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ABSTRACT

This paper considers a criterion for the stability of linear, finite dimensional, time-varying control systems with an arbitrary finite number of feedback memoryless nonlinearities. To establish stability, diagonal matrices must be determined such that a particular function involving these matrices, the parameters of the linear (time-varying) part of the system, and the sector bounds of the nonlinearities be a covariance. The covariance criterion reduces to the positive real criterion previously established for the time invariant case. This in turn reduces to the Popov criterion for the single nonlinearity case.

1. INTRODUCTION

The purpose of this paper is to develop a criterion suitable for establishing the stability of linear, finite-dimensional, time-varying systems containing an arbitrary finite number of feedback memoryless nonlinearities.

Since the introduction of the Popov criterion [1], which gives a sufficient condition for the stability in the sense of Lyapunov of linear, time-invariant, stable systems with a memoryless nonlinearity in the feedback loop, various extensions have been developed. For example, the case when a time-varying nonlinearity is involved [2],[3] and the multiple nonlinearity case [4],[5] have been considered. However, extensions to the case when the linear part of the system is time varying have not been previously considered. In the next section, it is shown that a recent result in optimal control [6] makes such an extension possible. In section 3, an example is given to illustrate the application of the criterion.

2. STABILITY CRITERION

Since zero input stability is to be investigated, without loss of generality the system S of Fig. 1 will be considered. The linear, time-varying, finite-dimensional subsystem W is represented by the state-space equations

$$\dot{x} = Fx + Gu \tag{1a}$$

$$y = H'x \tag{1b}$$

where the dimension of both the input vector u and the output vector y is n. The impulse response matrix $w(t,\tau)$ of the subsystem W is given by

$$w(t,\tau) = H'(t)\Phi(t,\tau)G(\tau)1(t-\tau) \tag{2}$$

where

$$\dot{\Phi}(t,\tau) = F\Phi(t,\tau); \qquad \Phi(\tau,\tau) = I_n \tag{3}$$

and $1(t)$ is the unit step function. We assume that the following system properties are satisfied.

(S1) [F, G] is uniformly completely controllable [7].(Note: If F and G are constant, this reduces to ordinary complete controllability). The matrices F, G and H are bounded with H being differentiable and $\dot{H}$ bounded and F is asymptotically stable.

The feedback memoryless nonlinearities $\mu_i(y_i)$ $(i = 1, 2, \ldots, n)$ are assumed to satisfy

(S2) $\mu'(y)\, y \leq y'Ky$

where $\mu'(y) = (\mu_1(y_1), \mu_2(y_2), \ldots, \mu_n(y_n))$ and $K = \text{diag.}\ \{k_1, k_2, \ldots, k_n\}$ and is a constant positive definite matrix.

Note that the nonlinearities are time-invariant, in the sense that μ_i depends explicitly only on y_i, rather than y_i and t.

To develop a stability criterion for the above system S, a hypothetical system Z is considered having an impulse response matrix

$$z(t,\tau) = A(t)K^{-1}\delta(t-\tau) + A(t)w(t,\tau) + B(t)\frac{d}{dt}w(t,\tau) \tag{4}$$

where $\delta(t)$ is the Dirac delta function and

(Z1) $A(t) = \text{diag.}\ \{a_1(t), a_2(t), \ldots, a_n(t)\}$ and $B(t) = \text{diag.}\ \{b_1(t), b_2(t), \ldots, b_n(t)\}$, where $B(t)$ is differentiable and $A(t)$, $B(t)$ and $-\dot{B}(t)$ are nonnegative definite and bounded for all t.

Before proceeding with the development of the stability theorem, the multidimensional Popov criterion is reviewed for the case when F, G, H, A and B are constant [4], [5].

<u>Theorem 1</u> If matrices $A = \text{diag.}\ \{a_1, a_2, \ldots, a_n\}$ and $B = \text{diag.}\ \{b_1, b_2, \ldots, b_n\}$ can be found such that $a_i > 0$, $b_i > 0$, $a_i + b_i > 0$, $-a_i/b_i$ is not a pole of the Laplace transform $W(s)$ of $w(t-\tau)$, and if further the Laplace transform of $z(t-\tau)$ i.e.

$$Z(s) = AK^{-1} + (A + Bs)W(s) \tag{5}$$

is positive real, then the system of Fig. 1 with $[F, G]$ completely controllable, F asymptotically stable and (S2) satisfied is stable in the sense of Lyapunov. The time domain formulation of the requirement that $Z(s)$ be positive real is

$$\int_{T_1}^{T_2} \int_{T_1}^{T_2} q'(t)z(t-\tau)q(\tau)dtd\tau \geq 0 \tag{6}$$

where $q(\cdot)$ is an arbitrary vector function defined over $[T_1, T_2]$ with T_1 and T_2 arbitrary.

