Solvability Conditions for 4-block $H^\infty$ Control Problems with Infinite and Finite $j\omega$-Axis Zeros

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Abstract

The 4-block $H^\infty$ control problem with infinite and finite $j\omega$-axis is discussed in this paper. Via the eigenstructures related to the infinite and finite $j\omega$-zeros, this paper extends the DGKF’s approach to the $H^\infty$ control problem without the constraints on the infinite or finite $j\omega$-axis. The necessary and sufficient conditions are proposed for checking its solvability by solving two reduced-order Riccati equations and examining matrix norm conditions related to $j\omega$-axis zeros.

1 Introduction

Consider a generalized plant described as

$$\begin{bmatrix}
  z \\
  y
\end{bmatrix} = P(s) \begin{bmatrix}
  w \\
  u
\end{bmatrix} = \begin{bmatrix}
  P_{11} & P_{12} \\
  P_{21} & P_{22}
\end{bmatrix} \begin{bmatrix}
  w \\
  u
\end{bmatrix},$$

where $z \in \mathbb{R}^m$, $y \in \mathbb{R}^q$, $w \in \mathbb{R}^r$ and $u \in \mathbb{R}^s$ are the controlled error, the observation output, the exogenous input and the control input, respectively. The $H^\infty$ control problem is to find a proper control law $u(s) = K(s)y(s)$ which internally stabilizes the closed-loop system and satisfies $\|\Phi(s)\|_\infty < 1$, where $\Phi(s)$ is the closed-loop transfer function from $w$ to $z$ given by

$$\Phi(s) = F_1(P; K) := P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}. \quad (2)$$

It is well known that the standard $H^\infty$ control problems has been solved [1] when plant (1) satisfies the following assumptions:

(A1) $(A, B_2, C_2)$ is stabilizable and detectable.
(A2) rank $D_{12} = p$, rank $D_{21} = q$.
(A3) $P_{12}(s)$ and $P_{21}(s)$ have no $j\omega$-axis invariant zeros.

The assumption (A1) is necessary for the close-loop stability. An $H^\infty$ control problem is called non-standard or singular if (A2) and/or (A3) do not hold. The above non-standard $H^\infty$ control problem, which is often encountered in many practical cases, has attracted considerable research interests [2]-[9].

In this paper, instead of assumptions (A2) and (A3), we assume that

(A4) $P_{12}(s)$ and $P_{21}(s)$ have full normal column and row ranks, respectively.

The above assumption can allow $P_{12}(s)$ and $P_{21}(s)$ to have invariant zeros on $j\omega$-axis including the infinity (denoted as $\infty$). The purpose of this paper is to extend the DGKF’s approach [1] to the $H^\infty$ control problem without the constraints on the infinite or finite $j\omega$-axis zeros and to provide the necessary and sufficient conditions for its solvability in terms of solutions of Riccati equations or generalized eigenvalue problems. In this paper, $\Omega$-eigenstructure of a tall pencil with full normal rank discussed in [11] and the lossless factorization for $P(s)$ similar to [3] and [10] play an important role.

Notations: The open left and right half complex plane are denoted by $C_{\infty}$ and $C_{\infty}^+$, respectively. The $j\omega$-axis is denoted by $\Omega$. The set of all $m \times r$ constant real matrices is denoted by $\mathbb{R}^{m \times r}$. $I_r$ denotes the identity matrix of size $r \times r$. $RH^\infty_{m \times r}$ denotes the set of all $m \times r$ rational stable proper matrices, and $BH^\infty_{m \times r}$ denotes the subset of $RH^\infty_{m \times r}$ with $H^\infty$-norm less than 1. $\sigma(A)$ denotes the set of all eigenvalues of matrix $A$. $\rho(X)$ is the maximum eigenvalue of $X$. Im $A$ and Ker $A$ denote the image space and null space of matrix $A$, respectively. We denote $G^\ast(s) := G^T(-s)$ and express the star product of $M_1$ and $M_2$ by $M_1 \ast M_2$ so that $F_1(M_1, F_1(M_2, K)) = F_1(M_1 \ast M_2, K)$ holds.

