QUADRATIC MINIMIZATION, POSITIVE REAL MATRICES AND SPECTRAL FACTORIZATION†

by

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SUMMARY

Connections between quadratic loss minimization problems, positive real matrices and spectral factorization are explored. Procedures for carrying out spectral factorizations of polynomials and matrices are developed which are based on the determination of the limiting solution of a Riccati equation. The procedures lead to spectral factors with stable inverses.

1. INTRODUCTION

The aim of this paper is to indicate interrelations between quadratic loss minimization problems, [1 - 3], positive real matrices [4], particularly when viewed in system theoretic terms, [5 - 7], and the spectral factorization problem, [8 -11]. Byproducts of the results are new techniques for carrying out a spectral factorization, and for generating certain matrices which are useful in some problems of nonlinear control system stability [12, 13] and network synthesis [14,15].

In section 2, quadratic loss minimization problems are considered where the loss function involves a term quadratic in the control together with a cross-product term between the control and the state; there is however no term quadratic in the state. It is shown by modifying procedures suggested in [3] that if and only if a certain matrix is positive real, a class of optimal control problems have solutions, and a certain matrix Riccati equation has a limiting solution with attendant special properties.

The treatment of that part of the optimization theory which has appeared elsewhere [1-3] is intentionally cursory; differences between the results here and those elsewhere are emphasised however.

In section 3, we consider the stability properties of closed loop systems with feedback law suggested by the material of section 2; a byproduct of this section is the solution of an infinite time interval quadratic loss problem.

In section 4, the results of the earlier sections are applied to deduce a spectral factorization procedure; this is related to other procedures discussed in the literature [8 - 11], and to earlier algebraic descriptions of the positive real property [5 - 7]. The new procedure relies on finding the limiting solution of a matrix Riccati equation, and leads to a spectral factor which is stable, together with its inverse.

2. A CLASS OF QUADRATIC MINIMIZATION PROCEDURES

Systems described by

$$\dot{x} = Fx + Gu \tag{1}$$

will be considered, together with performance indices of the form

$$V(x_0,u,t_0,t_1) = \int_{t_0}^{t_1} (u'Ru + 2x'Hu)dt \tag{2}$$

where the superscript prime denotes matrix transposition. The matrices F, G, H and $R = R'$ are constant and, as the notation implies, the value of the performance index is a function of the initial state x_0, the control u, and the endpoints of the integration interval t_0, t_1. Observe though that $V(x_0,u,t_0,t_1)$ is a function of $t_1 - t_0$ in the sense that with $v(t) = u(t-t_0)$,

$$V(x_0,u,t_0,t_1) = V(x_0,v,0,t_1-t_0) \tag{3}$$

In the sequel, a great deal of use will be made of the concept of a rational positive real matrix: a matrix $Z(s)$ of real rational functions of a complex variable s is termed (rational) positive real if [4]

(a) The elements of $Z(s)$ are analytic in $\text{Re } s > 0$, and have only simple poles on the line $\text{Re } s = 0$, with the matrix of residues at any such pole being nonnegative definite hermitian.

(b) The matrix $Z(j\omega) + Z'^{*}(j\omega)$ is nonnegative definite hermitian for all real ω such that $j\omega$ is not a pole of any element of $Z(\cdot)$.

A well-known result in network theory is that the impedance matrix, if it exists, of an electric network of passive elements comprising resistors, inductors, capacitors, transformers and gyrators is rational positive real. Moreover, a necessary and sufficient condition for a square matrix $Z(s)$ to be positive real can be stated as a "passivity condition", [4]: let $y_0(\cdot)$ be a vector function such that

$$y_0(t) = \int_{t_{-1}}^{t} z(t-\tau)u(\tau)d\tau \qquad \text{for all } t \geq t_{-1} \tag{4}$$

where $z(\cdot)$ is the inverse Laplace transform of $Z(s)$, and $u(\cdot)$ is an arbitrary finite-valued vector function. Then $Z(s)$ is positive real if and only if

$$\int_{t_{-1}}^{T} y_0'(t)u(t)dt \geq 0 \qquad \text{for all } u(\cdot),\ t_{-1},\ T \geq t_{-1} \tag{5}$$

The optimization problem of minimizing (2) given (1) is related to the concept of a positive real matrix via

<u>Lemma 1</u> Suppose the system (1) is completely controllable for all t_0, t_1 and x_0; then the minimum value with respect to u of the performance index (2), assuming the minimum exists, is nonpositive for all t_0, t_1 and x_0. A necessary and

sufficient condition for the performance index to be bounded below for all t_0, t_1, x_0 and u, independently of t_0, t_1, and u, is that

$$Z(s) = \frac{R}{2} + H'(sI-F)^{-1}G \tag{6}$$

be positive real.

<u>Proof</u> Setting u identically zero in (1) and (2) leads to a performance index of zero, independently of t_0, t_1 and x_0. Thus the minimum value, if it exists, is nonpositive.

As a preliminary to the proof of the remainder of the lemma, define

$$y = \frac{R}{2} u + H'x \tag{7}$$

Then y has the property, following from (1) and (6), that with $x(t_{-1})$ zero, equations (4) and (5) hold with $y(t) = y_0(t)$. Also,

$$u'Ru + 2x'Hu = 2y'u \tag{8}$$

allows rewriting of (2).

Now suppose $Z(s)$ is not positive real. We shall show that (2) may be made arbitrarily negative by appropriate choice of t_0, t_1 and u. Let x_0 be arbitrary and let u_1 be a control taking the state

x_0 at time t_0 to the state zero at time $t_\alpha \geq t_0$. Existence of u_1 and t_α follows by complete controllability. Let u_2 be a control such that, for some t_1

$$\int_{t_\alpha}^{t_1} y_2'(t)u_2(t)dt < 0 \qquad (9)$$

where $y_2(\cdot)$ is related to $u_2(\cdot)$ via (4), with t_α replacing t_{-1}. Such a $u_2(\cdot)$ and t_1 must exist, else by the remarks above, $Z(s)$ would be positive real. Denote by u the concatenation of u_1 and ku_2, where k is a constant, and denote by y_1 the quantity $\frac{R}{2}u_1 + H'x$ on the interval $[t_0,t_\alpha]$. Then

$$V(x_0,u,t_0,t_1) = \int_{t_0}^{t_\alpha} (2y_1'u_1)dt + k^2 \int_{t_\alpha}^{t_1} (2y_2'u_2)dt \qquad (10)$$

The first term is independent of k, while the second may be made arbitrarily negative, because of (9), by appropriate choice of k. So $V(x_0,u,t_0,t_1)$ is not bounded below independently of u, t_0 and t_1.

