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Solvability Conditions for 4-block H^∞ Control Problems with Infinite and Finite $j\omega$-Axis Zeros

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Abstract

The 4-block H^∞ control problem with infinite and finite $j\omega$-axis is discussed in this paper. Via the eigenstructures related to the infinite and finite $j\omega$-zeros, this paper extends the DGKF's approach to the H^∞ control problem without the constraints on the infinite or finite $j\omega$-axis zeros. The necessary and sufficient conditions are proposed for checking its solvability by solving two reduced-order Riccati equations and examining matrix norm conditions related to $j\omega$-axis zeros.

1 Introduction

Consider a generalized plant described as

$$\begin{aligned}\begin{bmatrix} z \\ y \end{bmatrix} &= P(s)\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]\begin{bmatrix} w \\ u \end{bmatrix}, \end{aligned} \qquad (1)$$

where $z \in R^m$, $y \in R^q$, $w \in R^r$ and $u \in R^p$ are the controlled error, the observation output, the exogenous input and the control input, respectively. The H^∞ control problem is to find a *proper* control law $u(s) = K(s)y(s)$ which internally stabilizes the closed-loop system and satisfies $\|\Phi(s)\|_\infty < 1$, where $\Phi(s)$ is the closed-loop transfer function from w to z given by

$$\Phi(s) = F_l(P\,;\,K) := P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1}P_{21}. \quad (2)$$

It is well known that the *standard* H^∞ control problems has been solved [1] when plant (1) satisfies the following assumptions:

(A_1) (A, B_2, C_2) is stabilizable and detectable.
(A_2) rank $D_{12} = p$, rank $D_{21} = q$.
(A_3) $P_{12}(s)$ and $P_{21}(s)$ have no $j\omega$-axis invariant zeros.

The assumption (A_1) is *necessary* for the close-loop stability. An H^∞ control problem is called *non-standard* or *singular* if (A_2) and/or (A_3) do/does not hold. The above non-standard H^∞ control problem, which is often encountered in many practical cases, has attracted considerable research interests [2]–[9].

In this paper, instead of assumptions (A_2) and (A_3), we assume that

(A_4) $P_{12}(s)$ and $P_{21}(s)$ have full normal column and row ranks, respectively.

The above assumption can allow $P_{12}(s)$ and/or $P_{21}(s)$ to have invariant zeros on $j\omega$-axis including the infinity (denoted as Ω_e). The purpose of this paper is to extend the DGKF's approach [1] to the H^∞ control problem without the constraints on the infinite or finite $j\omega$-axis zeros and to provide the necessary and sufficient conditions for its solvability in terms of solutions of *Riccati equations* or *generalized eigenvalue problems*. In this paper, Ω_e-eigenstructure of a tall pencil with full normal rank discussed in [11] and the lossless factorization for $P(s)$ similar to [3] and [10] play an important role.

Notations: The open left and right half complex plane are denoted by C_- and C_+, respectively. The $j\omega$-axis is denoted by Ω. The set of all $m \times r$ constant real matrices is denoted by $R^{m\times r}$. I_r denotes the identity matrix of size $r \times r$. $RH^\infty_{m\times r}$ denotes the set of all $m \times r$ rational stable proper matrices, and $BH^\infty_{m\times r}$ denotes the subset of $RH^\infty_{m\times r}$ with H^∞-norm less than 1. $\sigma(A)$ denotes the set of all eigenvalues of matrix A. $\rho(X)$ is the maximum eigenvalue of X. Im A and Ker A denote the image space and null space of matrix A, respectively. We denote $G^\sim(s) := G^T(-s)$ and express the star product of M_1 and M_2 by $M = M_1 * M_2$ so that $F_l(M_1, F_l(M_2, K)) = F_l(M_1 * M_2, K)$ holds.

2 Preliminaries

2.1 Infinite eigenstructures

Denote the system matrix pencils of $P_{12}(s)$ and $P_{21}^T(s)$ as $-sP_E + P_A$ and $-s\tilde{P}_E + \tilde{P}_A$, respectively, where

$$P_E := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad P_A := \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix}. \qquad (3)$$

$$\tilde{P}_E := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{P}_A := \begin{bmatrix} A^T & C_2^T \\ B_1^T & D_{21}^T \end{bmatrix}. \quad (4)$$

According to assumption (A_4), the above two pencils has full normal column ranks.