2 Preliminaries

2.1 Infinite eigenstructures

Denote the system matrix pencils of $P_{12}(s)$ and $P_{21}(s)$ as $-sP_E + P_A$ and $-sP_E + P_A$, respectively, where

$$P_E := \begin{bmatrix}
  I_n & 0 \\
  0 & 0
\end{bmatrix}, \quad P_A := \begin{bmatrix}
  A & B_2 \\
  C_1 & D_{12}
\end{bmatrix}. \quad (3)$$
\( \hat{P}_E := \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{P}_A := \begin{bmatrix} AT & CT \\ BT & DT \end{bmatrix}. \)

According to assumption (A4), the above two pencils have full normal column ranks.

Let \((v_1^1, \ldots, v_1^n)\) be a base of \(\text{Ker } \hat{P}_E\). Then the infinite eigenvectors are defined by

\[
P_E v_j^1 = 0, \quad j = 1, \ldots, p, \tag{5}
\]

\[
P_E v_j^{k+1} = P_A v_j^k, \quad k = 1, \ldots, k_j - 1, \tag{6}
\]

where \(v_j^k\) is the last one of each infinite eigenvector chain satisfying \(P_A v_j^k \notin \text{Im } \hat{P}_E\). Now construct

\[ V_\infty := [ V_r \ V_h ], \tag{7} \]

where \(V_h \in \mathbb{R}^{(n+p) \times p}\) contains all the last infinite eigenvectors and \(V_r\) are the remainders. Therefore, the complete infinite eigenstructure of \(-s\hat{P}_E + \hat{P}_A\) is defined by

\[
(-s\hat{P}_E + \hat{P}_A) V_\infty = P_A V_\infty (sN + I), \tag{8}
\]

where \(N\) is a nilpotent matrix. From (6), we know that \([ \begin{array}{c} C_1 \\ D_1 \end{array} ] V_r = 0\), then decompose

\[
P_A V_\infty = \begin{bmatrix} A & C_1 \\ B & D_1 \end{bmatrix} [ V_r : V_h ] := \begin{bmatrix} T & \hat{B}_2 \\ 0 & \hat{D}_{12} \end{bmatrix}, \tag{9}
\]

which yields

\[
T := [ A \ B_2 ] V_r, \quad \hat{B}_2 := [ A \ B_2 ] V_h, \tag{10}
\]

\[
\hat{D}_{12} := [ \begin{array}{c} C_1 \\ D_1 \end{array} ] V_h. \tag{11}
\]

Note that \(\hat{D}_{12}\) is injective [11].

Dually consider \(\hat{P}_E^T(s)\). Now arrange all the infinite eigenvectors of \(-s\hat{P}_E + \hat{P}_A\) as

\[ V_\infty := [ V_r \ V_h ], \tag{12} \]

where \(V_h \in \mathbb{R}^{(n+p) \times q}\) contains all the last infinite eigenvectors and \(V_r\) are the remainders. From \(\hat{P}_A V_\infty\), define

\[
\hat{T} := [ A^T \ C_2^T ] V_r, \quad \hat{C}_2^T := [ A^T \ C_2^T ] V_h, \tag{13}
\]

\[
\hat{D}_{21} := [ B^T \ D_{21}^T ] V_h, \tag{14}
\]

which follows that \(\hat{D}_{21}\) is also injective [11].