Now suppose $Z(\cdot)$ is positive real. Let u_1 now be a control taking the zero state at any arbitrary time $t_{-1} < t_0$ to the state x_0 at time t_0; existence of u_1 follows by complete controllability. Then if u is equal to u_1 on $[t_{-1}, t_0]$ and is otherwise arbitrary,

$$V(x_0,u,t_0,t_1) = \int_{t_{-1}}^{t_1} 2y'udt - \int_{t_{-1}}^{t_0} 2y'udt \qquad (11)$$

The second term on the right hand side depends on u_1, but is independent of the values assumed by u on $[t_0,t_1]$. Since u_1 depends on x_0, we can in fact write

$$V(x_0,u,t_0,t_1) = \int_{t_{-1}}^{t_1} 2y'u\,dt + K(x_0) \tag{12}$$

where $K(x_0)$ is nonpositive, by virtue of (5), for any x_0. Also it follows from (5) that the first term on the right of (12) is nonnegative. Hence

$$V(x_0,u,t_0,t_1) \geq K(x_0) \tag{13}$$

which is the desired result.

We now turn to the minimization of (2). For loss functions of the form $x'Qx + u'Ru$ with $Q = Q' \geq 0$ (nonnegative definite) and $R = R' > 0$ (positive definite) the problem has been extensively discussed [1 - 3], and the solution is well known. For this reason, we shall only give a brief sketch of the similar procedure applying to (2).

We define, with p the adjoint variable,

$$H(x,p,t,u) = u'Ru + 2x'Hu + 2p'Fx + 2p'Gu \tag{14}$$

and observe that, if R is positive definite symmetric,

$$u^0(t) = -R^{-1}G'p - R^{-1}H'x \tag{15}$$

minimizes (14) for fixed x,p,t and variable u. With this minimizing $u(\cdot)$ inserted into (14), there obtains the Hamiltonian

$$\mathcal{H}^0(x,p,t) = -p'GR^{-1}G'p - x'HR^{-1}H'x + 2p'(F-GR^{-1}H')x \tag{16}$$

with which may be associated a Hamilton-Jacobi equation

$$\frac{\partial V^0(x(t),t)}{\partial t} + \mathcal{H}^0(x(t), \frac{\partial V^0(x(t),t)}{\partial x}, t) = 0 \tag{17a}$$

with initial condition

$$V^0(x(t_1),t_1) = 0 \qquad \text{for all} \quad x(t_1) \tag{17b}$$

It is then easily checked that (17) has a solution of the form

$$V^0(x(t),t) = x'(t)\Pi(t,t_1)x(t) \tag{18}$$

if and only if $\Pi(\cdot,t_1)$ satisfies the Riccati differential equation

$$-\dot{\Pi} = \Pi(F-GR^{-1}H') + (F'-HR^{-1}G')\Pi - \Pi GR^{-1}G'\Pi - HR^{-1}H' \tag{19a}$$

with initial condition

$$\Pi(t_1,t_1) = 0 \tag{19b}$$

As the Hamilton-Jacobi theory points out, $V^0(x(t),t)$ will yield the minimum value of $V(x(t),u,t,t_1)$ with respect to $u(\cdot)$.

At this stage, lemma 1 and the preceding remarks can be used to establish the following result:

Theorem 1 With the same hypothesis as lemma 1 and with R positive definite symmetric, a necessary and sufficient condition for the minimum value of the performance index (2) to be finite-valued and attainable for all finite t_0, t_1 is that $Z(s)$ as defined in lemma 1 be positive real. The performance index is then given by

$$V^0(x_0,t_0,t_1) = x_0'\Pi(t_0,t_1)x_0 \tag{20}$$

where $\Pi(t_0,t_1)$ is the value at time t_0 of the solution of the differential equation (19a) with initial condition (19b).

Proof If $Z(s)$ is not positive real, it follows from lemma 1 that for some x_0, t_0, t_1, the performance index can be made arbitrarily negative by appropriate choice of $u(\cdot)$. A finite-valued minimum is then not obtained for all t_0, t_1. So suppose $Z(s)$ is positive real. If (19) has a well-defined solution for all $t \leq t_1$, the theorem follows through the Hamilton-Jacobi theory [3]; equation (15) with $p(t)$ replaced by $\Pi(t,t_1)x(t)$ gives the optimal control (see [3]) and (18) with t set equal to t_0 gives the optimum performance index.

The only difficulty that may arise is that the solution of (19) may become infinite at some time $t < t_1$; however the bounds obtained in lemma 1 can be used to show that this situation is impossible, analogously again to [3]. Thus with $Z(s)$ positive real, the solution of (19) is well-defined for all $t \leq t_1$, and all t_1.