Let $(v_1^1, \cdots, v_p^1)$ be a base of Ker P_E. Then the infinite eigenvectors are defined by

$$P_E v_j^1 = 0, \quad j = 1, \cdots, p, \quad (5)$$

$$P_E v_j^{k+1} = P_A v_j^k, \quad k = 1, \cdots, k_j - 1, \quad (6)$$

where $v_j^{k_j}$ is the last one of each infinite eigenvector chain satisfying $P_A v_j^{k_j} \notin \operatorname{Im} P_E$. Now construct

$$V_\infty := [\ V_r \quad V_h\ ], \quad (7)$$

where $V_h \in R^{(n+p)\times p}$ contains all the *last* infinite eigenvectors and V_r are the remainders. Therefore, the complete infinite eigenstructure of $-sP_E + P_A$ is defined by

$$(-sP_E + P_A)V_\infty = P_A V_\infty(-sN + I), \quad (8)$$

where N is a nilpotent matrix. From (6), we know that $[\ C_1 \quad D_{12}\ ] V_r = 0$, then decompose

$$P_A V_\infty = \begin{bmatrix} A & B_2 \\ \hdashline C_1 & D_{12} \end{bmatrix} [\ V_r \ \vdots\ V_h\ ] =: \begin{bmatrix} T & \hat{B}_2 \\ 0 & \hat{D}_{12} \end{bmatrix}, \quad (9)$$

which yields

$$T := [\ A \quad B_2\ ] V_r, \quad \hat{B}_2 := [\ A \quad B_2\ ] V_h, \quad (10)$$

$$\hat{D}_{12} := [\ C_1 \quad D_{12}\ ] V_h. \quad (11)$$

Note that $\hat{D}_{12}$ is *injective* [11].

Dually consider $P_{21}^T(s)$. Now arrange all the infinite eigenvectors of $-s\tilde{P}_E + \tilde{P}_A$ as

$$\tilde{V}_\infty := [\ \tilde{V}_r \quad \tilde{V}_h\ ], \quad (12)$$

where $\tilde{V}_h \in R^{(n+q)\times q}$ contains all the *last* infinite eigenvectors and $\tilde{V}_r$ are the reminders. From $\tilde{P}_A \tilde{V}_\infty$, define

$$\tilde{T} := [\ A^T \quad C_2^T\ ] \tilde{V}_r, \quad \hat{C}_2^T := [\ A^T \quad C_2^T\ ] \tilde{V}_h, \quad (13)$$

$$\hat{D}_{21}^T := [\ B_1^T \quad D_{21}^T\ ] \tilde{V}_h, \quad (14)$$

which follows that $\hat{D}_{21}^T$ is also *injective.*

2.2 Finite $j\omega$-axis eigenstructures

Let the $j\omega$-axis eigenspaces of $-sP_E + P_A$ and $-s\tilde{P}_E + \tilde{P}_A$ be spanned by real $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ and $\begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \end{bmatrix}$, respectively. It follows that there exist Λ_j and $\tilde{\Lambda}_j$ such that $\sigma(\Lambda_j) \subset \Omega$ and $\sigma(\tilde{\Lambda}_j) \subset \Omega$ hold, and

$$(-sP_E + P_A) \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix} (-sI + \Lambda_j), \quad (15)$$

$$(-s\tilde{P}_E + \tilde{P}_A) \begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \end{bmatrix} = \begin{bmatrix} \tilde{T}_1 \\ 0 \end{bmatrix} (-sI + \tilde{\Lambda}_j). \quad (16)$$

2.3 Stable eigenstructures

Denote the system matrices of $P_{12}^T(-s)P_{12}(s)$ and $P_{21}(-s)P_{21}^T(s)$ as:

$$W_{12}(s) := \begin{bmatrix} -sI + A & 0 & B_2 \\ -C_1^T C_1 & -sI - A^T & -C_1^T D_{12} \\ D_{12}^T C_1 & B_2^T & D_{12}^T D_{12} \end{bmatrix}, \quad (17)$$

$$W_{21}(s) := \begin{bmatrix} -sI + A^T & 0 & C_2^T \\ -B_1 B_1^T & -sI - A & -B_1 D_{21}^T \\ D_{21} B_1^T & C_2 & D_{21} D_{21}^T \end{bmatrix}. \quad (18)$$