2.2 Finite \(j_\omega\)-axis eigenstructures

Let the \(j_\omega\)-axis eigenspaces of \(-s\hat{P}_E + \hat{P}_A\) be spanned by real \([ \begin{array}{c} T_1 \\ T_2 \end{array} ]\) and \([ \begin{array}{c} \hat{T}_1 \\ \hat{T}_2 \end{array} ]\), respectively. It follows that there exist \(\Lambda_j\) and \(\hat{\Lambda}_j\) such that \(\sigma(\Lambda_j) \subset \Omega\) and \(\sigma(\hat{\Lambda}_j) \subset \Omega\) hold, and

\[
(-s\hat{P}_E + \hat{P}_A) \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix} (-sI + \Lambda_j), \tag{15}
\]

\[
(-s\hat{P}_E + \hat{P}_A) \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \end{bmatrix} = \begin{bmatrix} \hat{T}_1 \\ 0 \end{bmatrix} (-sI + \hat{\Lambda}_j). \tag{16}
\]

2.3 Stable eigenstructures

Denote the system matrices of \(P_1^T(-s)P_1(s)\) and \(P_2^T(-s)P_2(s)\) as:

\[
W_{12}(s) := \begin{bmatrix} -sI + A & 0 \\ -C_1^T C_1 & -sI + A^T & -C_1^T D_{12} \end{bmatrix}, \tag{17}
\]

\[
W_{21}(s) := \begin{bmatrix} -sI + A^T & 0 \\ -B_1 C_2^T & -sI + A & -B_1 D_{21}^T \end{bmatrix}. \tag{18}
\]

Let \([ \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array} ]\) and \([ \begin{array}{c} \hat{U}_1 \\ \hat{U}_2 \end{array} ]\) spanned stable eigenspace of \(W_{12}(s)\) and \(W_{21}(s)\), respectively. There exist stable \(\Lambda_{12}\) and \(\Lambda_{21}\) such that

\[
W_{12}(s) \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ 0 \end{bmatrix} (-sI + \Lambda_{12}), \tag{19}
\]

\[
W_{21}(s) \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} = \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \\ 0 \end{bmatrix} (-sI + \Lambda_{21}). \tag{20}
\]

From [11], we can obtain

**LEMMA 1** Under the assumptions (A1) and (A4). Then \(\hat{S}\) and \(S\) are nonsingular, where

\[
S := [ U_1 \ T_1 \ T ], \quad \hat{S} := [ \hat{U}_1 \ \hat{T}_1 \ \hat{T} ]. \tag{21}
\]

3 Solvability Conditions

We are ready to state the main results of this paper.

**THEOREM 1** Under the assumptions (A1) and (A4), the \(H^\infty\) control problem for plant \(P(s)\) in (1) is solvable if and only if the following statement holds.

(i) The following Riccati equation has a stabilizing solution \(X_\infty \geq 0\),

\[
(A_r - \hat{B}_2 E_{12}^{-1} \hat{D}_{12}^T C_{12}^T) X_r + X_r (A_r - \hat{B}_2 E_{12}^{-1} \hat{D}_{12}^T C_{12}) + X_r (B_r C_{12}^T - \hat{B}_2 E_{12}^{-1} \hat{D}_{12}) X_r + C_{12}^T (I - \hat{D}_{12} E_{12}^{-1} \hat{D}_{12}^T) C_{12} = 0, \tag{22}
\]

where \(E_{12} := \hat{D}_{12}^T \hat{D}_{12},\) and

\[
A_r := L_1 A U_1, \quad B_{r1} := L_1 B_1, \tag{23}
\]

\[
\hat{B}_2 := L_1 \hat{B}_2, \quad C_{r1} := C_1 U_1, \tag{24}
\]

\[
[ \begin{array}{c} L_1^T \\ L_2^T \end{array} ]^T := \begin{bmatrix} U_1 & T_1 & T \end{bmatrix}^{-1}. \tag{25}
\]