Notice that the optimal control $u_0(t)$, obtained by replacing p in (15) by $\frac{\partial V^0(x(t),(t)}{\partial x}$, is

$$u_0(t) = -R^{-1}[G'\Pi(t,t_1) + H']x \tag{21}$$

Of interest is the limiting behaviour of solutions of (19). We have

<u>Theorem 2</u> With the same hypothesis as lemma 1, with $Z(s)$ as defined in lemma 1 positive real and with R positive definite symmetric, the following limit exists

$$\lim_{t_1\to\infty} \Pi(t,t_1) = \bar{P} \tag{22}$$

and is independent of t; moreover

$$\bar{P}(F-GR^{-1}H') + (F'-HR^{-1}G')\bar{P} - \bar{P}GR^{-1}G'\bar{P} - HR^{-1}H' = 0 \tag{23}$$

<u>Proof</u> Let u_{01} be the control which is optimum for initial state x_0, and range of integration $[t_0,t_1]$. Let the final state of the resulting trajectory be x_1, and let u_{12} be the control that is optimum for initial state x_1, and range of integration $[t_1,t_2]$. Let u_{012} be the concatenation of u_{01} and u_{12}. Then

$$\begin{aligned} x_0'\Pi(t_0,t_2)x_0 &= V^0(x_0,t_0,t_2) < V(x_0,u_{012},t_0,t_2) \\ &= V^0(x_0,t_0,t_1) + V^0(x_1,t_1,t_2) \\ &= x_0'\Pi(t_0,t_1)x_0 + V^0(x_1,t_1,t_2) \end{aligned} \tag{24}$$

The term $V^0(x_1,t_1,t_2)$ in (24) is zero or negative by lemma 1. Thus

$$x_0'\Pi(t_0,t_2)x_0 \leq x_0'\Pi(t_0,t_1)x_0 \tag{25}$$

Inequality (25) and the existence of a lower bound for arbitrary x_0 on $x_0'\Pi(t_0,t_1)x_0$ from lemma 1 and theorem 1 then imply after a slightly nontrivial argument that $\lim_{t_1\to\infty}\Pi(t_0,t_1)$ exists. Moreover since Π is actually a function of $t_1 - t_0$, it is possible to write

$$\lim_{t_1\to\infty}\Pi(t_0,t_1) = \lim_{t_0\to-\infty}\Pi(t_0,t_1) = \bar{P} \tag{26}$$

where $\bar{P}$ is a constant matrix. To show that $\bar{P}$ satisfies a limiting version of (19), i.e. that (23) holds, we proceed as follows. Let $\Pi(t,t_1,A)$ denote a solution of (19a) with initial condition $\Pi(t_1,t_1,A) = A$, for A any symmetric matrix. The notation $\Pi(t,t_1)$ will continue to mean $\Pi(t,t_1,0)$. Then the semigroup property of differential equations leads to

$$\Pi(t,t_1,\Pi(t_1,t_2)\ ) = \Pi(t,t_2) \tag{27}$$

Since any solution of (19a), if it exists, is continuous in the initial condition, it follows that

$$\begin{aligned}\bar{P} &= \lim_{t_2\to\infty}\Pi(t,t_2) = \lim_{t_2\to\infty}\Pi(t,t_1,\Pi(t_1,t_2)\ ) \\ &= \Pi(t,t_1,\ \lim_{t_2\to\infty}\Pi(t_1,t_2)\ ) = \Pi(t,t_1,\bar{P})\end{aligned} \tag{28}$$

Thus an initial condition $P(t_1) = \bar{P}$ leads to a constant solution of (19a) viz. $\bar{P}$, and thus (23) holds.

From lemma 1 and theorem 1 it follows that $\Pi(t,t_1)$ is nonpositive definite symmetric for all $t < t_1$; the same property must then be true of $\bar{P}$. A condition guaranteeing strict negative definiteness is as follows:

Theorem 3 With the same hypothesis as theorem 2, a necessary and sufficient condition for the matrix $\bar{P}$ defined in Theorem 2 to be negative definite is that $[F,H]$ be completely observable.

Proof The proof relies on the result, established in [1], that the optimal control for (1) which minimizes (2) is unique. (Note: though this reference considers the case where the loss function is of the form $x'Qx + u'Ru$, with $Q = Q' \geq 0$, the proof goes through unaltered.)

First, suppose $\bar{P}$ is singular. Then there exists a nonzero x_0 for which $x_0'\bar{P}x_0 = 0$; theorem 1 and equation (25) yield that $\bar{P} \leq \Pi(t_0,t_1) \leq 0$ for all t_0, t_1 and so $x_0\Pi(t_0,t_1)x_0$ is zero for all t_0, t_1. Moreover, since the optimal performance index for (2) is zero, and the control $u = 0$ leads to a performance index of zero, the control $u = 0$ must be the optimal control since it is unique. The state of the system at time t is then $x(t) = e^{F(t-t_0)}x_0$, and as the optimal control over $[t,t_1]$ is zero, it follows that

$$\min_u V(e^{F(t-t_0)}x_0,u,t,t_1) = x_0'e^{F'(t-t_0)}\Pi(t,t_1)e^{F(t-t_0)}x_0 \qquad (29)$$

is zero, which in turn implies that $\Pi(t,t_1)e^{F(t-t_0)}x_0$ is zero. Now the optimal control at time t is given, see (21) by

$$w_0(t) = -R^{-1}[G'\Pi(t,t_1) + H']e^{F(t-t_0)}x_0 \tag{30}$$

and thus, since this is zero and $\Pi(t,t_1)e^{F(t-t_0)}x_0$ is zero, it follows that $H'e^{F(t-t_0)}x_0$ is zero for all t, establishing lack of complete observability.

Conversely, suppose $[F,H]$ is not completely observable; then there exists x_0 such that $H'e^{F(t-t_0)}x_0$ is zero for all t, and thus $y(t)$ in (9) will be identical, for arbitrary u, with the $y(t)$ which would be observed if x_0 were zero. Then $V(x_0,u,t_0,t_1) = \int_{t_0}^{t_1} 2y'u\,dt$ for arbitrary u will be identical with $V(0,u,t_0,t_1)$. The minimum value of the latter is obviously zero, and is achieved when u is identically zero. Hence $x_0'\Pi(t_0,t_1)x_0$ is zero, and since this is independent of t_0 and t_1, it follows that $x_0'\bar{P}x_0$ is zero, proving the theorem.

It will be noticed that theorems 1 and 2 provide a new test for a square matrix $Z(\cdot)$ with $Z(\infty)$ positive definite and each entry rational to be positive real. From a completely controllable (and, if desired, completely observable) realization of $Z(\cdot)$, the Riccati equation (19) is formed and solved backwards. Divergence of the solution indicates $Z(\cdot)$ is not positive real, boundedness (and indeed convergence) indicates $Z(\cdot)$ is positive real.