Let $\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$ and $\begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{bmatrix}$ spanned stable eigenspace of $W_{12}(s)$ and $W_{21}(s)$, respectively. There exist stable Λ_{12} and Λ_{21} such that

$$W_{12}(s) \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix} (-sI + \Lambda_{12}), \quad (19)$$

$$W_{21}(s) \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{bmatrix} = \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ 0 \end{bmatrix} (-sI + \Lambda_{21}). \quad (20)$$

From [11], we can obtain

LEMMA 1 *Under the assumptions (A_1) and (A_4). Then S and $\tilde{S}$ are nonsingular, where*

$$S := [\ U_1 \quad T_1 \quad T\ ], \quad \tilde{S} := [\ \tilde{U}_1 \quad \tilde{T}_1 \quad \tilde{T}\ ]. \quad (21)$$

3 Solvability Conditions

We are ready to state the main results of this paper.

THEOREM 1 *Under the assumptions (A_1) and (A_4), the H^∞ control problem for plant $P(s)$ in (1) is solvable if and only if the following statement holds.*

(i) The following Riccati equation has a stabilizing solution $X_r \geq 0$,

$$(A_r - \hat{B}_{r2}E_{12}^{-1}\hat{D}_{12}^T C_{r1})^T X_r + X_r(A_r - \hat{B}_{r2}E_{12}^{-1}\hat{D}_{12}^T C_{r1})$$
$$+ X_r(B_{r1}B_{r1}^T - \hat{B}_{r2}E_{12}^{-1}\hat{B}_{r2}^T)X_r + C_{r1}^T(I - \hat{D}_{12}E_{12}^{-1}\hat{D}_{12}^T)C_{r1} = 0, \quad (22)$$

where $E_{12} := \hat{D}_{12}^T \hat{D}_{12}$, and

$$A_r := L_1 A U_1, \quad B_{r1} := L_1 B_1, \quad (23)$$

$$\hat{B}_{r2} := L_1 \hat{B}_2, \quad C_{r1} := C_1 U_1, \quad (24)$$

$$[\ L_1^T \quad L_2^T \quad L_3^T\ ]^T := [\ U_1 \quad T_1 \quad T\ ]^{-1}, \quad (25)$$

where T, $\hat{B}_2$ are given by (10), $\hat{D}_{12}$ is given by (11), T_1 is given by (15), and U_1 is given by (19), respectively.

(ii) The following Riccati equation has a stabilizing solution $Y_r \geq 0$,

$$Y_r(\tilde{A}_r - \tilde{B}_{r1}\hat{D}_{21}^T E_{21}^{-1}\hat{C}_{r2})^T + (\tilde{A}_r - \tilde{B}_{r1}\hat{D}_{21}^T E_{21}^{-1}\hat{C}_{r2})Y_r$$
$$+Y_r(\tilde{C}_{r1}^T\tilde{C}_{r1} - \hat{C}_{r2}^T E_{21}^{-1}\hat{C}_{r2})Y + \tilde{B}_{r1}(I - \hat{D}_{21}^T E_{21}^{-1}\hat{D}_{21})\tilde{B}_{r1}^T = 0, \quad (26)$$

where $E_{21} := \hat{D}_{21}\hat{D}_{21}^T$, and

$$\tilde{A}_r := \tilde{U}_1^T A \tilde{L}_1^T, \quad \tilde{B}_{r1} := \tilde{U}_1^T B_1, \quad (27)$$

$$\hat{C}_{r2} := \hat{C}_2 \tilde{L}_1^T, \quad \tilde{C}_{r1} := \tilde{U}_1^T C_1, \quad (28)$$

$$\left[\ \tilde{L}_1^T \quad \tilde{L}_2^T \quad \tilde{L}_3^T\ \right]^T := \left[\ \tilde{U}_1 \quad \tilde{T}_1 \quad \tilde{T}\ \right]^{-1}, \quad (29)$$

where $\tilde{T}$, $\hat{C}_2$ are given by (13), $\hat{D}_{21}$ is given by (14), $\tilde{T}_1$ is given by (16), and $\tilde{U}_1$ is given by (20), respectively.