where \(T, \hat{T}_2\) are given by (10), \(\hat{D}_{12}\) is given by (11), \(T_1\) is given by (15), and \(U_1\) is given by (19), respectively.
(ii) The following Riccati equation has a stabilizing solution $Y_r \geq 0$,
\[ Y_r(A_r - B_r D_{21} E_2^{-1} C_r) + (A_r - B_r D_{21} E_2^{-1} C_r) Y_r + Y_r(C_r^T C_r - C_r^T E_2^{-1} C_r) Y_r + B_r (I - D_{21} E_2^{-1} D_{21}) B_r^T = 0, \]
where $E_{21} := D_{21} D_{21}^T$, and
\[ A_r := U_1^T A L_1^T, \quad B_r := U_1^T B_1, \]
\[ C_r := C_2 L_1^T, \quad C_r := U_2 C_1, \]
\[ \begin{bmatrix} \bar{L}_1^T & \bar{L}_2^T & \bar{L}_3^T \end{bmatrix}^T := \begin{bmatrix} \bar{U}_1 \bar{T}_1 \bar{T}_2 \end{bmatrix}^{-1}, \]
where $\bar{T}_2$, $C_2$ are given by (13), $\bar{T}_{21}$ is given by (14), $\bar{T}_1$ is given by (16), and $U_1$ is given by (20), respectively.

(iii) $p(XY) < 1$, where
\[ X := L_1^T X L_1, \quad Y := L_2^T Y L_1. \]

(iv) $U_{12}^T U_{21} > X_{12}^T B_n^T B_n^T X_{12}$,
\[ U_{11}^T U_{21} > X_{11}^T C_n^T C_n^T X_{11}, \]
where $X_{12}$, $U_{12}$, $X_{11}$ and $U_{21}$ satisfy
\[ \begin{bmatrix} -jw I + A_n^T & C_n^T \\ B_n^T & N_{12} \end{bmatrix} \begin{bmatrix} X_{12} \\ U_{12} \end{bmatrix} = 0, \]
\[ \begin{bmatrix} -jw I + A_n & B_n \\ C_n & N_{21} \end{bmatrix} \begin{bmatrix} X_{11} \\ U_{11} \end{bmatrix} = 0, \]
where $A_n$, $B_n$, $B_{n1}$, $B_{n2}$, $C_n$, $C_{n1}$, $C_{n2}$ are defined by the following new plant as:
\[ P_n(s) = \begin{bmatrix} P_{n1} & P_{n2} \\ P_{n1}^T & P_{n2}^T \end{bmatrix} = \begin{bmatrix} A_n & B_{n1} & B_{n2} \\ C_{n1} & 0 & N_{12} \\ C_{n2} & N_{21} & 0 \end{bmatrix}, \]
with its matrices defined by
\[ A_n := A + B_1 B_1^T X + Z Y F_2 T_2 F_{nc}, \]
\[ B_{n1} := -Z L_{nc}, \quad B_{n2} := B_2 - Z Y F_2^T N_{12}, \]
\[ C_{n1} := -F_{nc}, \quad C_{n2} := (C_2 - N_{21} L_{nc}^T X) Z^{-1}, \]
\[ Z := (I - Y X)^{-1}, \]
\[ F_{nc} := -E_{12}^{-1/2} (B_2^T X + D_{12} C_1), \]
\[ L_{nc} := (-Y C_2^T + B_1 D_{21}^T) E_{12}^{-1/2}, \]
\[ N_{12} := E_{12}^{-1/2} D_{12}^T D_{12}, \]
\[ N_{21} := (E_{12}^{-1/2} D_{21} D_{21}^T)^T. \]

Moreover, if the $H^\infty$ control problem for plant $P(s)$ in (1) is solvable, then $(A_n, B_{n1}, C_n)$ is stabilizable and detectable.