To conclude this section, several examples will be given. We consider the transfer function.

$$Z(s) = \frac{1}{2} + \frac{a}{s+b} \tag{31}$$

which is positive real so long as $b \geq 0$ and $a \geq -\frac{1}{2}b$. Make the identifications $F = -b$, $G = 1$, $H = a$ and $R = 1$. Then the Riccati equation (19) may be solved analytically. Some typical results are:

(a) For $a = 1$, $b = -1$ (thus $Z(\cdot)$ is not positive real), $\Pi(t,t_1) = -\tan(t_1-t)$ which is not bounded below for all $t < t_1$.

(b) For $a = 1$, $b = 0$ (thus $Z(\cdot)$ is positive real), $\Pi(t,t_1) = -1 + 1/(t_1-t+1)$, and $\bar{P} = -1$.

(c) For $a = 1$, $b = 1$, (thus $Z(\cdot)$ is positive real), $\Pi(t,t_1) = (2 + \sqrt{3})[\exp(2\sqrt{3}(t_1-t)) - 1][1-(2 + \sqrt{3}) \times (2 - \sqrt{3})^{-1}\exp(2\sqrt{3}(t_1-t)]^{-1}$ and $\bar{P} = -(2 - \sqrt{3})$.

(d) For $a = 1$, $b = 1$ (thus $Z(\cdot)$ is positive real), $\Pi(t,t_1) = 0$ and $\bar{P} = 0$.

In cases (b) and (c), the pair $[F,H]$ is completely observable and $\bar{P}$ is nonsingular; in case (d) the pair $[F,H]$ is not completely observable, and $\bar{P}$ is singular.

3. STABILITY PROPERTIES

The optimal control for the performance index (2) is

$$u(t) = -R^{-1}[G'\Pi(t,t_1) + H']x(t) \tag{21}$$

so that the closed loop system is

$$\dot{x} = [F-GR^{-1}(G'\Pi(t,t_1) + H')]x \tag{32}$$

In the limit as t_1 approaches infinity, the closed loop system is time-invariant being

$$\dot{x} = [F-GR^{-1}(G'\bar{P} + H')]x \tag{33}$$

and the question arises as to what are the stability properties of the closed loop system. This is relevant in considering the minimization of $V(x_0,u,t_0,\infty)$, or more accurately $\lim_{t_1\to\infty} V(x_0,u,t_0,t_1)$, for, as will be shown, asymptotic stability of the closed loop system implies that $x_0'\bar{P}x_0$ is the minimum value of the infinite time performance index, with optimal control law given by replacing $\Pi(t,t_1)$ in (21) by $\bar{P}$.

We shall first demonstrate conditions for (33) to be stable, as distinct from asymptotically stable, and then present extra conditions on $Z(s)$ to ensure asymptotic stability.

<u>Theorem 4</u> With the same hypothesis as theorem 2, with $\bar{P}$ defined as in theorem 2, and with $[F,H]$ completely observable, the system (33) is stable.

<u>Proof</u> For convenience, we shall denote the matrix $F-GR^{-1}(G'\bar{P}+H')$ by $\hat{F}$. It then follows from (19) and (23) that

$$\frac{d}{dt}(\Pi-\bar{P}) = -(\Pi-\bar{P})\hat{F} - \hat{F}'(\Pi-\bar{P}) + (\Pi-\bar{P})GR^{-1}G'(\Pi-\bar{P})$$

with $\Pi(t_1,t_1) - \bar{P} = -\bar{P}$. Since $\Pi(t,T)$ approaches $\bar{P}$ as t approaches $-\infty$, it follows that the equation

$$\dot{W} = -W\hat{F} - \hat{F}'W + WGR^{-1}G'W \tag{34c}$$

with

$$W(t_1) = -\bar{P} \tag{34b}$$

has the property that $\lim_{t\to-\infty} W(t) = 0$. Consequently the equation

$$\dot{X} = X\hat{F} + \hat{F}'X - XGR^{-1}G'X \tag{35a}$$

with

$$X(0) = -\bar{P} \tag{35b}$$

has the property that $\lim_{t\to+\infty} X(t) = 0$. Theorem 3 guarantees existence of $(-\bar{P})^{-1}$ as a positive definite matrix, and it then follows that $Y(t) = X^{-1}(t)$ exists for all t and is the solution of

$$\dot{Y} = -\hat{F}Y - Y\hat{F}' + GR^{-1}G' \tag{36a}$$

(derived from (35a)), with initial condition

$$Y(0) = (-\bar{P})^{-1} \tag{36b}$$

Now for any real vector m, it is true that $[m'X^{-1}(t)m][m'X(t)m] \geq (m'm)^2$, see [16 p.69], and since $\lim_{t\to+\infty} X(t) = 0$, it follows that $m'Y(t)m$ diverges as $t\to+\infty$ for any real m.

We shall now show that if $\hat{F}$ has an eigenvalue with positive real part, then there exists m such that $m'Y(t)m$ is bounded. This will then establish by contradiction the stability of $\hat{F}$. First suppose there is a real positive eigenvalue λ of $\hat{F}$; let m be an associated eigenvalue. Then multiplying (36a) on the left by m' and the right by m, and setting $y(t) = m'Y(t)m$ and $q = m'GR^{-1}G'm$, there obtains

$$\dot{y}(t) = -2\lambda y(t) + q \tag{37a}$$

with

$$y(0) = -m'\bar{P}^{-1}m \tag{37b}$$

The solution of (37) is obviously bounded for all t.

If $\hat{F}$ has no real positive eigenvalue but a complex eigenvalue with positive real part, one can proceed similarly to show that there exists a complex m such that $[\mathrm{Re}\, m']Y(t)[\mathrm{Re}\, m]$ and $[\mathrm{Im}\, m']Y(t)[\mathrm{Im}\, m]$ are bounded. This proves the theorem.