(iii) $\rho(XY) < 1$, where

$$X := L_1^T X_r L_1, \quad Y := \tilde{L}_1^T Y_r \tilde{L}_1. \quad (30)$$

(iv)

$$U_{12i}^* U_{12i} > X_{12i}^* B_{n1} B_{n1}^T X_{12i}, \quad (31)$$

$$U_{21i}^* U_{21i} > X_{21i}^* C_{n1}^T C_{n1} X_{21i}, \quad (32)$$

where X_{12i}, U_{12i}, X_{21i} and U_{21i} satisfy

$$\begin{bmatrix} -j\omega_i I + A_n^T & C_{n1}^T \\ B_{n2}^T & N_{12}^T \end{bmatrix} \begin{bmatrix} X_{12i} \\ U_{12i} \end{bmatrix} = 0, \quad (33)$$

$$\begin{bmatrix} -j\omega_i I + A_n & B_{n1} \\ C_{n2} & N_{21} \end{bmatrix} \begin{bmatrix} X_{21i} \\ U_{21i} \end{bmatrix} = 0, \quad (34)$$

where A_n, B_{n1}, B_{n2}, C_{n1}, C_{n2} are defined by the following new plant as:

$$P_n(s) = \begin{bmatrix} P_{n11} & P_{n12} \\ P_{n21} & P_{n22} \end{bmatrix} = \left[\begin{array}{c|cc} A_n & B_{n1} & B_{n2} \\ \hline C_{n1} & 0 & N_{12} \\ C_{n2} & N_{21} & 0 \end{array}\right] \quad (35)$$

with its matrices defined by

$$A_n := A + B_1 B_1^T X + ZYF_\infty^T F_\infty, \quad (36)$$

$$B_{n1} := -ZL_\infty, \quad B_{n2} := B_2 - ZYF_\infty^T N_{12}, \quad (37)$$

$$C_{n1} := -F_\infty, \quad C_{n2} := (C_2 - N_{21} L_\infty^T XZ)Z^{-1}, \quad (38)$$

$$Z := (I - YX)^{-1}, \quad (39)$$

$$F_\infty := -E_{12}^{-1/2}(\hat{B}_2^T X + \hat{D}_{12}^T C_1), \quad (40)$$

$$L_\infty := -(Y\hat{C}_2^T + B_1\hat{D}_{21}^T)E_{21}^{-1/2}, \quad (41)$$

$$N_{12} := E_{12}^{-1/2}\hat{D}_{12}^T D_{12}, \quad (42)$$

$$N_{21} := (E_{21}^{-1/2}\hat{D}_{21} D_{21}^T)^T. \quad (43)$$

Moreover, if the H^∞ control problem for plant $P(s)$ in (1) is solvable, then (A_n, B_{n2}, C_{n2}) is stabilizable and detectable.

Remark 3.1 Riccati equation (22) is of size $n - (n_\infty - p) - n_j$, where $n_\infty := \sum_{j=1}^{p} k_j \geq p$ and n_j are the dimensions of $\{\infty\}$ and $j\omega$-axis eigenspaces of $-sP_E + P_A$, respectively. Similar analysis can be given to Riccati equation (22). Note that ω_i satisfying (33) and/or (34) are the invariant zeros of $P_{n12}(s)$ and/or $P_{n21}(s)$. It can be shown such ω_i are also the invariant zeros of $P_{12}(s)$ and/or $P_{21}(s)$. Therefore, all conditions in Theorem 1 can be checked easily by solving two reduced-order Riccati equations and checking static conditions related to $j\omega$-axis zeros of $P_{12}(s)$ and/or $P_{21}(s)$.

Remark 3.2 If assumption (A_2) holds, from (10), (11), (13) and (14), choose $V_\infty = V_h = [\ 0 \quad I_p\ ]^T$ and $\tilde{V}_\infty = \tilde{V}_h = [\ 0 \quad I_q\ ]^T$, we obtain $\hat{B}_2 = B_2$, $\hat{D}_{12} = D_{12}$, $\hat{C}_1 = C_1$, $\hat{D}_{21} = D_{21}$. If assumption (A_3) holds, (A_4) holds trivially, and Condition *(iv)* no longer exists. If both assumption (A_2) and assumption (A_3) hold, we can choose $L_1 = U_1 = \tilde{L}_1 = \tilde{U}_1 = I_n$. Theorem 1 is reduced to the results of the standard H^∞ control problems [1].

To establish the relation of Theorem 1 with [3], [9], in what follows, we obtain the explicit solution to the QMIs with rank constraints in those papers.