We now prove that the criterion, corresponding to (5) or (6), for establishing the stability of the time-varying nonlinear system S having the properties indicated by Fig. 1 and conditions (S1) and (S2) is

(Z2) $[z(t,\tau) + z(\tau,t) - 2\eta I_n\delta(t-\tau)]$ is a covariance for some positive η and some $A(t)$ and $B(t)$ satisfying (Z1).

We note that (Z2) is equivalent to the condition

$$\int_{T_1}^{T_2} \int_{T_1}^{T_2} q'(t)[z(t,\tau) - \eta I_n\delta(t-\tau)]q(\tau)dtd\tau \geq 0 \tag{7}$$

where $q(\cdot)$ is an arbitrary vector function defined over $[T_1, T_2]$ with T_1 and T_2 arbitrary.

From (2) and (4) the impulse response of the modified system Z may be written as

$$z(t,\tau) = \tfrac{1}{2}R_z(t)\delta(t-\tau) + H_z'(t)\Phi(t,\tau)G(\tau)1(t-\tau) \tag{8}$$

where

$$R_z = 2AK^{-1} + BH'G + G'HB \tag{9a}$$

$$H_z = HA + (F'H + \mathring{H})B \tag{9b}$$

Under the conditions (S1) and (Z2) and the further condition

(Z3) $[F, H_z']$ is uniformly completely observable,

a recent control theory result [6] may be used to define a matrix function $P(t)$ such that for all t

$$0 < \alpha_1 I_n \leq P(t) \leq \alpha_2 I_n < \infty; \qquad P = P' \tag{10}$$

for some positive constants α_1 and α_2, with the time derivative of P given by

$$-\mathring{P} = P(F-GR_z^{-1}H_z') + (F'-H_zR_z^{-1}G')P + PGR_z^{-1}G'P + H_zR^{-1}H_z' \tag{11}$$

for all t. (The "uniformly" part of condition (Z3) [(S1)] guarantees the lower [upper] limit of (10)).

Consider now as a tentative Lyapunov function for the system S of Fig. 1

$$V(x,t) = x'P(t)x + 2\sum_i b_i(t)\int_0^{y_i} \mu_i(\rho_i)d\rho_i \tag{12}$$

Differentiating (12) gives

$$\dot{V}(x,t) = 2\dot{x}'P(t)x + x'\dot{P}(t)x + 2\sum_i b_i(t)\mu_i\dot{y}_i + 2\sum_i \dot{b}_i(t) \int_0^{y_i}\mu_i(\rho_i)d\rho_i \quad (13)$$

Noting that in (1) $u = -\mu(y)$, the appropriate substitutions of (1), (9) and (11) into (13) and a rearrangement of terms yields

$$V(x,t) = -[x'(H_z-PG)R_z^{-1} - \mu'(y)]R_z[R_z^{-1}(H_z'-G'P)x - \mu(y)]$$

$$- 2\mu'(y)A[y - K^{-1}\mu(y)] + 2\sum_i \dot{b}_i \int_0^{y_i}\mu_i(\rho_i)d\rho_i \quad (14)$$

We conclude that (S1), (S2), (Z1), (Z2) and (Z3) are sufficient conditions for $V(x,t)$ of (12) to be a Lyapunov function. Clearly, since conditions (S1), (Z2) and (Z3) imply the result (10) and (Z1) ensures that B is nonnegative, the right side of (12) has the required characteristics. The covariance condition (Z2) ensures that $R_z - \eta I_n$ is nonnegative definite and that R_z^{-1} is bounded, and thus the first term of (14) is plainly nonpositive. Condition (Z1) ensures that A and $-\dot{B}$ are nonnegative definite with $\dot{B}$ bounded. This together with (S2) ensures that the second and third terms of (14) are nonpositive and thus the right side of (14) has the required characteristics.

The case when $B(t) = 0$ is of interest since $V(x,t) = x'P(t)x$ is a Lyapunov function even when the nonlinearities are time-varying i.e. μ_i depends not only on y_i, but also on t. In this instance the matrix K in (S2) may also be taken to be time-varying.

It will be noted that no condition requiring $A(t) + B(t)$ to be positive definite has been used explicitly despite its appearance in theorem 1. Actually (9a) and the requirement that $R_z - \eta I_n$ be nonnegative definite subsume the condition on $A(t) + B(t)$.

Extensions to the case when the degree of stability is of interest are straightforward and follow closely sections of [5]. It is required that $(F - \sigma_0 I_n)$ be asymptotically stable (see (S1)), and in (9) F is replaced by $(F - \sigma_0 I_n)$. In (14) an additional term $2\sigma_0[x'Px + \mu'(y)By]$ must be considered. A detailed development for this case is not given but the results of such a development are included in the summarizing theorem now presented.