If $[F,H]$ is not completely observable, it will be noted that the above proof breaks down, since P^{-1} cannot be formed; indeed it would be expected that if unobservable states were present, stability of the closed loop system would not follow unless the unobservable states were stable.

To secure a result regarding asymptotic stability, it is necessary to restrict $Z(\cdot)$ further: elements of $Z(\cdot)$ are required to be analytic in $\text{Re } s \geq 0$ rather than merely $\text{Re } s > 0$, and $Z(j\omega) + Z'^*(j\omega)$, which is now defined for all real ω, is required to be positive definite rather than merely nonnegative definite for all real ω. We then have

Theorem 5 With the same hypothesis as theorem 4 and the above two constraints, the system (33) is asymptotically stable.

Proof The proof proceeds in several stages; first, we show that F is asymptotically stable, secondly that there exists a positive η such that $Z(s) - \frac{\eta}{2} I$ is positive real, thirdly that the control obtained by replacing Π in (21) by $\bar{P}$ is square integrable on $[t_0, \infty)$, and finally that the desired result holds.

The asymptotic stability of F follows immediately from the fact that no element of $Z(s)$ has poles in $\text{Re } s \geq 0$, and simultaneously $[F,G]$ is completely controllable, and $[F,H]$ completely observable.

Referring to the positive real definition and the constraint on poles of elements of $Z(s)$, it will be noted that the only way $Z(s) - \frac{\eta}{2} I$ can fail to be positive real is that $Z(j\omega) + Z'(-j\omega) - \eta I$ fails to be nonnegative definite for some real ω. Since $\lim_{\omega \to \infty} Z(j\omega) = \frac{R}{2}$, which is positive definite symmetric, it follows that there exists some real ω_0 such that for all real ω with $|\omega| > \omega_0$, $Z(j\omega) + Z'(-j\omega) - \frac{R}{2}$ is nonnegative definite. For all ω with

$|\omega| < \omega_0$, $Z(j\omega) + Z'(-j\omega)$ is known to be positive definite, and since $[-\omega_0, \omega_0]$ is a finite interval, it follows that there exists $\epsilon > 0$ such that $Z(j\omega) + Z'(-j\omega) - \epsilon I$ is nonnegative definite for all ω such that $|\omega| < \omega_0$. Then $\eta = \min\{\epsilon, \lambda_{min}(\frac{R}{2})\}$ is such that $Z(s) - \frac{\eta}{2} I$ is positive real.

Now denote by u_{0t_1} the control in (21) and by $u_{0\infty}$ the control obtained by replacing $\Pi(t,t_1)$ by $\bar{P}$. Let u_c be a control taking the zero state at an arbitrary time $t_{-1} < t_0$ to the state x_0 at time t_0. Let u denote the concatenation of u_c and u_{0t_1}. Then because $\frac{1}{2}(R-\eta I) + H'(sI-F)^{-1}G$ is positive real,

$$\int_{t_{-1}}^{t_1} [u'(R-\eta)u + 2x'Hu]dt \geq 0$$

or

$$\int_{t_{-1}}^{t_0} (u_c'Ru_c + 2x'Hu_c)dt + x_0'\Pi(t_0,t_1)x_0$$
$$\geq \int_{t_0}^{t_1} \eta u_{01}'u_{01}dt$$

For any $t_\alpha < t_1$, clearly

$$\int_{t_{-1}}^{t_0} (u_c'Ru_c + 2x'Hu_c)dt + x_0'\Pi(t_0,t_1)x_0$$
$$\geq \eta \int_{t_0}^{t_\alpha} u_{01}'u_{01}dt \tag{38}$$

Now let t_1 approach infinity. Because the range of integration on the right side of (38) is finite, the limiting value is readily computed. Likewise the limit of the left side is readily computed to yield

$$\int_{t_{-1}}^{t_0} (u_c'Ru_c + 2x'Hu_c)dt + x_0'\bar{P}x_0 \geq \eta \int_{t_0}^{t_\alpha} u_{0\infty}'u_{0\infty}dt \tag{39}$$

Since t_α can be taken arbitrarily large, it follows that $u_{0\infty}$ is square integrable.

The closed loop system may be written as

$$\dot{x} = Fx + Gu_{0\infty} \tag{40}$$

whence it follows from the asymptotic stability of F and the square integrability of $u_{0\infty}$ that it is asymptotically stable.

Illustrations of theorems 4 and 5 are provided by the examples at the end of section 2. For example (b), $Z(s) = \frac{1}{2} + \frac{1}{s}$ and has a pole on $\text{Re } s = 0$; hence stability rather than asymptotic stability is expected, and with $\bar{P} = -1$, it is verified that the closed loop system is $\dot{x} = 0$. For example (c), $Z(s) = \frac{1}{2} + \frac{1}{1+s}$, and theorem 5 predicts asymptotic stability; the closed loop system resulting here is $\dot{x} = -\sqrt{3}x$. A further example is provided by $Z(s) = \frac{1}{2} - \frac{1/2}{1+s}$, for which $\text{Re } z(0) = 0$. Solution of the Riccati equations leads to $\bar{P} = -\frac{1}{2}$; the closed loop system is then $\dot{x} = 0$.

The natural question arises as to whether in theorem 5 the two conditions imposed on $Z(\cdot)$ in addition to those required by theorem 4 are necessary, as well as sufficient, to guarantee asymptotic stability of the closed-loop system. Indeed this is the case, as will be seen subsequently.

Meanwhile theorem 5 allows consideration of the infinite time minimization problem:

<u>Corollary to theorem 5</u> With the same hypothesis as in theorem 5, the optimal control minimizing

$$V(x_0,u,t_0,\infty) = \lim_{t_1\to\infty} \int_{t_0}^{t_1} (u'Ru + 2x'Hu)dt \tag{41}$$

is obtained by replacing Π in (24) by $\bar{P}$, and the optimal performance index is $x_0'\bar{P}x_0$.