LEMMA 2 *Under the assumptions (A_1) and (A_4), if Conditions (i)–(iv) in Theorem 1 hold, then*

(i) X, F_∞ and N_{12} in (30), (40) and (42) satisfy

$$\begin{bmatrix} XA + A^T X + XB_1B_1^T X + C_1^T C_1 & XB_2 + C_1^T D_{12} \\ B_2^T X + D_{12}^T C_1 & D_{12}^T D_{12} \end{bmatrix}$$
$$= \begin{bmatrix} -F_\infty^T \\ N_{12}^T \end{bmatrix} [\ -F_\infty \quad N_{12}\ ] \geq 0. \quad (44)$$

$$\text{rank} \begin{bmatrix} -sI + A + B_1 B_1^T X & B_2 \\ -F_\infty & N_{12} \end{bmatrix} = n + p, \quad s \in C_+, \quad (45)$$

which implies that $P_{n12}(s)$ has no invariant zeros in C_+. Moreover, $P_{n12}(s)$ and $P_{12}(s)$ have the same finite $j\omega$-axis invariant zeros, and (A_n, B_{n2}) is stabilizable.

(ii) Y, L_∞ and N_{21} in (30), (41) and (43) satisfy

$$\begin{bmatrix} AY + YA^T + YC_1^T C_1 Y + B_1 B_1^T & YC_2^T + B_1 D_{21}^T \\ C_2 Y + D_{21} B_1^T & D_{21} D_{21}^T \end{bmatrix}$$
$$= \begin{bmatrix} -L_\infty \\ N_{21} \end{bmatrix} [\ -L_\infty^T \quad N_{21}^T\ ] \geq 0, \quad (46)$$

$$\text{rank} \begin{bmatrix} -sI + A + YC_1^T C_1 & -L_\infty \\ C_2 & N_{21} \end{bmatrix} = n + p, \quad s \in C_+, \quad (47)$$

which implies that $P_{n21}(s)$ has no invariant zeros in C_+. Moreover, $P_{n21}(s)$ and $P_{21}(s)$ have the same finite $j\omega$-axis invariant zeros, and (A_n, C_{n2}) is detectable.

4 Proof of Necessary Conditions

We briefly introduce the following steps in the proof of the necessity of Theorem 1:

Step 1 Prove Condition (i) via the solvability of the full information (FI) problem for $P(s)$.

Step 2 Perform lossless factorization $P(s) = \Theta(s) * P_{tmp}(s)$ to get a 2-block plant $P_{tmp}(s)$, where $\Theta(s)$ is an inner matrix.

Step 3 Prove Conditions (ii) and (iii) via the solvability of the FI problem corresponding to $P_{tmp}(s)$. This step is just a copy of Step 1.

Step 4 Perform the lossless factorization $P_{tmp}^T(s) = \Psi^T(s) * P_n^T(s)$ to get 1-block plant $P_n(s)$. This step is just a copy of Step 2.

Step 5 Prove Condition (iv) via the static solvability conditions related to the $j\omega$-axis zeros of $P_n(s)$.

4.1 FI Problem for $P(s)$

Since the H^∞ control problem for the FI case of $P(s)$ in (1)

$$P_{FI}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \end{array}\right] \tag{48}$$

is solvable, we have

LEMMA 3 *Suppose the 4-block H^∞ control problem is solvable. Then the FI H^∞ control problem for*

$$P_{FIr}(s) = \left[\begin{array}{c|cc} A_r & B_{r1} & \hat{B}_{r2} \\ \hline C_{r1} & 0 & \hat{D}_{12} \end{array}\right] \tag{49}$$

is solvable, where A_r, B_{r1} are defined in (27), $\hat{B}_{r2}$ is defined in (24), and $\hat{D}_{12}$ is defined (11). Moreover,

$$P_{12r}(s) := \left[\begin{array}{c|c} A_r & \hat{B}_{r2} \\ \hline C_{r1} & \hat{D}_{12} \end{array}\right] \tag{50}$$

is a stabilizable realization and has no finite $j\omega$-axis invariant zeros.

Proof of Condition (i) of Theorem 1 is a direct consequence of Lemma 3 and the result of standard H^∞ control problem in [1].

Now we can construct the following Riccati equation of size n, which will be used later.