Theorem 2 Part (a) The nonlinear time-varying system S of Fig. 1 with the linear subsystem W [see (1) (2) and (3)] satisfying (S1) and the memoryless feedback elements satisfying (S2) is stable in the sense of Lyapunov if a hypothetical system Z defined in terms of the system S and diagonal matrices $A(t)$ and $B(t)$ satisfying (Z1)[see (4),(8) and (9)] can be found such that

(i) (Z2) and (Z3) are satisfied

or (ii) [and (i) implies (ii)] the limit

$$P(t) = \lim_{t_1 \to \infty} -\Pi(t, t_1) \tag{15}$$

satisfies both (10) and (11) for all t, where $\Pi(t, t_1)$ (see [6]) is given from the matrix Riccati differential equation

$$-\dot{\Pi} = \Pi(F-GR_z^{-1}H_z') + (F'-H_zR_z^{-1}G')\Pi - \Pi GR_z^{-1}G'\Pi - H_zR_z^{-1}H_z' \tag{16a}$$

$$\Pi(t_1, t_1) = 0 \tag{16b}$$

A Lyapunov function for the system S is

$$V(x) = x'P(t)x + 2\sum_i b_i(t)\int_0^{y_i}\mu_i(\rho_i)d\rho_i \tag{17}$$

<u>Part (b)</u> For the case when the subsystem W of system S satisfies the further condition that $(F - \sigma_0 I_n)$ is asymptotically stable for some real positive σ_0, then Part (a) still applies with F replaced by $(F - \sigma_0 I_n)$ in (Z3), (11) and (16). Moreover if the nonlinearities satisfy

$$\mu_i(y_i)y_i \geq \int_0^{y_i}\mu_i(\rho_i)d\rho_i \quad (i = 1, 2, \cdots, n) \tag{18}$$

there exists a Lyapunov function V for which

$$\frac{\dot{V}}{V} \leq \sigma_0 \tag{19}$$

and if the nonlinearities satisfy

$$\mu_i(y_i)y_i \geq 2\int_0^{y_i}\mu_i(\rho_i)d\rho_i \quad (i = 1, 2, \cdots, n) \tag{20}$$

there exists a Lyapunov function V for which

$$\frac{\dot{V}}{V} \leq 2\sigma_0 \tag{21}$$

(Note: Condition (S2) implies (18) for sufficiently small y_i and for all monotonic nonlinearities; (S2) implies (20) for any nonlinearity concave upwards).

Corollary For the case when $B(t)$ is chosen as the zero matrix, Theorem 2 holds for time-varying nonlinearities satisfying (S2).

We note that theorem 2 reduces to that given in [5] for the case when F, G, H are constant which in turn reduces to the single loop Popov criterion [1] when $n = 1$.

3. EXAMPLE

Consider as an example the stability of the first order system

$$\dot{x} + [f_1(t) + f_2(x)]x = 0 \tag{22}$$

where $f_2(x)$ is a memoryless nonlinearity.

This equation corresponds to the equation describing the electrical network of Fig. 2 which contains a fixed capacitor, a time-varying resistor, and a nonlinear resistor.

We first identify

$$F = -f_1(t); \quad G = 1; \quad H = 1; \quad \mu(x) = f_2(x)x \tag{23}$$

and from (23) and (9) derive (with a and b yet to be specified)

$$R_z = 2ak^{-1} + 2b; \qquad H_z = a - f_1(t)b \tag{24}$$

where it is assumed, see (S2) that

$$0 \leq f_2(x) \leq k < \infty, \quad k \text{ arbitrary} \tag{25}$$

This restriction amounts to requiring that the conductance G_2 should be nonnegative and bounded.

The following condition, with an obvious interpretation in terms of restrictions on the conductance G_1, will also be assumed:

$$0 < \alpha \le f_1(t) \le \beta < \infty \quad \text{for all } t \tag{26}$$

This condition guarantees that F in (23) is asymptotically stable. Setting $b = 0$, the function $z(t,\tau)$ is calculated from (9) as

$$z(t,\tau) = ak^{-1} + a \exp\left[-\int_\tau^t f_1(\lambda)d\lambda\right] 1(t-\tau) \tag{27}$$

We now observe that the sufficient conditions for stability [(S1), (S2), (Z1), (Z2) and (Z3)] will be satisfied provided that a is an arbitrary positive constant and provided that the function

$$r(t,\tau) = a \exp\left[-\int_\tau^t f_1(\lambda)d\lambda\right] 1(t-\tau)$$

$$+ a \exp\left[-\int_t^\tau f_1(\lambda)d\lambda\right] 1(\tau-t) \tag{28}$$

is a covariance. This latter function may be written as

$$r(t,\tau) = c(t)\left[\frac{d(\tau)}{c(\tau)} 1(t-\tau) + \frac{d(t)}{c(t)} 1(\tau-t)\right]c(\tau) \tag{29}$$

where

$$c(t) = a \exp\left[-\int_{t_0}^t f_1(\lambda)d\lambda\right] \tag{30a}$$

and

$$d(t) = \exp\left[\int_{t_0}^t f_1(\lambda)d\lambda\right] \tag{30b}$$

The application of a theorem from [9] gives the result that $r(t,\tau)$ is a covariance provided

$$\text{(i)} \quad c(t) \neq 0 \tag{31a}$$

$$\text{(ii)} \quad \frac{d}{dt}\left[\frac{d(t)}{c(t)}\right] \geq 0 \tag{31b}$$