<u>Proof</u> Consider the formula

$$V(x_0,u_{0_\infty},t_0,t_1) = x_0'\bar{P}x_0 - x'(t_1)\bar{P}x(t_1)$$

which follows from (23), (1), (2), and the definition of u_{0_∞} after a little manipulation. Letting t_1 approach infinity and using the stability of the closed loop yields $\lim_{t_1\to\infty} V(x_0,u_{0_\infty},t_0,t_1) = x_0'\bar{P}x_0$. To demonstrate optimality of u_{0_∞}, suppose there exists u^* for which $\lim_{t_1\to\infty} V(x_0,u^*,t_0,t_1) < x_0'\bar{P}x_0$. Then for sufficiently large t_1,

$$\begin{aligned} V(x_0,u^*,t_0,t_1) &< x_0'\bar{P}x_0 \leq x_0'\Pi(t_0,t_1)x_0 \\ &= \min_u V(x_0,u,t_0,t_1) \end{aligned} \tag{43}$$

where the second inequality follows by letting t_2 approach infinity in (25). Equation (43) is a contradiction, and thus the corollary is established.

Notice that if the closed loop system is stable rather than asymptotically stable, the limit in (41) with u set equal to u_{0_∞} may not even be well defined.

4. CONNECTIONS WITH ALGEBRAIC CHARACTERIZATION OF POSITIVE REAL MATRICES AND SPECTRAL FACTORIZATION

In this section, we discuss the relation of the preceding two sections to material in a number of other references [5 - 11]. To begin with, we note

Lemma 2 Let $Z(s)$ be a positive real matrix with $Z(s) = \frac{R}{2} + H'(sI-F)^{-1}G$, the pair $[F,G]$ completely controllable, and $R = R' > 0$. Let $\bar{P}$ be as defined in Theorem 2. Then a matrix $W(s)$ satisfying

$$Z(s) + Z'(-s) = W'(-s)W(s) \tag{44}$$

is given by

$$W(s) = R^{\frac{1}{2}} + R^{-\frac{1}{2}}(\bar{P}G + H)'(sI-F)^{-1}G \tag{45}$$

where $R^{\frac{1}{2}}$ is the unique positive definite square root of R.

Proof With $W(s)$ as in (45), we have

$$\begin{aligned} W'(-s)W(s) = R &+ (\bar{P}G + H)'(sI-F)^{-1}G + G'(-sI-F')(\bar{P}G + H) \\ &+ G'(-sI-F')^{-1}(P\bar{G} + H)R^{-1}(P\bar{G} + H)'(sI-F)^{-1}G \end{aligned}$$

Now observe that from (23), it follows that

$$(\bar{P}G + H)R^{-1}(\bar{P}G + H)' = \bar{P}F + F'\bar{P}$$

so that

$$\begin{aligned} G'(-sI-F')^{-1}(\bar{P}G + H)R^{-1}(PG + H)'(sI-F)^{-1}G \\ = G'(-sI-F')^{-1}[-\bar{P}(sI-F) - (-sI-F')\bar{P}](sI-F)^{-1}G \\ = -G'(-sI-F')^{-1}\bar{P}G - G\bar{P}(sI-F)^{-1}G \end{aligned}$$

Then

$$\begin{aligned} W'(-s)W(s) &= R + H'(sI-F)^{-1}G + G'(-sI-F')^{-1}H \\ &= Z'(-s) + Z(s) \end{aligned}$$

as required.

The "spectral factorization" of (44), that is, the determination of $W(s)$ from $Z(s)$, has been extensively discussed by Youla [8] amongst others from the frequency domain point of view. State-space interpretations of some of these results may be found in [5], [6] and [7].

The first of these references uses Youla's result that $W(s)$ is uniquely defined to within multiplication on the left by an arbitrary constant orthogonal matrix if $W(s)$ is required to have entries analytic in $\text{Re } s > 0$ and to be of constant rank in $\text{Re } s > 0$. Then [5] shows that for an arbitrary positive real $Z(s) = J + H'(sI-F)^{-1}G$ with the pair F, H completely observable, but J not necessarily symmetric positive definite, the particular $W(s)$ described above has the form $W_0 + L'(sI-F)^{-1}G$ for some L and W_0.

The second of these references [6], in effect does the same thing; however there is an error in the statement of the main result which is as follows. Equation (44) is considered not merely for positive real $Z(\cdot)$, but for any transfer function $Z(\cdot)$ such that $Z(j\omega) + Z'(-j\omega)$ is nonnegative definite for all real ω such that $j\omega$ is not a pole of any element of $Z(\cdot)$. With $Z(s) = J + H'(sI-F)^{-1}G$, it is in effect claimed that there is a solution of (44) of the form $W(s) = W_0 + L'(sI-F)^{-1}G$ for some L and W_0 which has elements analytic in the half plane $\text{Re } s > 0$. But since $Z(\cdot)$ is permitted to have an element with a pole in $\text{Re } s > 0$, this is plainly impossible.

The third reference [7] considers nonpositive real $Z(\cdot)$ for which $Z(j\omega) + Z'(-j\omega)$ is still nonnegative definite for all real ω such that $j\omega$ is not a pole of any element of $Z(\cdot)$. There it is shown that with $Z(s) = J + H'(sI-F)^{-1}G$ and $[F,G]$ completely controllable, there is a solution of (44) of the form $W(s) = W_0 + L'(sI-F)^{-1}G$ for some L and W_0. Note that it is not required that $[F, H]$ be completely observable, and that the result of reference [5] discussed above is extended in the sense of exhibiting a certain solution to (44) when $[F, H]$ is not completely observable; however it is no longer guaranteed that $W(s)$ has constant rank in $\text{Re } s > 0$.

In [5] and [7], as a consequence of the state-space descriptions of $Z(\cdot)$ and $W(\cdot)$ discussed above, the following result is established.