Remark 3.1 Riccati equation (22) is of size $n - (n_1 - p) - n_j$, where $n_\infty := \sum_{j=1}^p k_j \geq p$ and $n_j$ are the dimensions of $\{\infty\}$ and $jw$-axis eigenspaces of $-s P_g + P_A$, respectively. Similar analysis can be given to Riccati equation (22). Note that $\omega_i$ satisfying (33) and/or (34) are the invariant zeros of $P_{n12}(s)$ and/or $P_{n21}(s)$. It can be shown such $\omega_i$ are also the invariant zeros of $P_{12}(s)$ and/or $P_{21}(s)$. Therefore, all conditions in Theorem 1 can be checked easily by solving two reduced-order Riccati equations and checking static conditions related to $jw$-axis zeros of $P_{12}(s)$ and/or $P_{21}(s)$.

Remark 3.2 If assumption (A2) holds, from (10), (11), (13) and (14), choose $V_\infty = \tilde{V}_h = \begin{bmatrix} 0 & I_p \end{bmatrix}$ and $\tilde{V}_\infty = \tilde{V}_h = \begin{bmatrix} 0 & I_q \end{bmatrix}$, we obtain $\tilde{B}_2 = B_2, \tilde{D}_{12} = D_{12}, \tilde{C}_1 = C_1, \tilde{D}_{21} = D_{21}$. If assumption (A3) holds, $(A_n)$ holds trivially, and Condition (iv) no longer exists. If both assumption (A2) and assumption (A3) hold, we can choose $L_1 = \tilde{L}_1 = \tilde{U}_1 = \tilde{L}_1$. Theorem 1 is reduced to the results of the standard $H^\infty$ control problems [1].

To establish the relation of Theorem 1 with [3], [9], in what follows, we obtain the explicit solution to the QMIs with rank constraints in those papers.

**Lemma 2** Under the assumptions (A1) and (A4), if Conditions (i)–(iv) in Theorem 1 hold, then

(i) $X, F_\infty$ and $N_{12}$ in (30), (40) and (42) satisfy
\[ \begin{bmatrix} X A + A^T X + X B_1 B_1^T X + C_1^T C_1 X B_2 + C_1^T D_{12} \\ B_1^T X + D_{12}^T C_1 \end{bmatrix} \begin{bmatrix} D_{12} \end{bmatrix}^T = \begin{bmatrix} -F_\infty^T \\ N_{12} \end{bmatrix}, \]
\[ \begin{bmatrix} -s I + A + B_1 B_1^T X & B_2 \\ -F_\infty & N_{12} \end{bmatrix} = n + p, \quad s \in C_+, \]
which implies that $P_{n12}(s)$ has no invariant zeros in $C_+$. Moreover, $P_{n12}(s)$ and $P_{12}(s)$ have the same finite $jw$-axis invariant zeros, and $(A_n, B_{n1})$ is stabilizable.

(ii) $Y, L_{nc}$ and $N_{21}$ in (30), (41) and (43) satisfy
\[ \begin{bmatrix} A Y + Y A^T + Y C_2^T C_1 Y & B_1 B_1^T Y C_2^T + B_1 D_{21}^T \\ C_2 Y + D_{21} B_1^T & D_{21} \end{bmatrix} \begin{bmatrix} D_{21} \end{bmatrix}^T = \begin{bmatrix} -L_\infty^T \\ N_{21} \end{bmatrix}, \]
\[ \begin{bmatrix} -s I + A + Y C_2^T C_1 & -L_\infty \\ C_2 & N_{21} \end{bmatrix} = n + p, \quad s \in C_+, \]
which implies that $P_{n21}(s)$ has no invariant zeros in $C_+$. Moreover, $P_{n21}(s)$ and $P_{21}(s)$ have the same finite $jw$-axis invariant zeros, and $(A_n, C_{n2})$ is detectable.

4 Proof of Necessary Conditions

We briefly introduce the following steps in the proof of the necessity of Theorem 1:

$$1967$$
Step 1 Prove Condition (i) via the solvability of the full information (FI) problem for $P(s)$.