LEMMA 4 *Suppose the 4-block H^∞ control problem is solvable. Then*

$$(A - \hat{B}_2 E_{12}^{-1}\hat{D}_{12}^T C_1)^T X + X(A - \hat{B}_2 E_{12}^{-1}\hat{D}_{12}^T C_1)$$

$$+X(B_1 B_1^T - \hat{B}_2 E_{12}^{-1}\hat{B}_2^T)X + C_1^T(I - \hat{D}_{12}E_{12}^{-1}\hat{D}_{12}^T)C_1 = 0, \tag{51}$$

has a solution

$$X = L_1^T X_r L_1 = S^{-T}\begin{bmatrix} X_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1}, \tag{52}$$

which yields

$$XT_1 = 0, \quad XT = 0. \tag{53}$$

4.2 Lossless Factorization of $P(s)$

LEMMA 5 *Suppose the 4-block H^∞ control problem is solvable. Then $P(s)$ can be factorized as*

$$P(s) = \Theta(s) * P_{tmp}(s), \tag{54}$$

where

$$\Theta(s) = \begin{bmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_r + \hat{B}_{r2}\hat{F}_r & B_{r1} & \hat{B}_{r2}E_{12}^{-1/2} \\ \hline C_{r1} + \hat{D}_{12}\hat{F}_r & 0 & \hat{D}_{12}E_{12}^{-1/2} \\ -B_{r1}^T X_r & I_r & 0 \end{array}\right], \tag{55}$$

$$P_{tmp}(s) = \left[\begin{array}{c|cc} A + B_1 B_1^T X & B_1 & B_2 \\ \hline -F_\infty & 0 & N_{12} \\ C_2 + D_{21}B_1^T X & D_{21} & 0 \end{array}\right], \tag{56}$$

where $\hat{F}_r := -E_{12}^{-1}(\hat{B}_{r2}^T X_r + \hat{D}_{12}^T C_{r1})$. And $\Theta(s) = I$ is a lossless matrix, i.e., $\Theta^\sim(s)\Theta(s) = I$, $\Theta(s) \in RH^\infty$ and $\Theta_{21}^{-1}(s) \in RH^\infty$. Moreover, $(A + B_1 B_1^T X, B_2, C_2 + D_{21}B_1^T)$ is stabilizable and detectable.

According to Lemma 15 in [1], the solvability of H^∞ control problems for $P(s)$ and $P_{tmp}(s)$ is equivalent with the same controller.

4.3 FI Problem for $P_{tmp}^T(s)$

Now we apply Condition (i) in Theorem 1 to

$$P_{tmp}^T(s) = \left[\begin{array}{c|cc} A^T + XB_1B_1^T & -F_\infty^T & C_2^T + XB_1D_{21}^T \\ \hline B_1^T & 0 & D_{21}^T \\ B_2^T & N_{12}^T & 0 \end{array}\right]. \tag{57}$$

To this end, we have to study the Ω_e eigenstructure of $-s\bar{P}_E + \bar{P}_A$ with $\bar{P}_E = \tilde{P}_E$

$$\bar{P}_A := \begin{bmatrix} A^T + XB_1B_1^T & C_2^T + XB_1D_{21}^T \\ B_1^T & D_{21}^T \end{bmatrix}, \tag{58}$$

which yields

$$-s\bar{P}_E + \bar{P}_A = \begin{bmatrix} I & XB_1 \\ 0 & I \end{bmatrix}(-s\tilde{P}_E + \tilde{P}_A), \tag{59}$$

where $\tilde{P}_E$ and $\tilde{P}_A$ are defined in (4). It follows that $\tilde{V}_\infty$ in (12) contains all the infinite eigenvectors of $-s\bar{P}_E + \bar{P}_A$. From (13) and (14), we obtain

$$\bar{P}_A\tilde{V}_\infty = \begin{bmatrix} \bar{T} & \bar{C}_2^T \\ 0 & \hat{D}_{21}^T \end{bmatrix}, \tag{60}$$

where $\bar{C}_2 := \hat{C}_2 + \hat{D}_{21}B_1^T X$. From (59) and (16), we get

$$(-s\bar{P}_E + \bar{P}_A)\begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \end{bmatrix} = \begin{bmatrix} \tilde{T}_1 \\ 0 \end{bmatrix}(-sI + \tilde{\Lambda}_j). \tag{61}$$