<u>Lemma 3</u> Let $Z(\cdot)$ be a matrix of rational function of the complex variable s with $Z(\infty) < \infty$. Suppose $Z(s) = J + H'(sI-F)^{-1}G$, with $[F, G]$ completely controllable. Then if $Z(\cdot)$ is positive real, there exist real matrices $P = P' \geq 0$, L and W_0 such that

$$PF + F'P = -LL' \tag{46a}$$

$$PG = H - LW_0 \tag{46b}$$

$$W_0'W_0 = J + J' \tag{46c}$$

Moreover in (46) a necessary and sufficient condition for P to be nonsingular is that $[F, H]$ be completely observable.

For the case when $Z(\infty)$ is positive definite symmetric (and, by a straightforward extension, when $Z(\infty) + Z'(\infty)$ is positive ... symmetric), the above lemma can be recovered from theorem 2 by setting $P = -\bar{P}$, $L = R^{-\frac{1}{2}}(\bar{P}G + H)$ and $W_0 = R^{\frac{1}{2}}$, with theorem 3 guaranteeing that P is nonsingular if and only if $[F, H]$ is completely observable.

Further relation with [5] and [8] is provided by examining the rank of $W(s)$ in $\text{Re } s > 0$. It is straightforward to verify that $W^{-1}(s)$ exists and is given by

$$W^{-1}(s) = R^{-\frac{1}{2}} - R^{-1}(\bar{P}G + H)[sI-F + GR^{-1}(PG + H)']^{-1}GR^{-\frac{1}{2}} \tag{47}$$

from which it is evident that any pole of an element of $W^{-1}(s)$ is an eigenvalue of $F-GR^{-1}(\bar{P}G + H)'$.

Moreover, if each element of $W^{-1}(s)$ is written as a polynomial in s divided by $\det[sI-F + GR^{-1}(PG + H)']$, without cancellations being made, then any eigenvalue of $F-GR^{-1}(PG + H)'$ is a pole of each element of $W^{-1}(s)$.

From lemma 2 and theorems 4 and 5 it then follows that

Theorem 6 Let $Z(\cdot)$ be a positive real matrix with $Z(s) = \frac{R}{2} + H'(sI-F)^{-1}G$, the pair $[F, G]$ completely controllable, the pair $[F, H]$ completely observable and $\bar{P}$ defined as in theorem 2. Then

$$W(s) = R^{\frac{1}{2}} + R^{-\frac{1}{2}}(\bar{P}G + H)'(sI-F)^{-1}G \tag{45}$$

which is a solution of (44), is nonsingular throughout $\text{Re } s > 0$; if $Z(\cdot)$ has no element with a pole in $\text{Re } s \geq 0$ and if $Z(j\omega) + Z'(-j\omega)$ is positive definite for all real ω, then $W(s)$ is nonsingular in $\text{Re } s \geq 0$.

From Youla's results, we know that $W(s)$ is unique to within multiplication on the left by an orthogonal matrix, and so, at least in the case where $Z(\infty) + Z'(\infty)$ is positive definite symmetric, we have a new procedure for calculating this $W(s)$, (together with the matrices P, L and W_0 of lemma 3).

Theorem 5 established sufficient conditions for $F-GR^{-1}(\bar{P}G + H)'$ to have eigenvalues of negative real part. We can now show these conditions to be sufficient.

<u>Theorem 5′</u> Let $Z(\cdot)$ be a positive real matrix with $Z(s) = \frac{R}{2} + H'(sI-F)^{-1}G$, the pair $[F, G]$ completely controllable, the pair $[F, H]$ completely observable and $\bar{P}$ defined as in theorem 2. If either

(i) $Z(j\omega) + Z'(-j\omega)$ is singular for some real ω, with $j\omega$ not a pole of any element of $Z(\cdot)$, or

(ii) $Z(\cdot)$ has an element with a pole on $\text{Re } s = 0$

then $F-GR^{-1}(\bar{P}G + H)'$ has an eigenvalue with zero real part.

<u>Proof</u> (i) Suppose $Z(j\omega_0) + Z'(-j\omega_0)$ is singular with ω_0 real (and $j\omega_0$ not a pole of any element of $Z(\cdot)$). Then from (44) $W'(-j\omega_0)W(j\omega_0)$ and thus $W(j\omega_0)$ is singular. From (47), $j\omega_0$ is an eigenvalue of $F-GR^{-1}(\bar{P}G + H)'$.

(ii) Suppose $Z(\cdot)$ has at least one element with a pole on $\text{Re } s = 0$. Then $Z(\cdot)$ may be decomposed as

$$Z(s) = Z_1(s) + Z_2(s) \qquad (48)$$

with $Z_1(s)$ positive real and no element possessing a pole on $\text{Re } s = 0$, and $Z_2(s)$ lossless positive real, so that $Z_2(s) + Z_2'(-s) = 0$. This is a well-known result of network theory, see [4]. Then there exists $W_1(s)$ such that (i) the set of poles of elements of $W_1(\cdot)$ is the same as the set of poles of elements of $Z_1(\cdot)$, (all of which are in $\text{Re } s < 0$),

(ii) $W_1(s)$ is nonsingular in $\text{Re } s \geq 0$, and (iii)

$$W_1'(-s)W_1(s) = Z_1(s) + Z_1'(-s) = Z(s) + Z'(-s) \tag{49}$$

the second equality following from (48) and the lossless nature of $Z_2(\cdot)$. Let $W(s)$ be as in (45). Then from Youla's result, $W_1(s)$ and $W(s)$ differ (at least when all elements in each matrix are expressed in fractions with no common factors in numerator and denominator) by multiplication on the left by a constant orthogonal matrix. Now

$$\begin{aligned}\det[sI-F + GR^{-1}(\bar{P}G + H)']^{-1} &= \det[sI-F]\det(I + (sI-F)^{-1}GR^{-1}(\bar{P}G + H)'] \\ &= \det[sI-F]\det[I + R^{-1}(PG + H)'(sI-F)^{-1}G] \\ &= \det[sI-F]\det R^{\frac{1}{2}} \det W(s) \qquad \text{[see (45)]} \\ &= K \det[sI-F] \det W_1(s)\end{aligned}$$

where K is $\pm \det R^{\frac{1}{2}}$, the sign depending on the orthogonal matrix relating $W(s)$ to $W_1(s)$. Let $j\omega_0$ with ω_0 real be an eigenvalue of F. Then since $W_1(\cdot)$ is analytic at $j\omega_0$, it follows that $j\omega_0$ is an eigenvalue of $F-GR^{-1}(\bar{P}G + H)'$. This proves the theorem.