Step 2 Perform lossless factorization $P(s) = \Theta(s) \ast P_{\text{tmp}}(s)$ to get a 2-block plant $P_{\text{tmp}}(s)$, where $\Theta(s)$ is an inner matrix.

Step 3 Prove Conditions (ii) and (iii) via the solvability of the FI problem corresponding to $P_{\text{tmp}}(s)$. This step is just a copy of Step 1.

Step 4 Perform the lossless factorization $P_{\text{tmp}}^T(s) = \Psi^T(s) \ast P_{\text{tmp}}^T(s)$ to get 1-block plant $P_{\text{n}}(s)$. This step is just a copy of Step 2.

Step 5 Prove Condition (iv) via the static solvability conditions related to the $jw$-axis zeros of $P_{\text{n}}(s)$.

4.1 FI Problem for $P(s)$

Since the $H^\infty$ control problem for the FI case of $P(s)$ in (1)

$$P_{FI}(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \end{bmatrix}$$

is solvable, we have

**LEMMA 3** Suppose the 4-block $H^\infty$ control problem is solvable. Then the FI $H^\infty$ control problem for

$$P_{FI}(s) = \begin{bmatrix} A_r & B_1 & B_2 \\ C_1 & 0 & D_{12} \end{bmatrix}$$

is solvable, where $A_r$, $B_1$ are defined in (27), $B_2$ is defined in (24), and $D_{12}$ is defined (11). Moreover,

$$P_{12}(s) = \begin{bmatrix} A_r & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

is a stabilizable realization and has no finite $jw$-axis invariant zeros.

Proof of Condition (i) of Theorem 1 is a direct consequence of Lemma 3 and the result of standard $H^\infty$ control problem in [1].

Now we can construct the following Riccati equation of size $n$, which will be used later.

**LEMMA 4** Suppose the 4-block $H^\infty$ control problem is solvable. Then

$$(A - B_2 E_{12}^{-1} \hat{D}_{12} C_1)X + X(A - B_2 E_{12}^{-1} \hat{D}_{12} C_1)^T + X(B_1 B_1^T - B_2 E_{12}^{-1} \hat{B}_2^T X + C_1^T (I - \hat{D}_{12} E_{12}^{-1} \hat{D}_{12}) C_1 = 0,$$

has a solution

$$X = L_1^T X L_1 = S^{-T} \begin{bmatrix} X_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1},$$

which yields

$$XT_1 = 0, \quad XT = 0.$$

4.2 Lossless Factorization of $P(s)$

**LEMMA 5** Suppose the 4-block $H^\infty$ control problem is solvable. Then $P(s)$ can be factorized as

$$P(s) = \Theta(s) \ast P_{\text{tmp}}(s),$$

where

$$\Theta(s) = \begin{bmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{bmatrix}$$

is a lossless matrix, i.e., $\Theta(s)^T \Theta(s) = I$, $\Theta(s) \in RH^\infty$ and $\Theta(s) \in RH^\infty$. Moreover, $(A + B_1 B_1^T X, B_2 + D_{12} B_1^T)$ is stabilizable and detectable.

According to Lemma 15 in [1], the solvability of $H^\infty$ control problems for $P(s)$ and $P_{\text{tmp}}(s)$ is equivalent with the same controller.

4.3 FI Problem for $P_{\text{tmp}}^T(s)$

Now we apply Condition (i) in Theorem 1 to

$$P_{\text{tmp}}^T(s) = \begin{bmatrix} A + B_1 B_1^T X & B_1 & B_2 \\ C_1 + D_{12} B_1^T X & 0 & D_{12} E_{12}^{-1/2} \\ I_r & 0 \end{bmatrix},$$