By applying Lemma 4 to $P_{tmp}^T(s)$, we know that

$$W(A + B_1B_1^T X - B_1\hat{D}_{21}^T E_{21}^{-1}\bar{C}_2)^T + (A + B_1B_1^T X$$

$$-B_1\hat{D}_{21}^T E_{21}^{-1}\bar{C}_2)W + W(F_\infty^T F_\infty - \bar{C}_2^T E_{21}^{-1}\bar{C}_2)W$$

$$+ B_1(I - \hat{D}_{21}^T E_{21}^{-1}\hat{D}_{21})B_1^T = 0 \quad (62)$$

has solution $W \geq 0$ with

$$W\tilde{T}_1 = 0, \quad W\tilde{T} = 0. \quad (63)$$

Consider

$$Y(A - B_1\hat{D}_{21}^T E_{21}^{-1}\hat{C}_2)^T + (A - B_1\hat{D}_{21}^T E_{21}^{-1}\hat{C}_2)Y$$

$$+Y(C_1^T C_1 - \hat{C}_2^T E_{21}^{-1}\hat{C}_2)Y + B_1(I - \hat{D}_{21}^T E_{21}^{-1}\hat{D}_{21})B_1^T = 0. \quad (64)$$

Let H_Y and H_W be the Hamiltonian matrices corresponding to Riccati equations (64) and (62), respectively. By direct calculation, we have

$$H_W = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} H_Y \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}. \quad (65)$$

Let

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ W \end{bmatrix}. \quad (66)$$

Therefore,

$$Y = Y_2 Y_1^{-1} = W(I + XW)^{-1} = (I + WX)^{-1}W \geq 0 \quad (67)$$

is a solution of (64). Since $I + WX = (I - YX)^{-1} = Z > 0$, we get $\rho(XY) < 1$. Thus, $Y = Z^{-1}W$. It yields from (63) that $Y\tilde{T}_1 = 0$ and $Y\tilde{T} = 0$. Then Y in (67) can be represented as

$$Y = \tilde{S}^{-T}\begin{bmatrix} Y_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\tilde{S}^{-1} = \tilde{L}_1^T Y_r \tilde{L}_1, \quad (68)$$

which follows that $Y_r \geq 0$ is a solution of (26). Moreover, we can show that Y_r is the stabilizing one. □

4.4 Lossless Factorization of $P_{tmp}^T(s)$

From (62), define G^∞ corresponding to F_∞ in (40) as

$$G_\infty^T := -E_{21}^{-1/2}(\bar{C}_2 W + \hat{D}_{21}B_1^T). \quad (69)$$

which follows that $G_\infty^T = L_\infty^T Z^T$, where L_∞^T is defined in (41). According to Lemma 5, the lossless factorization for $P_{tmp}^T(s)$ is

$$P_{tmp}^T(s) = \Psi^T(s) * P_n^T(s), \quad (70)$$

where $\Psi^T(s)$ is lossless matrix whose explicit form is omitted for the brevity, and $P_n(s)$ is given by (35). Observe from Lemma 5 that (A_n, B_{n2}, C_{n2}) is stabilizable and detectable. Based on two lossless factorizations (54) and (70), we have

THEOREM 2 *The solvability of H^∞ control problems for $P(s)$ and $P_n(s)$ is equivalent with the same controller $K(s)$.*

4.5 Static Conditions Related to Finite $j\omega$-axis Zeros

From Theorem 2, the H^∞ control problem for 1-block plant $P_n(s)$ in (35) is solvable. Let $s = j\omega_i$ $(i = 1 \sim k)$ be the invariant zeros of $P_{n12}(s)$ and/or $P_{n21}(s)$, i.e., (33) and (34) hold. Condition (iv) is a direct consequence of Theorem 6 in [4]. □

5 Proof of Sufficiency Conditions

We can first prove Lemma 2. Then, since $P_n(s)$ is 1-block plant, $P_{n12}(s)$ and $P_{n21}(s)$ have no invariant zeros in $\bar{C}_+$, according to Theorem 6 in [4], the matrix norm conditions related to $j\omega$-axis zeros are satisfied and two generalized Riccati equations have solutions of zero matrices. Therefore, the H^∞ control problem for plant $P_n(s)$ is solvable, so is for plant $P(s)$ according to Theorem 2. □

Finally, as to the parameterization of all H^∞ controllers, it will be reported in [12].

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