triangle inequality.

### 4 Properties of the metric

In this section we will actually work with
\[
\theta_{21} = \arcsin d_{21} = \arccos e_{21}, \quad (4.1)
\]
with \(0 \leq \theta_{21} \leq \frac{\pi}{2}\) with \(e_{21}\) and \(d_{21}\), respectively, defined in (3.9) and (3.10). We show that \(\theta_{21}\) satisfies the triangle inequality. Note that \(\theta_{21}\) is not a reciprocal measure in general, i.e. \(\theta_{21} \neq \theta_{\bar{2}1}\) in case \(G_1\) and \(G_2\) are nonlinear. However, it shows subsequently that one can construct a reciprocal measure that still satisfies the triangle inequality using \(\theta_{21}\). We also show that this symmetric measure satisfies a robust stability theorem of the type presented in the previous section.

We now show the triangle inequality property for the distance defined in (4.1). The proof of the next theorem has been omitted but can be obtained from the authors.

**Theorem 4.1** For any \( P_1, P_2 \) and \( P_3 \) of the type (2.19) with stable image representation defined as in (2.27), one has
\[
\theta_{21} \leq \theta_{23} + \theta_{31}, \quad (4.2)
\]
with \(\theta_{ij}\) defined in (4.1).

We now introduce a symmetric distance and we show the metric property by showing that this distance still obeys the triangle inequality property. Let us define the symmetric distance
\[
\phi_{12} = \max \{ \theta_{21}, \theta_{12} \} \quad (4.3)
\]
\[
\leq \max \{ \theta_{23} + \theta_{31}, \theta_{13} + \theta_{32} \}
\]
\[
\leq \max \{ \max \{ \theta_{23}, \theta_{32} \} + \theta_{13}, \max \{ \theta_{13}, \theta_{31} \} + \theta_{23} \}
\]
\[
= \max \{ \theta_{13} + \theta_{31} \} + \max \{ \theta_{23}, \theta_{32} \}
\]
\[
= \phi_{13} + \phi_{32}
\]
which shows that (4.3) satisfies the triangle inequality. Note that it follows immediately from the definitions that \(\phi_{21} = 0\) if, and only if, \(P_1 = P_2\), i.e. the distance defined in (4.3) is a metric.

Note that the following robust stability result holds:

**Corollary 4.1** Let \( P_0, P_1 \) and \( C \) be such that
- \([P_0, C]\) is stable and
- \(\phi_{01} < \arcsin b_{P_0,C}\).

Then the feedback loop \([P_1, C]\) is stable.

The proof of this corollary is trivial from Corollary 3.3.

### 5 Conclusions

In this paper we have presented a nonlinear extension of the \(\nu\)-gap metric which was originally presented for linear systems in [5]. We have shown that the robust stability results presented in [5] can be extended to a nonlinear framework through the use of alternative input-output definitions for the \(\nu\)-gap metric and the generalized stability margin and the notion of an image representation of a nonlinear system.

Future research topics include an alternative nonlinear measure based on \(C_2|G|\) and a comparison of the nonlinear \(\nu\)-gap presented in this paper with the nonlinear gap metric presented in [3].
A Proofs of Section 3

In this section, we give an outline of some of the proofs of Section 3.

Proof of Theorem 3.1

The result is proved using the next lemma. The proof of the lemma is omitted but can be obtained from the authors.

Proof of Corollary 3.2

We need to show that

\[ \sup_{u \in \mathcal{L}_2^+} \left\{ \sup_{v(u) \in \mathcal{L}_2^+} \frac{\langle K v(u), G_0 u \rangle}{\|K v(u)\|_T \|G_0 u\|_T} \right\} = 1 + O(\epsilon_2). \]

The proof of the lemma is omitted but can be obtained from the authors.

References