where $F_r := -E_{12}^{-1} (B_2 X_r + D_{12} C_r)$. And $\Theta(s) = I$ is a lossless matrix, i.e., $\Theta(s)^T \Theta(s) = I$, $\Theta(s) \in RH^\infty$ and $\Theta(s) \in RH^\infty$. Moreover, $(A + B_1 B_1^T X, B_2 + D_{12} B_1^T)$ is stabilizable and detectable.
\[
-B_1 \tilde{D}_1^T E_{21}^{-1} \tilde{C}_2 W + W (P_\infty^T F_\infty - C_2^T E_{21}^{-1} \tilde{C}_2) W
+ B_1 (I - \tilde{D}_2 E_{21}^{-1} \tilde{D}_2) B_1^T = 0
\]

has solution \( W \geq 0 \) with

\[
W \tilde{T}_1 = 0, \quad W \tilde{T} = 0.
\]

Consider

\[
Y(A - B_1 \tilde{D}_1^T E_{21}^{-1} \tilde{C}_2) + (A - B_1 \tilde{D}_1^T E_{21}^{-1} \tilde{C}_2) Y
+ Y(\tilde{C}_1^T - \tilde{C}_2^T E_{21}^{-1} \tilde{C}_2) Y + B_1 (I - \tilde{D}_2 E_{21}^{-1} \tilde{D}_2) B_1^T = 0.
\]

Let \( H_Y \) and \( H_W \) be the Hamiltonian matrices corresponding to Riccati equations (64) and (62), respectively. By direct calculation, we have

\[
H_W = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} H_Y \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}.
\]

Let

\[
\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} I & W \end{bmatrix}.
\]

Therefore,

\[
Y = Y_2 Y_1^{-1} = W(I + X W)^{-1} = (I + X W)^{-1} W \geq 0
\]

is a solution of (64). Since \( I + X W = (I - Y X)^{-1} = Z > 0 \), we get \( \rho(Y) \leq 1 \). Thus, \( Y = Z^{-1} W \). It yields from (63) that \( Y \tilde{T}_1 = 0 \) and \( Y \tilde{T} = 0 \). Then \( Y \) in (67) can be represented as

\[
Y = \tilde{S}^{-T} \begin{bmatrix} Y_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tilde{S}^{-1} = \tilde{L}_1^T Y_2 \tilde{L}_1,
\]

which follows that \( Y_2 \geq 0 \) is a solution of (26). Moreover, we can show that \( Y_2 \) is the stabilizing one.

\[ \square \]

4.4 Lossless Factorization of \( P_{\text{tmp}}^T (s) \)

From (62), define \( G^\infty \) corresponding to \( F_\infty \) in (40) as

\[
G_\infty^T := -E_{21}^{-1/2} (C_2 W + \tilde{D}_2 B_1^T).
\]

which follows that \( G^\infty_\infty = L_\infty^T Z^T \), where \( L_\infty^T \) is defined in (41). According to Lemma 5, the lossless factorization for \( P_{\text{tmp}}^T (s) \) is

\[
P_{\text{tmp}}^T (s) = \Psi^T (s) * P_n^T (s),
\]

where \( \Psi^T (s) \) is lossless matrix whose explicit form is omitted for the brevity, and \( P_n (s) \) is given by (35). Observe from Lemma 5 that \( (A_n, B_n, C_n) \) is stabilizable and detectable. Based on two lossless factorizations (54) and (70), we have

\[ \square \]

THEOREM 2 The solvability of \( H^\infty \) control problems for \( P(s) \) and \( P_n(s) \) is equivalent with the same controller \( K(s) \).

5 Proof of Sufficiency Conditions

We can first prove Lemma 2. Then, since \( P_n(s) \) is 1-block plant, \( P_{n12}(s) \) and \( P_{n21}(s) \) have no invariant zeros in \( C_+ \), according to Theorem 6 in [4], the matrix norm conditions related to \( j\omega \)-axis zeros are satisfied and two generalized Riccati equations have solutions of zero matrices. Therefore, the \( H^\infty \) control problem for \( P_n(s) \) is solvable, so is for plant \( P(s) \) according to Theorem 2.

\[ \square \]

References