From (30a) it is seen that (31a) is satisfied and since

$$\frac{d}{dt}\left[\frac{d(t)}{c(t)}\right] = \frac{1}{a} \exp \left[2 \int_{t_0}^{t} f_1(\lambda)d\lambda\right] 2\, f_1(t)$$

the inequality (31b) is satisfied and we conclude that $r(t,\tau)$ is a covariance. Stability of (22) is thus guaranteed provided that (25) and (26) are satisfied.

4. CONCLUDING REMARKS

The theorem developed for testing the stability of time-varying systems with sector nonlinearities at the same time provides insight into the construction of Lyapunov functions for such systems. Unfortunately application of the procedure must prove considerably more difficult, even with a single nonlinearity, than in the time-invariant case. The introduction of time-variation of course prevents effective application of Fourier transform techniques, while it introduces a further degree of functional dependence into the various parameters appearing in the system descriptions. Intuitive understanding is thereby made more difficult. The theory is not entirely without meaningful application though: reference [10] gives a significant result on the structural stability of linear time-varying systems, by showing that, under appropriate conditions of boundedness, controllability, etc. small amounts of sector nonlinearity can always be tolerated.

5. REFERENCES

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FIGURE CAPTIONS

1. System Block Diagram.
2. Network of Section 3 example.

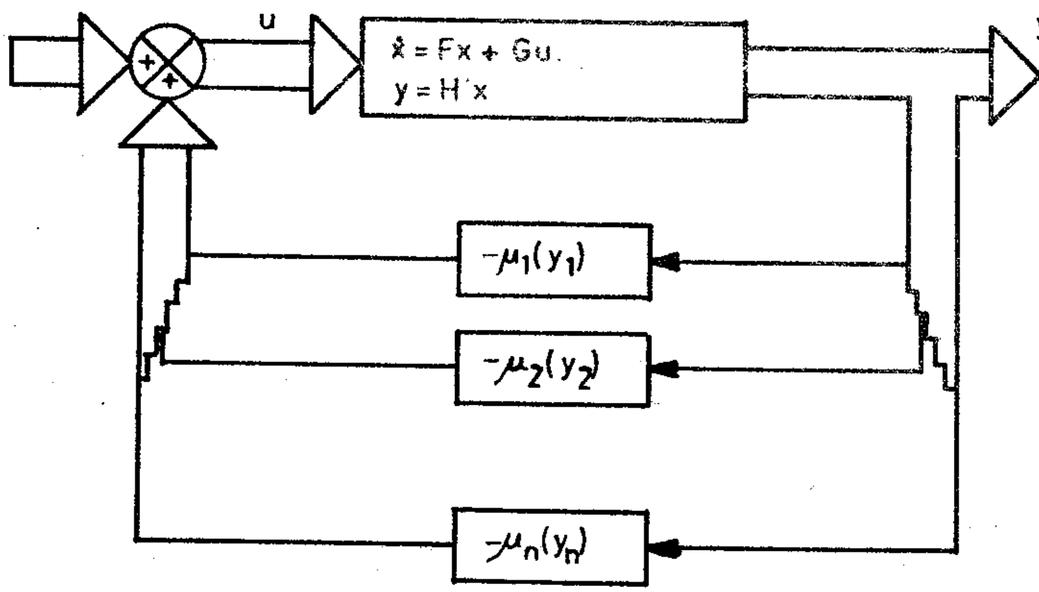

FIG. 1

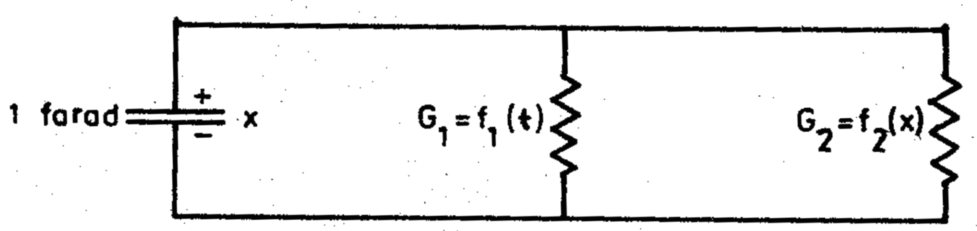

FIG. 2