In [9] a further approach is discussed to the determination of a $W(s)$ satisfying (44) and having the form $W_0 + L'(sI-F)^{-1}G$ when $Z(s) = \frac{R}{2} + H'(sI-F)^{-1}G$. Initially, no constraints are placed upon the analyticity in $\text{Re } s > 0$ or $\text{Re } s \geq 0$ of the elements of $W^{-1}(s)$.

The approach taken is to find any solution $\bar{P}$ of (23), i.e. $\bar{P}$ is not restricted to being the limiting solution of an associated Riccati equation. Then (45) provides a $W(s)$ satisfying (44), since as may be checked in the proof of lemma 2, the only requirement on $\bar{P}$ in order that (44) holds is that it satisfy the quadratic equation (23).

Algorithms are presented for solving (23), and for solving (23) subject to the additional constraints that the eigenvalues of $F-GR^{-1}(\bar{P}G + H)'$ should have nonpositive real part. A key step in the algorithms is the determination of the eigenvalues of a certain matrix with the property that if λ is an eigenvalue, so is $-\lambda$; in effect the algorithm requires spectral factorization of a polynomial. Thus the Riccati equation approach is the only technique of those mentioned which genuinely avoids polynomial factoring.

The treatment via modern optimal control methods of well-known problems tackled with Wiener-Hopf theory, and the appearance of Riccati equations in the modern methods, of course suggested the strong connection between spectral factorization (which underpins the Wiener-Hopf theory) and Riccati equations. An attempt to systematically use the Riccati equation to achieve a spectral factorization of a polynomial $C(\omega^2)$, nonnegative for all real ω, is made in [10]. In this reference, a problem of the type

$$\text{Minimize} \lim_{t_1\to\infty} \int_{t_0}^{t_1} (u^2 + x'Qx)dt \quad \text{subject to } \dot{x} = Fx + gu$$

$$x(t_0) = x_0, \quad x(t_1) = 0 \qquad (\text{where } Q = Q' \text{ and is indefinite})$$

is considered, and with F, g and Q appropriately chosen and depending in a certain way on $C(\omega^2)$; it is claimed that the optimization has a solution if and only if $C(\omega^2)$ is nonnegative for all real ω. The solution is derivable by finding the limiting solution of a Riccati equation, and knowledge of this solution yields a spectral factorization of $C(\omega^2)$, that is, a polynomial $d(s)$ such that $C(-s^2) = d(-s)d(s)$. It is moreover guaranteed that there is no zero of $d(s)$ in the right half plane.

Reference [11] also discusses connections between polynomial spectral factorization and quadratic minimization, but does not consider the solution of a polynomial spectral factorization problem via solving a Riccati equation.

The theory of the preceding sections can be used simply to provide a spectral factorization of both a polynomial $C(\omega^2)$, nonnegative for all real ω, and a matrix $D(\omega^2)$ of rational functions of ω^2 nonnegative definite for all real ω. Spectral factorization of the latter, requiring the determination of a matrix $E(s)$ of rational functions of s such that $D(-s^2) = E'(-s)E(s)$, first requires a polynomial spectral factorization.

Polynomial factorization proceeds by choosing a square matrix F with eigenvalues in $\text{Re } s < 0$ of the same dimension as the degree of $C(\omega^2)$ in ω^2. Then one forms $C(-s^2)/\det(sI-F)\det(-sI-F)$ and performs a partial fraction expansion to obtain a polynomial $C_1(s)$ of degree equal to the dimension of F such that

$$\frac{C(-s^2)}{\det(sI-F)\det(-sI-F)} = \frac{C_1(s)}{\det(sI-F)} + \frac{C_1(-s)}{\det(-sI-F)} \tag{50}$$

The nonnegativity of $C(\omega^2)$ for all real ω and the eigenvalue restriction on F mean that $z(s) = C_1(s)/\det(sI-F)$ is positive real; the degree of $C_1(s)$ is such that $z(\infty)$ is finite. Using lemma 2, a rational function $w(s)$ can be found such that

$$\frac{C(-s^2)}{\det(sI-F)\det(-sI-F)} = w(-s)w(s) \tag{51}$$

and then

$$C(-s^2) = [\det(-sI-F)w(-s)][\det(sI-F)w(s)] \tag{52}$$

yields a spectral factorization.

The matrix spectral factorization problem is discussed in [9], where it is shown how to reduce the problem, using a polynomial spectral factorization of the lowest common denominator of the elements of a prescribed $D(\omega^2)$, to the problem of finding a $W(s)$ satisfying (44) when $Z(s)$ is a known positive real matrix, with $Z(\infty)$ positive definite symmetric. Lemma 2 then provides the requisite solution.

5. CONCLUSIONS

The first contribution to the paper is to present necessary and sufficient conditions for the solution of a class of quadratic loss optimal control problems, in terms of a transfer function matrix. Then the paper goes on to discuss properties of the associated optimal performance indices, especially limiting properties. Next, the stability and asymptotic stability are examined of a time-invariant closed-loop control system with feedback law suggested by the optimal control results.

The spectral factorization problem is then considered, and one solution deduced by application of the optimal control results. This solution is compared with others to the same problem, especially those which depend but little on frequency domain ideas.

Besides the interrelation of various works and ideas in the paper, which we will not discuss further here, we note the constructive procedures suggested at various points. First, in section 2, a procedure is given for testing a square matrix of rational functions for positive realness; second, in section 4, a procedure is given for carrying out a spectral factorization, with a spectral factor that is stable (sometimes asymptotically stable) together with its inverse; third, a procedure is given for carrying out a polynomial spectral factorization; finally, a procedure is given for generating a matrix (the matrix P in lemma 3) which has proved exceptionally useful in other areas [12 - 15].

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